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MODELING THE EFFECT OF TECHNOLOGY ON THE CONSERVATION OF FORESTRY RESOURCE BIOMASS DEPLETED BY INDUSTRIALIZATION, POPULATION PRESSURE AND TOXICANTS

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Abstract: In this paper, we investigate a mathematical model to conserve the resource biomass that is depleted by industrialization, population pressure and toxicants with the help of technology. The model equations are analyzed mathematically with regard to the nature of equilibrium points and their stabilities using the theory of nonlinear ordinary differential equations and numerical simulations. It is shown that under suitable conditions, there exists a unique locally as well as globally asymptotically stable positive equilibrium. It is concluded from the analysis that density of resource biomass increases if technology is used to conserve it. Moreover, it is observed that for large depletion rate of resource biomass due to population, resource biomass goes to extinction if no technology is used for its conservation. However, resource biomass does not become extinct for the same depletion rate of resource biomass due to population if technology is applied to conserve the resource biomass. It is found that concentration of toxicants in the environment can be reduced significantly if technology is applied to conserve resource biomass.

Keywords: Resource biomass, industrialization, toxicants, technology, population.

1. Introduction

Resources play a significant role in the development of any country. It is a matter of great concern that the resources such as forestry, wildlife, energy, minerals are continuously being depleted to meet the demands of overgrowing population. Depletion of forestry resources occurs for agricultural land, resettlement and colonization, cutting of trees for fuel, paper and fodder, etc. An example of extensive depletion of forestry resources is the depletion of forests of Doon valley in Uttarakhand. Forests have been depleted largely in Doon valley due to overgrowth of human and livestock populations, limestone quarrying, wood based industries and various kinds of industrial discharges and chemical spills in the forms of smoke and poisonous gas fumes (Munn and Fedorov, 1986; Shukla et al. 1989, Shukla and Dubey, 1996, 1997).

Many researchers have investigated the depletion of resource biomass by overgrowing population, toxicants and industrialization. Shukla and Dubey (1997) have studied the effects of population and pollution on the depletion of forestry resources. They found that if the population density and the emission rate of pollutant increase without control, the forestry resource may tend to extinction. Shukla et al. (1989) have proposed a mathematical model to study the effect of industrialization on the depletion of a resource biomass. Dubey and Dass (1999) have proposed and analyzed a mathematical model to study the survival of species dependent on a resource, which is depleted due to industrialization. Shukla et al. (1996) investigated the effect of changing habitat on survival of species due to industrialization. Dubey and Narayanan (2010) have studied a mathematical model to demonstrate the effect of industrialization, population, and pollution on the depletion of a renewable resource. They observed that if the density of industrialization, population, and pollution increase then the density of the resource biomass decreases. M. Agarwal et al. (2010) proposed a ratio dependent mathematical model on the depletion of forestry resource biomass due to industrialization pressure. They found that the density of forestry biomass decreases due to increase in industrialization pressure that decreases the density of wildlife species. Gakkhar and Sahani (2007) proposed a delay model to determine the effects of environmental toxicant on biological species. Thomas et al. (1997) studied the effect of environmental pollution on a single-species population and derived some criteria to restrict the amount of pollution in the environment to ensure the survival of the population. Shukla et al. (2003) studied the effects of primary and secondary toxicants on the resource biomass. They observed that the resource may even become extinct if emission rate of primary toxicant and its transformation rate to secondary toxicant are very large. Dubey and Hussain (2003) studied a model with diffusion. They considered competition between two species that compete with each other and depend on a common resource. The resource considered by them is affected by industrialization. They demonstrated that the positive equilibrium can be stabilized by increasing diffusion coefficients. Dubey et al. (2003) analyzed the effect of industrialization and pollution on forestry resource. They considered three types of rate of emission of pollutant into the environment: industrialization dependent, constant, zero, or periodic. They concluded that resource biomass may become extinct in case of industrialization-dependent emission rate of pollutants. Shukla et al. (2011) studied the effect of technology on the conservation of forestry resource biomass. However, they did not consider the effect of toxicants on the depletion of resource biomass. Role of toxicants on the depletion of renewable resources like forests is considerable. They concluded that the resource biomass density decreases due to over growing population and industrialization. Resource biomass density was observed to decrease further as the resource-dependent industrial migration increases. They found that resource does not become extinct if some technology is applied for its conservation. It may be pointed out that in all the above-mentioned studies; effect of technology on the conservation of resource biomass that is depleted by combined effect of industrialization, population pressure and toxicants is not studied. As we all know that forests occupy central position in nature. They restore ecological balance of all ecosystems, maintain biological diversity, act as catchments for soil and water conservation, prevent floods and safeguard future of tribal people. Hence, sustainable management of forestry resource biomass is desirable. Sustainable management of forests is possible by using modern technologies such as genetic engineering like tissue culture and clonal seedlings, root-trainers etc. for new varieties of trees for plantation in forests. It is therefore very essential to study the effect of technology on conservation of forestry resources (Reed and Heras, 1992). We modify the paper of Dubey and Narayanan (2010) by considering the effect of technology on the conservation of forestry resource biomass keeping in mind the importance and efficacy of using technology in the conservation of resource biomass.

We, therefore, analyze a nonlinear ordinary differential equation model to investigate the efficacy of technology on the conservation of resource biomass. The stability theory of nonlinear ordinary differential equations and fourth order Runge–Kutta method are used to analyze and predict the behavior of the model.

2. Mathematical Model

We formulate a nonlinear ordinary differential system of equations to study an ecosystem where forestry resource biomass is being continuously depleted due to industrialization, population and toxicants. We also study the impact of technological effort applied to conserve the resource biomass that is the main objective of our paper. Resource biomass, industrialization and population grow according to the logistic law. Intrinsic growth rate of resource biomass is assumed to be a negative function of population. It is further assumed that density of resource biomass decreases due to industrialization and increases due to technology applied for its conservation. Moreover, it is assumed that carrying capacity of resource biomass in the environment, that is, maximum density of resource biomass that the environment can sustain decrease with the increase in industrialization, population and toxicants. Industrialization increases due to resource biomass since establishment of industries need resource biomass. Industrialization also increases due to increase in population to meet their demands. In addition, growth rate of population increases with the increase in density of resource biomass. Carrying capacities of industrialization and population is assumed to be constant. Emission of toxicants in the environment is assumed to occur due to industrialization as well as due to human population. We incorporate the effort applied to conserve the resource biomass through technology to balance the ecosystem. The rate at which technological effort is applied depends upon the resource biomass left that can be conserved. Further, depletion of technology due to technology failure is also considered in the model. Keeping these things in mind, we have the following system of differential equations:

$$
\begin{aligned}
\frac{dB}{dt} &= r_B(P)B - \frac{r_{B0}B^2}{K_B(I,P,T)} - \alpha IB + \phi BM, \\
\frac{dI}{dt} &= r_2 I\left(1 - \frac{I}{L}\right) + \beta IB + \gamma_1 IP, \\
\frac{dP}{dt} &= r_P(B)P - \frac{r_{P0}P^2}{S}, \\
\frac{dT}{dt} &= Q(I,P) - \delta_0 T - \alpha_1 BT, \\
\frac{dM}{dt} &= \eta(K_{B0} - B) - \eta_0 M, \\
&B(0) \geq 0,\ I(0) > 0,\ P(0) > 0,\ T(0) \geq 0,\ M(0) \geq 0.
\end{aligned} \tag{2.1}
$$

Here B is the density of resource biomass, I is the density of industrialization, P is the density of population, T is the concentration of toxicant present in the environment and M is the technological effort applied to conserve resource biomass. α is the depletion rate coefficient of resource biomass due to industrialization, γ is the growth rate of coefficient of industrialization due to industries being set up by population to meet the increasing demands of overgrowing population. $0 < \phi < 1$ is the growth rate coefficient of resource biomass due to technological efforts. r_2 is the intrinsic growth rate of industrialization and β is the growth rate of industrialization due to resource biomass. L is the maximum density of industries which the environment can support. S is the maximum density of population which the environment can support. δ_0 is the natural depletion rate of toxicants. α_1 is the depletion rate coefficient of the toxicants due to its uptake by forestry resource biomass. constant η is the growth rate coefficient of technological efforts and η_0 is the depletion rate coefficient of technological efforts due to failure of technology.

In addition, $r_B(P)$ denotes the intrinsic growth rate of the resource biomass. It is assumed that $r_B(P)$ decreases as P increase. Hence we take,

$$r_B(0) = r_{B0} > 0,\ \frac{dr_B}{dP} < 0 \text{ for } P \geq 0. \tag{2.2}$$

$K_B(I,P,T)$ represents the maximum density of the resource biomass which the environment can support in the presence of industrialization, population and pollution, and it also decreases as I, P and T increase. Hence, we take

$$K_B(0,0,0) = K_{B0} > 0,\ \frac{\partial K_B(I,P,T)}{\partial I} < 0,\ \frac{\partial K_B(I,P,T)}{\partial P} < 0,\ \frac{\partial K_B(I,P,T)}{\partial T} < 0 \text{ for } I \geq 0,\ P \geq 0,\ T \geq 0. \tag{2.3}$$

$r_P(B)$ represents the intrinsic growth rate of population, and it is assumed that intrinsic growth rate of population increases as the density of the resource biomass increases. Hence,

$$r_P(0) = r_{P0} > 0, \ \frac{dr_P}{dB} \geq 0 \ \text{ for } \ B \geq 0. \tag{2.4}$$

The rate of introduction of pollutant into the environment is denoted by $Q(I,P)$ that increases as I and P increase. Hence, we take function $Q(I,P)$ of the following form:

$$Q(0,0) = Q_0 > 0, \ \frac{\partial Q(I,P)}{\partial I} > 0, \ \frac{\partial Q(I,P)}{\partial P} > 0 \ \text{ for } I \geq 0, \ P \geq 0. \tag{2.5}$$

All the functions considered in the model are assumed to be sufficiently smooth so that solutions to the initial value problem exist uniquely and are continuable for all positive times. We will analyse the model using stability theory of differential equations.

3. Boundedness

In this section, we show that solutions of model system (2.1) are bounded. Boundedness of the system is proved in the form of following lemma 1, which establishes the region of attraction (Freedman and So, 1985).

Lemma 1. The set

$$\Omega = \{(B,I,P,T,M): 0 \leq B \leq K_{B0}, \ 0 \leq I \leq I_{\max}, \ 0 \leq P \leq P_{\max}, \ 0 \leq T \leq T_{\max}, 0 \leq M \leq M_{\max}\}$$

where,

$$I_{\max} = \frac{(r_2 + \beta K_{B0} + \gamma_1 P_{\max})L}{r_2},$$

$$P_{\max} = \frac{r_P(K_{B0})S}{r_{P0}},$$

$$T_{\max} = \frac{Q(I_{\max}, P_{\max})}{\delta_0},$$

$M_{\max} = \dfrac{\eta K_{B0}}{\eta_0}$, is the region of attraction for system (2.1) that attracts all solutions initiating in the interior of positive orthant.

Proof. From fifth equation of the system (2.1) we have,

$$\frac{dM}{dt} \leq \eta(K_{B0}) - \eta_0 M$$

$$\Rightarrow 0 \leq M \leq \frac{\eta K_{B0}}{\eta_0} = M_{\max}.$$

Using this value of M in first equation of the system (2.1), we have,

$$\frac{dB}{dt} \leq r_{B0} B\left(1 - \frac{B}{K_{B0}}\right) + \phi BM, \tag{3.1}$$

We observe that when $B \to K_{B0}$, $M \to 0$ and all the terms of the right hand side of the (3.1) tend to zero suggesting that $\frac{dB}{dt} \leq 0$ for $B \geq K_{B0}$.

This implies that $B \to K_{B0}$ for large t. Thus, we have $0 \leq B \leq K_{B0}$.

From second equation of the system (2.1), we get

$$\frac{dI}{dt} \leq r_2 I\left(1 - \frac{I}{L}\right) + \beta I K_{B0} + \gamma_1 I P_{\max}$$

$$\Rightarrow \frac{dI}{Idt} \leq r_2 + \beta K_{B0} + \gamma_1 P_{\max} - \frac{r_2}{L} I$$

$$\Rightarrow 0 \leq I \leq \frac{(r_2 + \beta K_{B0} + \gamma_1 P_{\max})L}{r_2} = I_{\max}.$$

From third equation of the system (2.1) we have,

$$\frac{dP}{dt} \leq r_P(K_{B0})P - \frac{r_{P0}P^2}{S},$$

$$\Rightarrow 0 \leq P \leq \frac{r_P(K_{B0})S}{r_{P0}} = P_{\max}.$$

From fourth equation of system (2.1), we have

$$\frac{dT}{dt} \leq Q(I_{\max}, P_{\max}) - \delta_0 T,$$

$$\Rightarrow 0 \leq T \leq \frac{Q(I_{\max}, P_{\max})}{\delta_0} = M_{\max}.$$

Thus, we have proved the lemma 1 and hence bounded of the system .

4. Equilibrium Analysis

It is observed that the system (2.1) has eight non-negative equilibria, namely,

$E_0(0,0,0,T_0,M_0)$, $E_1(0,0,P_1,T_1,M_1)$, $E_2(0,I_2,0,T_2,M_2)$, $E_3(0,I_3,P_3,T_3,M_3)$,

$E_4(B_4,0,0,T_4,M_4)$, $E_5(B_5,0,P_5,T_5,M_5)$, $E_6(B_6,I_6,0,T_6,M_6)$, $E_7(B^*,I^*,P^*,T^*,M^*)$.

Here we have, $T_0 = \dfrac{Q_0}{\delta_0}$, $M_0 = M_1 = M_2 = M_3 = \dfrac{\eta K_{B0}}{\eta_0}$,

$P_1 = P_3 = S$, $T_1 = \dfrac{Q(0,S)}{\delta_0}$, $I_2 = L$,

$$T_2 = \frac{Q(L,0)}{\delta_0},\ I_3 = \frac{(r_2+\gamma_1 S)L}{r_2},\ T_3 = \frac{Q\left(\dfrac{(r_2+\gamma_1 S)L}{r_2}, S\right)}{\delta_0}.$$

Existence of $E_4(B_4,0,0,T_4,M_4)$,

Here B_4 T_4, M_4 are the positive solutions of the following algebraic equations:

$$r_{B0} - \frac{r_{B0}B}{K_B(0,0,T)} + \phi M = 0, \tag{4.1}$$

$$Q_0 - \delta_0 T - \alpha_1 BT = 0 \tag{4.2}$$

$$\eta(K_{B0} - B) - \eta_0 M = 0 \tag{4.3}$$

From (4.2) we have,

$$T = \frac{Q_0}{\delta_0 + \alpha_1 B} = f_{11}(B), \tag{4.4}$$

Similarly (4.3) gives,

$$M = \frac{\eta}{\eta_0}(K_{B0} - B) = f_{12}(B). \tag{4.5}$$

Putting the values of T and M in equation (4.1) we have a function $F_1(B)$ of the following form,

$F_1(B) = r_{B0}B - \{r_{B0} + \phi M\}K_B(0,0,T)$, such that

$F_1(0) = -\{r_{B0} + \phi M\}K_B(0,0,T) < 0$ and

$F_1(K_{B0}) = r_{B0}K_{B0} - r_{B0}K_B(0,0,T) > 0$ since $K_{B0} > K_B(I,P,T)$ from our assumption.

It is easy to observe that a unique equilibrium B_4 lies in $(0, K_{B0})$ if the following inequality holds:

$$F_1'(B) = r_{B0} - \{r_{B0} + \phi f_1(B)\}\frac{\partial K_B}{\partial T}\frac{df_2(B)}{dB} - K_B(0,0,f_2(B))\phi\frac{df_1(B)}{dB} > 0. \tag{4.6}$$

Putting the value of B_4 in (4.4) and (4.5), we can easily obtain the equilibrium values T_4 and M_4.

Existence of $E_5(B_5, 0, P_5, T_5, M_5)$,

Here B_5, P_5, T_5, M_5 are the positive solutions of the following algebraic equations:

$$r_{B0}B = \{\phi M + r_B(P)\}K_B(0,P,T), \tag{4.7}$$

$$P = \frac{r_P(B)S}{r_{P0}} = f_{14}(B), \tag{4.8}$$

$$T = \frac{Q(0, f_4(B))}{\delta_0 + \alpha_1 B} = f_{15}(B), \tag{4.9}$$

$$M = \frac{\eta}{\eta_0}(K_{B0} - B) = f_{12}(B). \tag{4.10}$$

Similar to the existence of $E_4(B_4, 0, 0, T_4, M_4)$, It is easy to observe that a unique equilibrium B_5, lies in $(0, K_{B0})$ if the following inequality holds:

$$\begin{aligned} F_2'(B) = r_{B0} - \{r_B(f_{14}(B)) + \phi f_{12}(B)\}\left\{\frac{\partial K_B}{\partial P}\frac{df_{14}(B)}{dB} + \frac{\partial K_B}{\partial T}\frac{df_{14}(B)}{dB}\right\} \\ + K_B(0, f_{14}(B), f_{15}(B))\left\{\phi\frac{df_{12}(B)}{dB} + \frac{\partial r_B}{\partial P}\frac{df_{14}}{dB}\right\} > 0. \end{aligned} \tag{4.11}$$

Putting the value of B_5 in (4.8), (4.9) and (4.10), we can easily obtain the equilibrium values P_5, T_5 and M_5.

Existence of $E_6(B_6, I_6, 0, T_6, M_6)$,

Here B_6, I_6, T_6, M_6 are the positive solutions of the following algebraic equations:

$$r_{B0}B = \{\phi M + r_B(P) - \alpha I\}K_B(I,0,T), \tag{4.12}$$

$$I = \frac{L}{r_2}\left(r_2 + \beta B\right) = f_{16}(B), \tag{4.13}$$

$$T = \frac{Q\left(f_6(B),0\right)}{\delta_0 + \alpha_1 B} = f_{17}(B), \tag{4.14}$$

$$M = \frac{\eta}{\eta_0}(K_{B0} - B) = f_{12}(B). \tag{4.15}$$

Similarly, equilibrium $E_6(B_6, I_6, 0, T_6, M_6)$ exists if

$$\begin{aligned} F_3'(B) = r_{B0} - \{r_{B0} + \phi f_{12}(B) - \alpha f_{16}(B)\}\left\{\frac{\partial K_B}{\partial I}\frac{df_{16}(B)}{dB} + \frac{\partial K_B}{\partial T}\frac{df_{17}(B)}{dB}\right\} \\ - K_B\left(f_{16}(B), 0, f_{17}(B)\right)\left\{\phi\frac{df_{12}(B)}{dB} - \alpha\frac{df_{16}}{dB}\right\} > 0. \end{aligned} \tag{4.16}$$

Existence of interior equilibrium $E_7(B^*, I^*, P^*, T^*, M^*)$,

Here B^*, I^*, T^*, M^* are the positive solutions of the following algebraic equations:

$$r_{B0}B = \{\phi M + r_B(P) - \alpha I\}K_B(I,P,T), \tag{4.17}$$

$$I = \frac{L}{r_2}\left(r_2 + \beta B + \gamma_1 f_{14}(B)\right) = f_{18}(B), \tag{4.18}$$

$$P = \frac{r_P(B)S}{r_{P0}} = f_{14}(B), \tag{4.19}$$

$$T = \frac{Q\left(f_{18}(B), f_{14}(B)\right)}{\delta_0 + \alpha_1 B} = f_{19}(B), \tag{4.20}$$

$$M = \frac{\eta}{\eta_0}(K_{B0} - B) = f_{12}(B). \tag{4.21}$$

Taking

$$F_4(B) = r_{B0}B - \{\phi M + r_B(P) - \alpha I\}K_B(I,P,T),$$

$$= r_{B0}B - \{r_B(f_{14}(B)) + \phi f_{12}(B) - \alpha f_{18}(B)\}K_B(f_{18}(B), f_{14}(B), f_{19}(B)) \tag{4.22}$$

It can be checked easily that the equilibrium $E_7(B^*, I^*, P^*, T^*, M^*)$ exists if

$$\begin{aligned} F_4'(B) = r_{B0} - \{r_B(f_{14}(B)) + \phi f_{12}(B) - \alpha f_{18}(B)\}\left\{\frac{\partial K_B}{\partial I}\frac{df_{18}(B)}{dB} + \frac{\partial K_B}{\partial P}\frac{df_{14}(B)}{dB} + \frac{\partial K_B}{\partial T}\frac{df_{19}(B)}{dB}\right\} \\ - K_B(f_{18}(B), f_{14}(B), f_{19}(B))\left\{\frac{\partial r_B(P)}{\partial P}\frac{df_{14}}{dB} + \phi\frac{df_{12}(B)}{dB} - \alpha\frac{df_{18}}{dB}\right\} > 0. \end{aligned} \tag{4.23}$$

$$r_B(S) + \phi\frac{\eta}{\eta_0}K_{B0} > \alpha\frac{L}{r_2}(r_2 + \gamma S) \tag{4.24}$$

$$r_{B0}K_{B0} + \alpha\frac{L}{r_2}\left\{r_2 + \beta K_{B0} + \gamma\frac{r_P(K_{B0})S}{r_{P0}}\right\} > r_B\left(\frac{r_P(K_{B0})S}{r_{P0}}\right). \tag{4.25}$$

5. Stability Analysis

The local stability analysis of each equilibrium point can be studied by computing eigen values of the corresponding variational matrix from the general variational matrix. The general variational matrix of the system (2.1) is given as:

$$V(E) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \phi B \\ \beta I & a_{22} & \gamma_1 I & 0 & 0 \\ \dfrac{\partial r_p(B)}{\partial B} & 0 & a_{23} & 0 & 0 \\ -\alpha_1 T & \dfrac{\partial Q(I,P)}{\partial I} & \dfrac{\partial Q(I,P)}{\partial P} & -\delta_0 - \alpha_1 B & 0 \\ -\eta & 0 & 0 & 0 & -\eta_0 \end{bmatrix},$$

where

$$a_{11} = r_B(P) - 2r_{B0}\frac{B}{K_B(I,P,T)} - \alpha I + \phi M,$$

$$a_{12} = r_{B0}\frac{B^2}{K_B^2(I,P,T)}\frac{\partial K_B(I,P,T)}{\partial I} - \alpha B,$$

$$a_{13} = \frac{\partial r_B(P)B}{\partial P} + r_{B0}\frac{B^2}{K_B^2(I,P,T)}\frac{\partial K_B(I,P,T)}{\partial P},$$

$$a_{22} = r_2\left(1-\frac{I}{L}\right) - r_2\frac{I}{L} + \beta B + \gamma_1 P,$$

$$a_{23} = r_p(B) - 2r_{p0}\frac{P}{S}.$$

Local stability conditions for various equilibrium points of the system are as follows:

1. E_0 is a saddle point with stable manifold in the $T-M$ plane and with unstable manifold in the $B-I-P$ space.
2. E_1 is a saddle point with stable manifold in the $P-T-M$ plane and with unstable manifold in the $B-I$ plane.
3. E_2 is a saddle point with stable manifold in the $I-T-M$ plane and with unstable manifold in the $B-P$ plane.
4. E_3 is a saddle point with stable manifold in the $I-P-T-M$ plane and with unstable manifold in the B - direction.
5. E_4 is a saddle point with stable manifold in $B-T-M$ space and the unstable manifold in $I-P$ plane if $2r_{B0}\frac{B_4}{K_B(0,0,T_4)} > \max\left\{r_{B0} + \phi M_4 - \phi B_4\frac{\eta}{\eta_0}, r_{B0} + \phi M_4 - \eta_0\right\}$.
6. E_5 is also a saddle point with unstable manifold in the I - direction.
7. E_6 is unstable equilibrium according to Routh Hurwitz criteria if any of the following conditions do not hold:

   $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ and $\tilde{E}$ are positive

   $\tilde{A}\tilde{B} - \tilde{C} > 0$

$$(\widetilde{A}\widetilde{B}-\widetilde{C})\widetilde{C}-(\widetilde{A}\widetilde{D}-\widetilde{E})\widetilde{A}>0$$

$$(\widetilde{A}\widetilde{B}-\widetilde{C})(\widetilde{C}\widetilde{D}-\widetilde{B}\widetilde{E})-(\widetilde{A}\widetilde{D}-\widetilde{E})^2>0$$

where $\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D}$ and $\widetilde{E}$ are the coefficients of characteristic equation:

$\lambda^5+\widetilde{A}\lambda^4+\widetilde{B}\lambda^3+\widetilde{C}\lambda^2+\widetilde{D}\lambda+E=0$ of the variational matrix about E_6 and are given as below:

$$\widetilde{A}=\delta_0+\alpha_1 B_6+\eta_0-r_P(B)+r_{B0}\frac{B_6}{K_B(I_6,0,T_6)}+\frac{r_2 I_6}{L},$$

$$\widetilde{B}=(\delta_0+\alpha_1 B_6)\eta_0+(\delta_0+\alpha_1 B_6+\eta_0)\left(-r_P(B_6)+r_{B0}\frac{B_6}{K_B(I_6,0,T_6)}+\frac{r_2 I_6}{L}\right)$$

$$-r_P(B_6)\left(r_{B0}\frac{B_6}{K_B(I_6,0,T_6)}+\frac{r_2 I_6}{L}\right)+\frac{r_{B0}B_6 r_2 I_6}{K_B(I_6,0,T_6)L}-\frac{r_{B0}B_6^2\beta I_6}{K_B^2(I_6,0,T_6)}\frac{\partial K_B(I_6,0,T_6)}{\partial I}$$

$$+\alpha B_6\beta I_6-\frac{r_{B0}B_6^2\alpha_1 T_6}{K_B^2(I_6,0,T_6)}\frac{\partial K_B(I_6,0,T_6)}{\partial T}+\phi B_6,$$

$$\widetilde{C}=(\delta_0+\alpha_1 B_6)\eta_0\left(-r_P(B_6)+r_{B0}\frac{B_6}{K_B(I_6,0,T_6)}+\frac{r_2 I_6}{L}\right)$$

$$-r_P(B_6)(\delta_0+\alpha_1 B_6+\eta_0)\left(r_{B0}\frac{B_6}{K_B(I_6,0,T_6)}+\frac{r_2 I_6}{L}\right)-r_{B0}\frac{r_P(B_6)B_6 r_2 I_6}{K_B(I_6,0,T_6)L}-(\delta_0+\alpha_1 B_6+\eta_0)\beta I_6$$

$$+\beta I_6 r_P(B_6)-\frac{\partial r_P(B_6)\gamma_1 I_6}{\partial B}-\frac{r_{B0}B_6^2\beta I_6}{K_B^2(I_6,0,T_6)}\frac{\partial K_B(I_6,0,T_6)}{\partial T}\frac{\partial Q(I_6,0)}{\partial I}+\phi B_6\left(\delta_0+\alpha_1 B_6+\frac{r_2 I_6}{L}-r_P(B_6)\right)$$

$$-\left(\eta_0+\frac{r_2 I_6}{L}-r_P(B_6)\right)\frac{r_{B0}B_6^2\alpha_1 T_6}{K_B^2(I_6,0,T_6)}\frac{\partial K_B(I_6,0,T_6)}{\partial T},$$

$$\widetilde{D}=(\delta_0+\alpha_1 B_6)\eta_0\left\{-r_P(B_6)\left(r_{B0}\frac{B_6}{K_B(I_6,0,T_6)}+\frac{r_2 I_6}{L}\right)+r_{B0}\frac{B_6 r_2 I_6}{K_B(I_6,0,T_6)L}\right\}$$

$$
-(\delta_0+\alpha_1 B_6+\eta_0)\frac{r_P(B_6)r_{B0}B_6 r_2 I_6}{K_B(I_6,0,T_6)L}+\phi B\left\{(\delta_0+\alpha_1 B_6)\left(\frac{r_2 I_6}{L}-r_P(B_6)\right)-\frac{r_P(B_6)r_2 I_6}{L}\right\}
$$

$$
-\left\{(\delta_0+\alpha_1 B_6+\eta_0)\left(-\beta I_6 r_P(B_6)+\frac{\partial r_P(B_6)}{\partial B}\gamma_1 I\right)-\beta I_6(\delta_0+\alpha_1 B_6)\eta_0\right\}\left(\frac{r_{B0}B_6^2}{K_B^2(I_6,0,T_6)}\frac{\partial K_B(I_6,0,T_6)}{\partial I}-\alpha B_6\right)
$$

$$
-\frac{r_{B0}B_6^2\beta I_6}{K_B^2(I_6,0,T_6)}\frac{\partial K_B(I_6,0,T_6)}{\partial T}\frac{\partial Q(I_6,0)}{\partial T}(\eta_0-r_P(B_6))-\frac{r_{B0}B_6^2\alpha_1 T_6}{K_B^2(I_6,0,T_6)}\frac{\partial K_B(I_6,0,T_6)}{\partial T}\left\{\eta_0\left(\frac{r_2 I_6}{L}-r_P(B_6)\right)-r_P(B)\frac{r_2 I_6}{L}\right\}
$$

$$
-\frac{r_{B0}B_6^2\gamma_1 I_6}{K_B^2(I_6,0,T_6)}\frac{\partial K_B(I_6,0,T_6)}{\partial T}\frac{\partial r_P(B_6)}{\partial B}\frac{\partial Q(I_6,0)}{\partial I},
$$

$$
\widetilde{E}=-(\delta_0+\alpha_1 B_6)\eta_0\frac{r_P(B_6)r_{B0}B_6 r_2 I_6}{K_B(I_6,0,T_6)L}+\frac{\eta_0 r_P(B_6)r_{B0}B_6^2\beta I_6}{K_B^2(I_6,0,T_6)}\frac{\partial K_B(I_6,0,T_6)}{\partial T}\frac{\partial Q(I_6,0)}{\partial I}
$$

$$
-(\delta_0+\alpha_1 B_6)\eta_0\left(\frac{r_{B0}B_6^2}{K_B^2(I_6,0,T_6)}\frac{\partial K_B(I_6,0,T_6)}{\partial I}-\alpha B\right)\left(-\beta I_6 r_P(B)+\frac{\partial r_P(B_6)\gamma_1 I_6}{\partial B}\right)
$$

$$
+\eta_0\frac{r_P(B_6)r_{B0}B_6^2 r_2 I_6}{K_B^2(I_6,0,T_6)L}\frac{\partial K_B(I_6,0,T_6)}{\partial T}\alpha_1 T_6-\eta_0\frac{r_{B0}B_6^2\gamma_1 I_6}{K_B^2(I_6,0,T_6)}\frac{\partial K_B(I_6,0,T_6)}{\partial T}\frac{\partial r_P(B_6)}{\partial B}\frac{\partial Q(I_6,0)}{\partial I}
$$

$$
-\phi B_6 r_P(B_6)\frac{r_2 I_6}{L}(\delta_0+\alpha_1 B_6).
$$

Theorem 1: Let the following inequalities hold:

$$
c_1\gamma_1^2<\frac{4}{9}c_2\frac{r_2}{L}\frac{r_{P0}}{S}, \tag{5.1}
$$

$$
c_3\left(\frac{\partial Q(I^*,P^*)}{\partial I}\right)^2<\frac{4}{9}c_1 r_2\alpha_1\frac{B^*}{L}, \tag{5.2}
$$

$$
c_3\left(\frac{\partial Q(I^*,P^*)}{\partial I}\right)^2<\frac{4}{9}c_2 r_{P0}\alpha_1\frac{B^*}{S}, \tag{5.3}
$$

where,

$$c_1 = \frac{1}{\beta}\left\{ -\frac{r_{B0}B^*}{K_B^2(I^*,P^*,T^*)} \frac{\partial K_B(I^*,P^*,T^*)}{\partial I} + \alpha \right\} > 0, \tag{5.4}$$

$$c_2 = -\frac{1}{\frac{\partial r_P(B^*)}{\partial B}} \left(\frac{r_{B0}B^* K_{B2}}{K_B^2(I^*,P^*,T^*)} \frac{\partial K_B(I^*,P^*,T^*)}{\partial P} \right) > 0, \tag{5.5}$$

$$c_3 = \frac{1}{\alpha_1 T^*} \frac{r_{B0}B^*}{K_B^2(I^*,P^*,T^*)} \frac{\partial K_B(I^*,P^*,T^*)}{\partial T} > 0, \tag{5.6}$$

$$c_4 = \frac{\phi}{\eta}. \tag{5.7}$$

Then E_7 is locally asymptotically stable equilibrium point.

Proof of this theorem is stated in Appendix A.

Now we explore the conditions of global stability of the system (2.1) in the form of following theorem:

Theorem2. In addition to assumptions (2.2) –(2.5), let $r_B(P)$, $r_P(B)$, $K_B(I,P,T)$ and $Q(I,P)$ satisfy the following conditions in the region of attraction Ω:

$$K_m \le K_B(I,P,T) \le K_{B0},\ 0 \le -\frac{\partial K_B(I,P,T)}{\partial I} \le l_1,\ 0 \le -\frac{\partial K_B(I,P,T)}{\partial P} \le l_2,$$

$$0 \le -\frac{\partial K_B(I,P,T)}{\partial T} \le l_3, \qquad 0 \le -\frac{\partial r_P(B)}{\partial B} \le \rho_1,$$

$$0 \le -\frac{\partial r_B(P)}{\partial P} \le \rho_2,$$

$$0 \le -\frac{\partial Q(I,P)}{\partial I} \le \rho_3, \qquad 0 \le -\frac{\partial Q(I,P)}{\partial P} \le \rho_4, \tag{5.8}$$

for some positive constants K_m, l_1, l_2, l_3, ρ_1, ρ_2, ρ_3, ρ_4.

Then if the following inequalities hold in Ω

$$\left(\frac{r_{B0}K_{B0}l_1}{K_m^2}+\alpha+\beta\right)^2<\frac{r_{B0}r_2}{3K_B(I^*,P^*,T^*)L}, \tag{5.9}$$

$$\left(\rho_1+\frac{r_{B0}K_{B0}l_2}{K_m^2}+\rho_2\right)^2<\frac{r_{B0}r_{P0}}{3K_B(I^*,P^*,T^*)S}, \tag{5.10}$$

$$\left(\frac{r_{B0}K_{B0}l_3}{K_m^2}+\frac{1}{9}\frac{r_2}{L}\frac{(\delta_0+\alpha_1B^*)}{\rho_3^2}\alpha_1T_{\max}\right)^2<\frac{r_{B0}r_2(\delta_0+\alpha_1B^*)^2}{27L\rho_3^2K_B(I^*,P^*,T^*)}, \tag{5.11}$$

$$\gamma_1^2<\frac{4}{9}\frac{r_2}{L}\frac{r_{P0}}{S}, \tag{5.12}$$

$$\frac{r_2}{L}\frac{\rho_4^2}{\rho_3^2}<4\frac{r_{P0}}{S}, \tag{5.13}$$

E_7 is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

The proof of this theorem is given in Appendix B.

The above theorem implies that if conditions (5.6) – (5.10) hold forestry resource may settle down to its equilibrium level. Further, it may be pointed out that if $\alpha, \alpha_1, \beta, \gamma_1$ are kept at lower level then these conditions may be satisfied easily. This implies that stability of our system is more feasible when depletion rate coefficient of resource biomass due to industrialization, growth rate coefficient of industrialization due to easily available forestry resource biomass, depletion rate coefficient of toxicants from the environment due to its uptake by the forestry resource biomass and growth rate coefficient of industrialization due to increased population is kept at lower level.

6. Numerical Simulation

In the previous sections, the qualitative analysis of the model is presented. To study the system numerically we consider following particular forms of functions used in the model (2.1) (Dubey and Narayanan, 2010):

$r_B(P)=r_{B0}-r_{B1}P,$

$$K_B(I,P,T) = K_{B0} - K_{B1}I - K_{B2}P - K_{B3}T,$$

$$r_P(B) = r_{P0} + r_{P1}B,$$

$$Q(I,P) = Q_0 + Q_1 I + Q_2 P. \tag{6.1}$$

Now, to see the dynamical behavior of the system (2.1), we integrate it numerically by fourth order Runge – Kutta Method using the following set of hypothetical parameters (Dubey and Narayanan, 2010):

$$r_{B0} = 10,\ r_{B1} = 0.5,\ K_{B0} = 12.5,\ K_{B1} = 0.2,\ K_{B2} = 0.1,\ K_{B3} = 0.2,\ r_{P0} = 20,\ r_{P1} = 0.6,$$
$$Q_0 = 10,\ Q_1 = 0.3,\ Q_2 = 0.2,\ \alpha = 0.03,\ \alpha_1 = 0.05,\ \beta = 0.5,\ \gamma_1 = 0.29,\ \delta_0 = 2,$$
$$r_2 = 10.5,\ L = 5,\ M = 10,\ k_m = 10,\ l_1 = 0.2,\ l_2 = 0.1,\ l_3 = 0.2,\ \rho_1 = 0.5,\ \rho_2 = 0.2,$$
$$\rho_3 = 0.4,\ \rho_4 = 0.5,\ \phi = 0.5,\ \eta = 0.05,\ \eta_0 = 0.03. \tag{6.2}$$

With the above parameter values, we get following equilibrium values of variables used in the model

$$B^* = 6.9569,\ \ I^* = 8.3256,\ \ P^* = 12.0871,\ T^* = 6.3527,\ \ M^* = 9.2385. \tag{6.3}$$

On comparing the numerical values of the equilibrium obtained in Dubey and Narayanan, (2010) with our model, it is observed that if we introduce technology in the system, density of resource biomass, industrialization and population increases and concentration of toxicants decrease. Thus, introduction of technology not only help in conservation of resource biomass but also leads to increase in biomass density. Increase in biomass density lead to increase in industrialization and due to industrialization population increases in the ecosystem but concentration of toxicant remains under control. Moreover, it is observed that all the conditions for local and global stability are satisfied for above set of parameter values given in (6.2) and for the equilibrium values of the variables used in the model given by (6.3).

We compare the dynamics of resource biomass, industrialization and toxicants with and without technological effort applied and investigate the effect of various parameters on them by plotting their graphs with respect to time. It is found in all the figures that equilibrium level of resource biomass is less when technological effort is not applied to conserve the resource biomass. However, when some technological effort is applied to conserve the resource biomass, the density of resource biomass in the system increases.

Figure 1 displays the dynamics of resource biomass for different values depletion rate coefficient of resource biomass due to industrialization α, with respect to time t. It is observed that as α increases, resource biomass decreases with and without application of technological effort applied to conserve it. When technological effort is applied, resource biomass first increases and then attains a steady equilibrium value and when no technology is used to conserve the resource biomass, it decreases first due to industrialization and then attains an equilibrium level.

Figure 2 is plotted between resource biomass and time to investigate the effect of resource biomass depletion due to population r_{B1}, both in the presence and absence of technological effort. It is observed from the figure that as r_{B1} increases resource biomass decreases. It is further observed that in the absence of technology the decrease in resource biomass is comparatively more and it declines significantly.

In figures 3, 4 and 5, variation of resource biomass with time for different rate of decrease of carrying capacity of resource biomass due to industrialization, population pressure and toxicants present in the environment $K_{B1,}$ K_{B2} and K_{B3} is displayed both in the presence and absence of technology. It is observed from the figure that resource biomass decreases with the increases in $K_{B1,}$ K_{B2} and K_{B3} respectively. It is further observed that due to these factors resource biomass decrease considerably and the condition gets worse when no technology is applied for the conservation of resource biomass.

Figure 6 shows the variation of industrialization with time for different growth rate of industrialization due to resource biomass β, in the presence and absence technological effort applied to conserve the resource biomass. It is observed from the figure that as β increases industrialization increases. When technological effort is applied, industrialization first increases and then attains an equilibrium value. However, without any technological effort industrialization first decrease and then attains equilibrium level.

Figure 7 is drawn to exhibit the variation of toxicants present in the atmosphere with time for different depletion rate coefficient of the pollutant from the environment due to the forestry resource biomass α_1. It is observed that as α_1 increases toxicant concentration decreases. Thus, toxicant concentration in the environment can be reduced if forests are present to absorb the pollutant concentration present in the atmosphere. In figures 8 and 9, global stability of the system is displayed graphically. It is observed from the figures that system always reach the equilibrium point whatever initial position is considered.

Figures

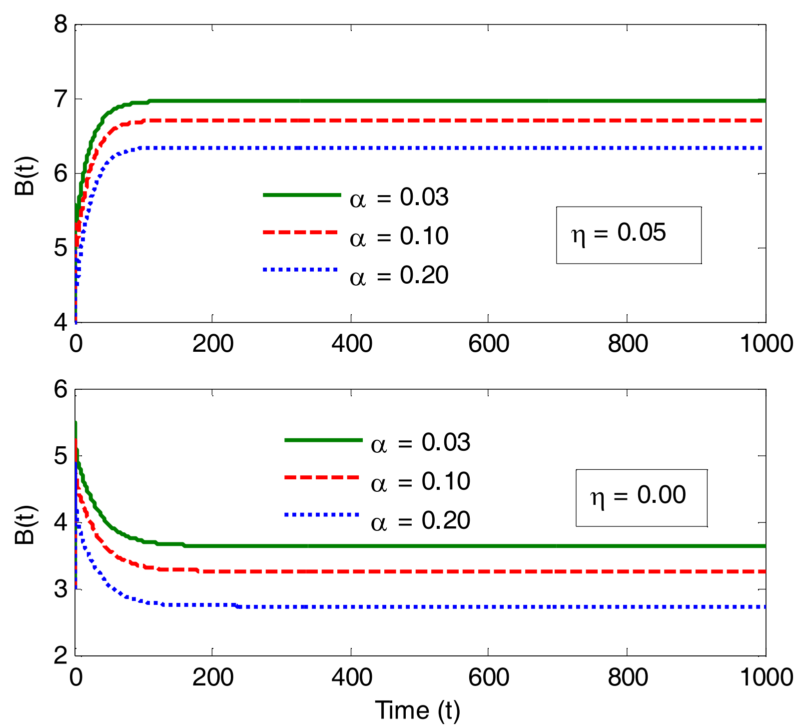

Figure 1. Dynamics of resource biomass for different values of depletion rate coefficient of resource biomass due to industrialization α, with respect to time t.

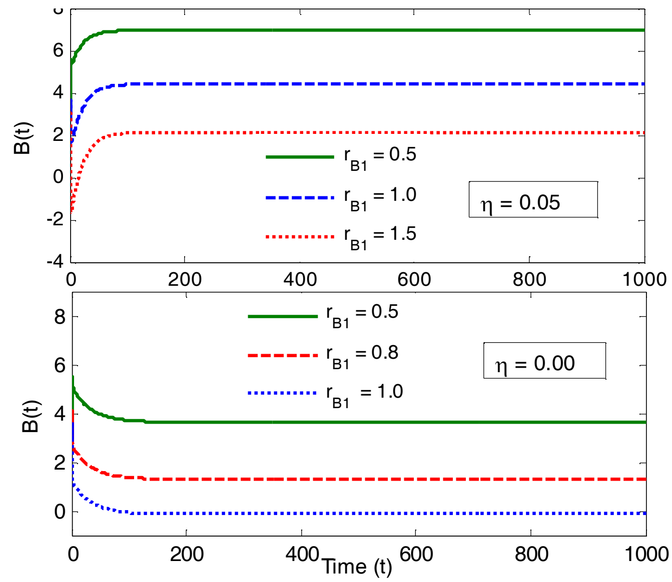

Figure 2. Dynamics of resource biomass for different values depletion rate coefficient of resource biomass due to population r_{B1}, with respect to time t.

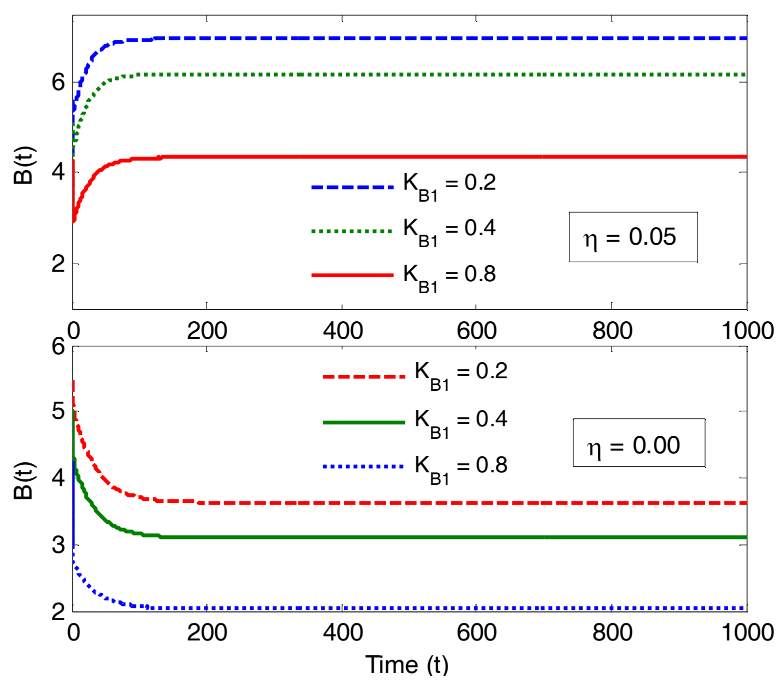

Figure 3. Dynamics of resource biomass for different values of depletion rate coefficient of carrying capacity of resource biomass due to industrialization K_{B1}, with respect to time t.

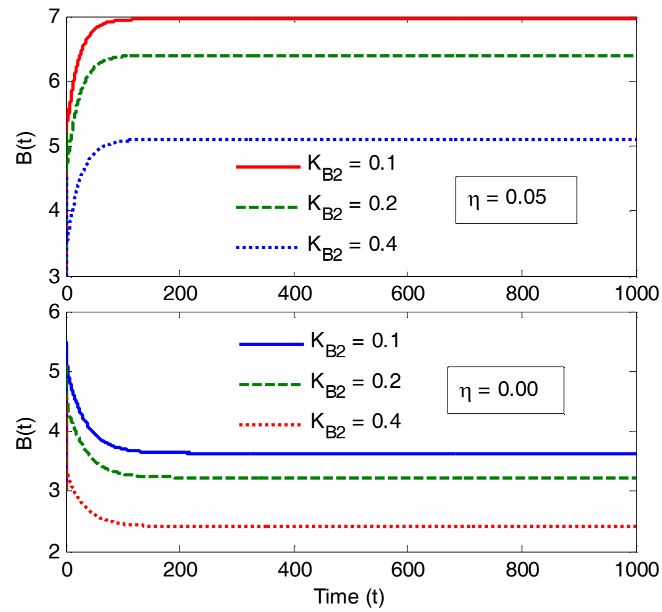

Figure 4. Dynamics of resource biomass for different values of depletion rate coefficient of carrying capacity of resource biomass due to population K_{B2}, with respect to time t.

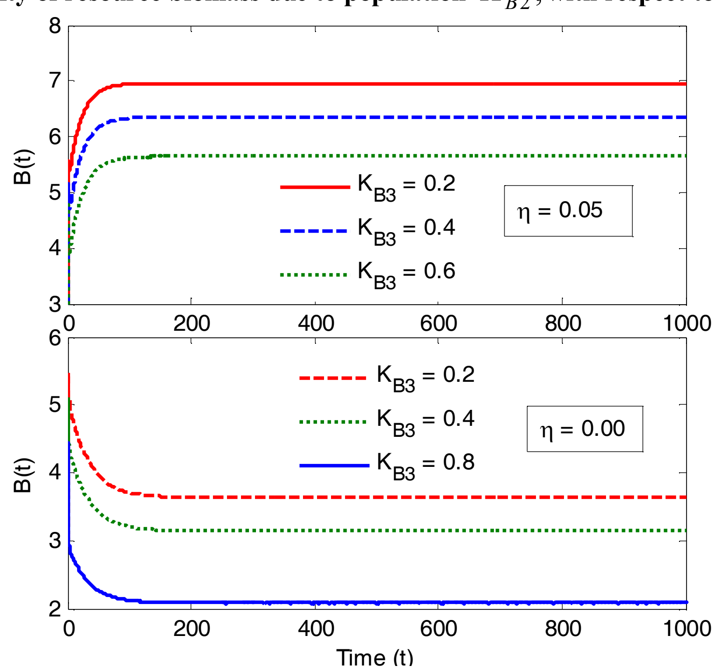

Figure 5. Dynamics of resource biomass for different values of depletion rate coefficient of carrying capacity of resource biomass due to toxicants K_{B3}, with respect to time t.

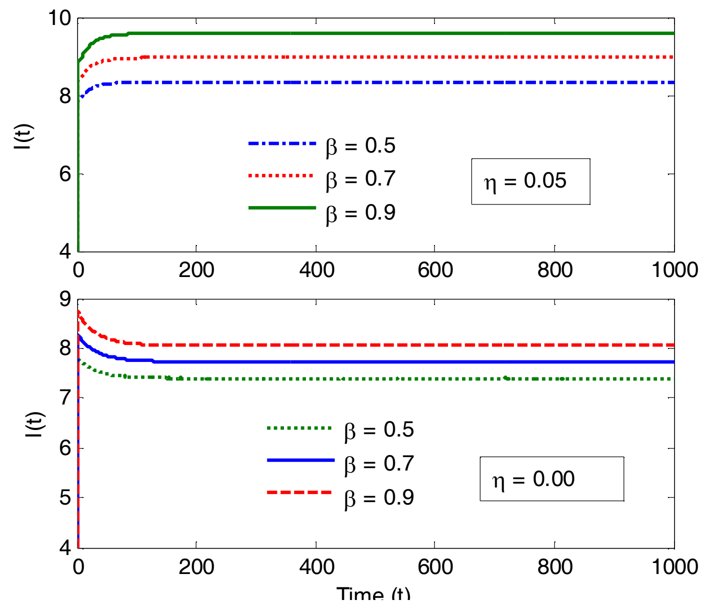

Figure 6. Dynamics of industrialization for different values of growth rate of industrialization due to resource biomass β, with respect to time t.

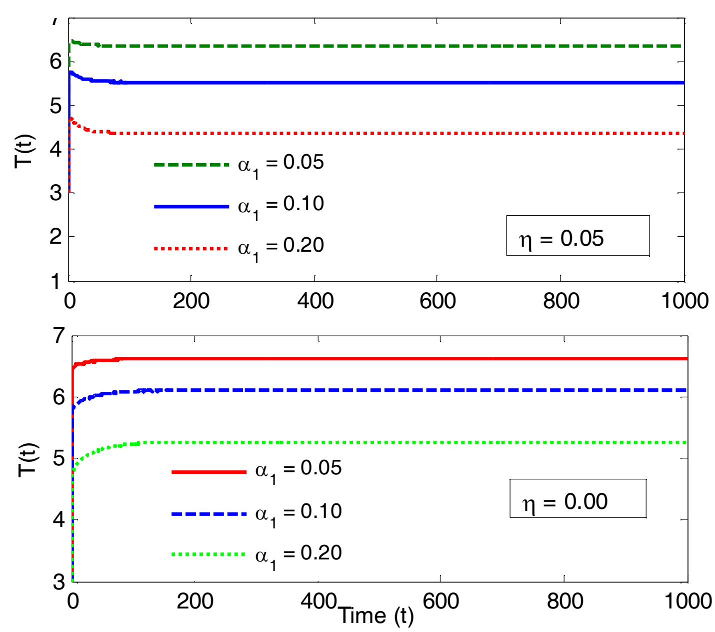

Figure 7. Dynamics of toxicant concentration for different values of depletion rate coefficient of toxicants due to uptake by resource biomass α_1, with respect to time t.

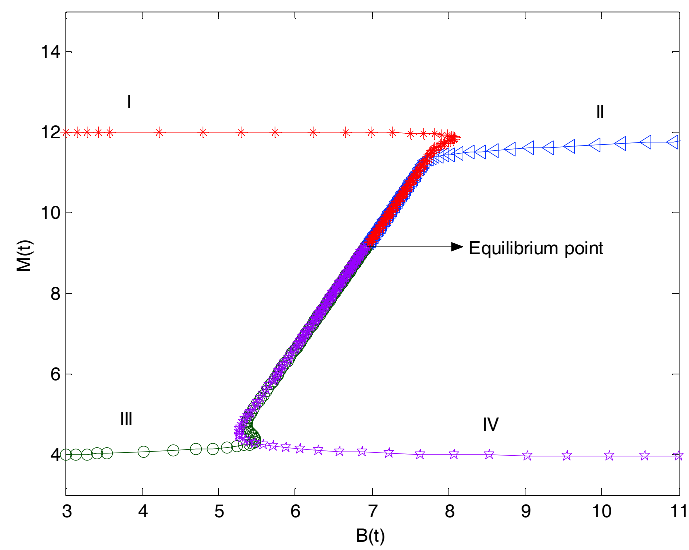

Figure 8. Variation of technological effort applied to conserve the resource biomass with resource biomass density

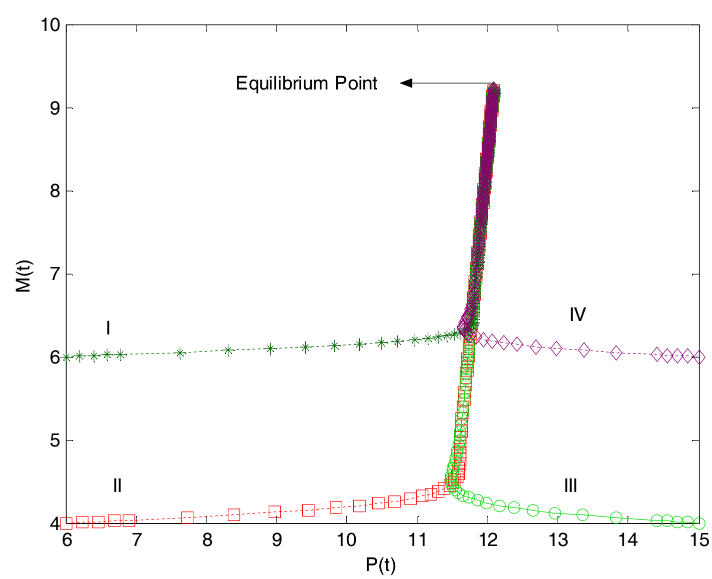

Figure 9. Variation of technological effort applied to conserve the resource biomass with population

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Conclusion

In this paper, a nonlinear mathematical model for determining the efficacy of technology in the conservation of resource biomass that is continuously being depleted due to industrialization, population and toxicants emitted in the environment is proposed and analyzed. The model is analyzed by using stability theory of differential equations and simulation. Both local and global stability of the system is studied and sufficient conditions for the equilibrium to be stable in both the local and global sense are determined. We have found from the analysis that density of resource biomass increases if some technology is used to conserve it. A little decrease in industrialization is also observed if no technology is applied to conserve resource biomass. This is obvious because industries use resource biomass for its establishment (e.g. wood and paper based industries). Further, we have observed that for large depletion rate coefficient of resource biomass due to population, resource biomass goes to extinction if no technology is used for its conservation. However, resource biomass does not become extinct for the same depletion rate of coefficient of resource biomass due to population if some technology is applied to conserve the resource biomass. It is observed that as depletion rate of resource biomass and its carrying capacity due to industrialization, toxicants and population growth increases resource biomass decreases. Moreover, we have observed that concentration of toxicants in the environment decreases due to resource biomass and if in addition new technology is applied to conserve the resource biomass concentration of toxicants in the environment can be reduced to a significant level.

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Appendix A.

Proof of theorem 1.

We consider small perturbations b, i, p, τ, u around the positive interior equilibrium, $E_7(B^*, I^*, P^*, T^*, M^*)$ and linearize system (2.1) by using following transformations

$$B = B^* + b,\ I = I^* + i,\ P = P^* + p,\ T = T^* + \tau,\ M = M^* + m. \tag{A.1}$$

To determine conditions for linearized stability we use the following positive definite function in the linearized model:

$$V = \frac{1}{2B^*}b^2 + \frac{c_1}{2I^*}i^2 + \frac{c_2}{2P^*}p^2 + \frac{c_3}{2}\tau^2 + \frac{c_4}{2}m^2 \tag{A.2}$$

Differentiating (A.2) with respect to time t, we have

$$\begin{aligned}\frac{dV}{dt} \le & -\frac{1}{4}a_{11}b^2 + a_{12}bi - \frac{1}{3}a_{22}i^2 - \frac{1}{4}a_{11}b^2 + a_{13}bp - \frac{1}{3}a_{22}p^2 \\ & -\frac{1}{4}a_{11}b^2 + a_{14}b\tau - \frac{1}{3}a_{44}\tau^2 - \frac{1}{4}a_{11}b^2 + a_{15}bm - a_{55}m^2 \\ & -\frac{1}{3}a_{22}i^2 + a_{23}ip - \frac{1}{3}a_{33}p^2 - \frac{1}{3}a_{22}i^2 + a_{24}i\tau - \frac{1}{3}a_{44}\tau^2 \\ & -\frac{1}{3}a_{33}p^2 + a_{34}p\tau - \frac{1}{3}a_{44}\tau^2\end{aligned} \tag{A.3}$$

where,

$$a_{11} = \frac{r_{B0}}{K_B(I^*, P^*, T^*)},\ a_{22} = \frac{c_1 r_2}{L},\ a_{33} = \frac{c_2 r_{P0}}{S},\ a_{44} = \alpha_1 B^*,\ a_{55} = c_4\eta_0,$$

$$a_{12} = \frac{r_{B0}B^*}{K_B^2(I^*, P^*, T^*)}\frac{\partial K_B(I^*, P^*, T^*)}{\partial I} - \alpha + c_1\beta,$$

$$a_{13} = \frac{r_{B0}B^*}{K_B^2(I^*, P^*, T^*)}\frac{\partial K_B(I^*, P^*, T^*)}{\partial P} + c_2\frac{\partial r_P(B)}{\partial B},$$

$$a_{14} = \frac{r_{B0}B^*}{K_B^2(I^*, P^*, T^*)}\frac{\partial K_B(I^*, P^*, T^*)}{\partial T} - c_3\alpha_1 T^*,$$

$a_{15} = \phi - \eta c_4$, $a_{23} = c_1\gamma_1$, $a_{34} = c_3 \dfrac{\partial Q(I^*, P^*)}{\partial P}$, $a_{24} = c_3 \dfrac{\partial Q(I^*, P^*)}{\partial I}$,

Sufficient conditions for $\dfrac{dV}{dt}$ to be negative definite are that the following conditions hold:

$$a_{12}^2 < \frac{1}{3} a_{11} a_{22}, \tag{A.4}$$

$$a_{13}^2 < \frac{1}{3} a_{11} a_{33}, \tag{A.5}$$

$$a_{14}^2 < \frac{1}{3} a_{11} a_{44}, \tag{A.6}$$

$$a_{15}^2 < \frac{1}{3} a_{11} a_{55}, \tag{A.7}$$

$$a_{23}^2 < \frac{4}{9} a_{22} a_{33}, \tag{A.8}$$

$$a_{24}^2 < \frac{4}{9} a_{22} a_{33}, \tag{A.9}$$

$$a_{34}^2 < \frac{4}{9} a_{33} a_{44}, \tag{A.10}$$

It may be easily seen that if conditions (5.4), (5.5), (5.6) and (5.7) hold then (A.4), (A.5), (A.6) and (A.7) respectively are satisfied automatically. We further observe that (5.1) implies (A.8), (5.2) implies (A.9) and (5.3) implies that (5.10) hold. It implies that $\dfrac{dV}{dt}$ is negative definite and hence equilibrium E_7 is locally asymptotically stable.

Appendix B.

Proof of theorem 2

To prove global stability of the equilibrium E_7 we consider the following positive definite function with some positive constants k_1 and k_2.

$$W = B - B^* - B^* \ln\frac{B}{B^*} + I - I^* - I^* \ln\frac{I}{I^*} + P - P^* - P^* \ln\frac{P}{P^*} + \frac{k_1}{2}(T - T^*)^2 + \frac{k_2}{2}(M - M^*)^2 \tag{B.1}$$

Differentiating (B.1) with respect to time t, we have

$$\begin{aligned}
\frac{dW}{dt} \le & -\frac{r_{B0}}{K_B(I^*, P^*, T^*)}(B - B^*)^2 - \frac{r_2}{L}(I - I^*)^2 - \frac{r_{p0}}{S}(P - P^*)^2 - k_1(\delta_0 + \alpha_1 B^*)(T - T^*)^2 \\
& - \eta_0 k_2 (M - M^*)^2 + \left[-r_{B0}\xi_{B1}(I, P, T)B - \alpha + \beta\right](B - B^*)(I - I^*) \\
& + \left[\eta_B(P) - r_{B0}\xi_{B2}(I^*, P, T)B + \eta_P(B)\right](B - B^*)(P - P^*) \\
& + \left[-r_{B0}\xi_{B3}(I^*, P^*, T)B - k_1\alpha_1 T\right](B - B^*)(T - T^*) + (\phi - k_2\eta)(B - B^*)(M - M^*) \\
& + \gamma_1(I - I^*)(P - P^*) + k_1\theta_{Q1}(I, P)(I - I^*)(T - T^*) + k_1\theta_{Q2}(I^*, P)(P - P^*)(T - T^*)
\end{aligned} \tag{B.2}$$

where,

$$\xi_{B1}(I, P, T) = \begin{cases} \left.\left(\dfrac{1}{K_B(I, P, T)} - \dfrac{1}{K_B(I^*, P, T)}\right)\right/ I - I^*, & I \ne I^* \\ -\dfrac{1}{K_B^2(I^*, P, T)}\dfrac{\partial K_B(I^*, P, T)}{\partial I}, & I = I^* \end{cases}$$

$$\xi_{B2}(I^*, P, T) = \begin{cases} \left.\left(\dfrac{1}{K_B(I^*, P, T)} - \dfrac{1}{K_B(I^*, P^*, T)}\right)\right/ P - P^*, & P \ne P^* \\ -\dfrac{1}{K_B^2(I^*, P^*, T)}\dfrac{\partial K_B(I^*, P^*, T)}{\partial P}, & P = P^* \end{cases}$$

$$\xi_{B3}(I^*, P^*, T) = \begin{cases} \left.\left(\dfrac{1}{K_B(I^*, P^*, T)} - \dfrac{1}{K_B(I^*, P^*, T^*)}\right)\right/ T - T^*, & T \ne T^* \\ -\dfrac{1}{K_B^2(I^*, P^*, T)}\dfrac{\partial K_B(I^*, P^*, T)}{\partial T}, & T = T^* \end{cases}$$

$$\eta_B(P) = \begin{cases} \left[r_B(P) - r_B(P^*)\right] / P - P^*, & P \neq P^* \\ r_B'(P^*), & P = P^* \end{cases}$$

$$\eta_P(B) = \begin{cases} \left[r_P(B) - r_p(B^*)\right] / B - B^*, & B \neq B^* \\ r_P'(B^*), & B = B^* \end{cases}$$

$$\theta_{Q1}(I,P) = \begin{cases} \left[Q(I,P) - Q(I^*,P)\right] / I - I^*, & I \neq I^* \\ \dfrac{\partial Q(I^*,P)}{\partial I}, & I = I^* \end{cases}$$

$$\theta_{Q2}(I^*,P) = \begin{cases} \left[Q(I^*,P) - Q(I^*,P^*)\right] / P - P^*, & P \neq P^* \\ \dfrac{\partial Q(I^*,P^*)}{\partial P}, & P = P^* \end{cases}$$

From (5.8) and mean value theorem we have,

$$\left|\xi_{B1}(I,P,T)\right| \le l_1 / k_m^2,\quad \left|\xi_{B2}(I,P,T)\right| \le l_2 / k_m^2,\ \left|\xi_{B3}(I,P,T)\right| \le l_3 / k_m^2,\ \left|\eta_B(P)\right| \le \rho_1,$$

$$\left|\eta_p(B)\right| \le \rho_2,\ \left|\theta_{Q1}(I,P)\right| \le \rho_3,\ \left|\theta_{Q2}(I^*,P)\right| \le \rho_4. \tag{B.3}$$

Now $\dfrac{dW}{dt}$ can be written as sum of quadratics as given below:

$$\begin{aligned} \frac{dW}{dt} = & -\frac{1}{4}b_{11}(B-B^*)^2 + b_{12}(B-B^*)(I-I^*) - \frac{1}{3}b_{22}(I-I^*)^2 \\ & -\frac{1}{4}b_{11}(B-B^*)^2 + b_{13}(B-B^*)(P-P^*) - \frac{1}{3}b_{33}(P-P^*)^2 \\ & -\frac{1}{4}b_{11}(B-B^*)^2 + b_{14}(B-B^*)(T-T^*) - \frac{1}{3}b_{44}(T-T^*)^2 \\ & -\frac{1}{4}b_{11}(B-B^*)^2 + b_{15}(B-B^*)(M-M^*) - b_{55}(M-M^*)^2 \end{aligned}$$

$$-\frac{1}{3}b_{22}(I-I^*)^2+b_{23}(I-I^*)(P-P^*)-\frac{1}{3}b_{33}(P-P^*)^2$$

$$-\frac{1}{3}b_{22}(I-I^*)^2+b_{24}(I-I^*)(T-T^*)-\frac{1}{3}b_{44}(T-T^*)^2$$

$$-\frac{1}{3}b_{33}(I-I^*)^2+b_{34}(I-I^*)(T-T^*)-\frac{1}{3}b_{44}(T-T^*)^2 \tag{B.4}$$

$$b_{11}=\frac{r_{B0}}{K_B(I^*,P^*,T^*)},\ b_{22}=\frac{r_2}{L},\ b_{33}=\frac{r_{P0}}{S},\ b_{44}=k_1(\delta_0+\alpha_1 B^*),\ b_{55}=k_2\eta_0,$$

$$b_{12}=-r_{B0}\xi_{B1}(I,P,T)B-\alpha+\beta,$$

$$b_{13}=\eta_B(P)-r_{B0}\xi_{B2}(I^*,P,T)B+\eta_P(B),$$

$$b_{14}=-r_{B0}\xi_{B3}(I^*,P^*,T)B-k_1\alpha_1 T,$$

$$b_{15}=\phi-\eta k_2,\ b_{23}=\gamma_1,\ b_{24}=k_1\theta_{Q1}(I,P),\ b_{34}=k_1\theta_{Q2}(I^*,P).$$

Sufficient conditions for $\frac{dW}{dt}$ to be negative definite are that the following conditions hold:

$$b_{12}^2<\frac{1}{3}b_{11}b_{22}, \tag{B.5}$$

$$b_{13}^2<\frac{1}{3}b_{11}b_{33}, \tag{B.6}$$

$$b_{14}^2<\frac{1}{3}b_{11}b_{44}, \tag{B.7}$$

$$b_{15}^2<\frac{1}{3}b_{11}b_{55}, \tag{B.8}$$

$$b_{23}^2<\frac{4}{9}b_{22}b_{33}, \tag{B.9}$$

$$b_{24}^2<\frac{4}{9}b_{22}b_{33}, \tag{B.10}$$

$$b_{34}^2<\frac{4}{9}b_{33}b_{44}, \tag{B.11}$$

It may be easily seen that if conditions (5.9), (5.10), (5.11), (5.12)and (5.13) hold then (B.5), (B.6), (B.7) (B.9) and (B.11) respectively are satisfied automatically. We further observe that (B.10) is satisfied automatically if $k_1 = \frac{1}{9}\frac{r_2}{L}\frac{(\delta_0 + \alpha_1 B^*)}{\rho_3^2}$ and (B.8) is satisfied automatically if $k_2 = \frac{\phi}{\eta}$. This demonstrates that $\frac{dW}{dt}$ is negative definite inside the region of attraction Ω if (5.9)-(5.13) hold ensuring global stability of interior equilibrium point E_7.