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ON COMPRESSION OF WEIGHTED SLANT HANKEL OPERATORS

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ABSTRACT. In this paper, some algebraic properties of the compression of weighted slant Hankel operators on the space $H^2(\beta)$, $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ being a sequence of positive numbers with $\beta_0 = 1$, are discussed. It is also shown that the Weyl's theorem holds if the compression of a weighted slant Hankel operator is compact.

1. INTRODUCTION

Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$, $\mathbb{Z}$ being the set of integers. Consider the space $L^2(\beta)$ of all formal Laurent series $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, for which

$$\|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 {\beta_n}^2 < \infty.$$

$L^2(\beta)$ is a Hilbert space with the norm $\|\cdot\|_\beta$ induced by the inner product

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} a_n\ \overline{b}_n {\beta_n}^2,$$

for $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=-\infty}^{\infty} b_n z^n$. The collection $\{e_n | n \in \mathbb{Z}\}$, where $e_n(z) = z^n/\beta_n$, forms an orthonormal basis for $L^2(\beta)$.

Let $H^2(\beta)$ denote the collection of all $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ (formal power series) for which $\|f\|_\beta^2 = \sum\limits_{n=0}^{\infty} |a_n|^2 {\beta_n}^2 < \infty$. It is a subspace of $L^2(\beta)$.

Let $L^\infty(\beta)$ denote the set of formal Laurent series $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$ such that $\phi L^2(\beta) \subseteq L^2(\beta)$ and there exists some $c > 0$ satisfying $\|\phi f\|_\beta \leq c\|f\|_\beta$ for each $f \in$

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$L^2(\beta)$. For $\phi \in L^\infty(\beta)$, define the norm $\|\phi\|_\infty$ as

$$\|\phi\|_\infty = \inf\{c > 0 : \|\phi f\|_\beta \leq c\|f\|_\beta \text{ for each } f \in L^2(\beta)\}.$$

$L^\infty(\beta)$ is a Banach space with respect to $\|\cdot\|_\infty$. Analogously, $H^\infty(\beta)$ denotes the set of formal Power series ϕ such that $\phi H^2(\beta) \subseteq H^2(\beta)$. We refer [3,7] as well as the references therein, for the details of the spaces $L^2(\beta)$, $H^2(\beta)$, $L^\infty(\beta)$ and $H^\infty(\beta)$. If $\beta_n = 1$ for each $n \in \mathbb{Z}$, and the functions under consideration are complex-valued measurable functions defined over the unit circle $\mathbb{T}$ then these spaces coincide with the classical spaces $L^2(\mathbb{T})$, $H^2(\mathbb{T})$, $L^\infty(\mathbb{T})$ and $H^\infty(\mathbb{T})$. The study over these spaces is more interesting as well as demandable because of the tendency of $H^2(\beta)$ to cover Bergman spaces, Hardy spaces and Dirichlet spaces (see[7]). If $\phi \in L^\infty(\beta)$ is given by $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ then define $\widetilde{\phi}$ as $\widetilde{\phi}(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^n$.

The multiplication operators or Laurent operators are the linear operators induced by the multiplication by the fixed function, having it's roots in the spectral theory and being pursued today in such guises as the theory of subnormal operators and the theory of Toeplitz and Hankel operators. Laurent operators on $L^2(\beta)$ are discussed in detail by Shields [7] in the year 1974 and we call them as weighted Laurent operators. In the year 2005, Lauric [5] introduced the notion of weighted Toeplitz operator $T_\phi^\beta = P^\beta M_\phi^\beta$ on $H^2(\beta)$, where $P^\beta : L^2(\beta) \mapsto H^2(\beta)$ denotes the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$ and M_ϕ^β is the weighted Laurent operator on $L^2(\beta)$ induced by $\phi \in L^\infty(\beta)$, that is $M_\phi^\beta f = \phi \cdot f$.

Let W be an operator on $L^2(\beta)$ given by

$$We_n(z) = \begin{cases} \frac{\beta_m}{\beta_{2m}} e_m(z) & \text{if } n = 2m \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

and J^β denote the reflection operator on $L^2(\beta)$ defined as $J^\beta f = \sum_{n=-\infty}^{\infty} a_n \beta_n e_{-n}$ for each $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ in $L^2(\beta)$. It is easy to verify that $P^\beta W = W P^\beta$.

In [2,3], the study is further extended to weighted Hankel operators and weighted slant Hankel operators, which are defined as follows.

Definition 1.1. [2] A weighted Hankel operator H_ϕ^β, $\phi \in L^\infty(\beta)$ on $H^2(\beta)$ is defined as $H_\phi^\beta = P^\beta J^\beta M_\phi^\beta$.

If $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ then for $j \geq 0$, $H_\phi^\beta e_j = \frac{1}{\beta_j} \sum_{n=0}^{\infty} a_{-n-j} \beta_{-n} e_n$.

Definition 1.2. [3] A weighted slant Hankel operator K_ϕ^β, $\phi \in L^\infty(\beta)$ on $L^2(\beta)$ is given by $K_\phi^\beta = J^\beta W M_\phi^\beta$.

If $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ then for each $j \in \mathbb{Z}$, $K_\phi^\beta e_j = \frac{1}{\beta_j} \sum_{n \in \mathbb{Z}} a_{-2n-j} \beta_{-n} e_n$.

In this paper, some properties of the compression of weighted slant Hankel operators K_ϕ^β to $H^2(\beta)$ are discussed. We also find a characterization for the product of weighted Toeplitz and the compression of a weighted slant Hankel operator to be compression of a weighted slant Hankel operator. It is also shown that the compression of weighted slant Hankel operators can not be Fredholm. Throughout the paper, the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is supposed to be a semi-dual sequence (that is $\beta_n = \beta_{-n}$ for each n) of positive numbers with $\beta_0 = 1$, $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$, for some $r > 0$.

2. Compression of Weighted Slant Hankel Operators

We denote the compression of weighted slant Hankel operators K^{β}_{ϕ} to $H^2(\beta)$ by L^{β}_{ϕ}. By the definition of compression, we have $L^{\beta}_{\phi} = P^{\beta}K^{\beta}_{\phi}|_{H^2(\beta)}$, equivalently, $L^{\beta}_{\phi}P^{\beta} = P^{\beta}K^{\beta}_{\phi}P^{\beta}$. As the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is assumed to be semi-dual sequence, so if $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$ then for each $j \geq 0$, $L^{\beta}_{\phi}e_j = \frac{1}{\beta_j}\sum\limits_{n=0}^{\infty} a_{-2n-j}\beta_n e_n$.

Consider the transformation V on $L^2(\beta)$ given by

$$Vf(z) = f(z^2) = \sum_{n=-\infty}^{\infty} a_n z^{2n},$$

for $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$ in $L^2(\beta)$. Then V is bounded if and only if $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. If $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded and $\phi \in L^{\infty}(\beta)$ then (see [3])

(1) $M^{\beta}_{\phi}W = WM^{\beta}_{\phi(z^2)}$.
(2) $VM^{\beta}_{\phi} = M^{\beta}_{\phi(z^2)}V$.
(3) $WM^{\beta}_{\phi}V = M^{\beta}_{\psi}$ where $\psi = W\phi \in L^{\infty}(\beta)$.

Using these observations, one can prove the following:

Proposition 2.1. *Let $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ be bounded and $\phi \in L^{\infty}(\beta)$. Then $H^{\beta}_{\phi}WP^{\beta} = L^{\beta}_{\phi(z^2)}$.*

A routine computation shows that $L^{\beta}_{\phi} = WH^{\beta}_{\phi}$. Hence if $\phi \in zH^2(\beta)$ then $L^{\beta}_{\phi} = 0$ as $H^{\beta}_{\phi} = 0$. Also, if $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded and $\phi \in L^{\infty}(\beta)$, then $T^{\beta}_{z^{-1}}L^{\beta}_{\phi} = T^{\beta}_{z^{-1}}WH^{\beta}_{\phi} = WT^{\beta}_{z^{-2}}H^{\beta}_{\phi} = L^{\beta}_{\phi}T_{z^2}$.

It is known [2, Theorem 3] that the product $H^{\beta}_{\phi}T^{\beta}_{\psi}$ of a non-zero weighted Hankel operator H^{β}_{ϕ} and a non-zero weighted Toeplitz operator T^{β}_{ψ} is a weighted Hankel operator if and only if $\psi \in H^{\infty}(\beta)$.

It is a natural problem to look for a similar result for the compression of weighted slant Hankel operators. A simple computation shows that for $\phi, \psi \in L^{\infty}(\beta)$, $L^{\beta}_{\phi}T^{\beta}_{\psi} = L^{\beta}_{\phi\psi}$. The condition $\psi \in H^{\infty}(\beta)$ does not seem to be necessary for $L^{\beta}_{\phi}T^{\beta}_{\psi}=L^{\beta}_{\phi\psi}$. For, if $\psi(z) = z^{-1}$ and $\phi(z) = 1$ then $L^{\beta}_{\phi}T^{\beta}_{\psi}(= L^{\beta}_{z^{-1}})$ is compression of the weighted slant Hankel operator K_{ψ} without being ψ in $H^{\infty}(\beta)$. However, we obtain a result in the following form.

Theorem 2.2. *Let $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ be bounded. Let $\phi, \psi \in L^{\infty}(\beta)$ be such that $L^{\beta}_{\phi(z^2)}$ and T^{β}_{ψ} are non-zero operators on $H^2(\beta)$. Then the product $L^{\beta}_{\phi(z^2)}T^{\beta}_{\psi}$ is the compression of a weighted slant Hankel operator if and only if $z\psi \in H^{\infty}(\beta)$.*

Proof. Let $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$ and $\psi(z) = \sum\limits_{n=-\infty}^{\infty} b_n z^n$ in $L^{\infty}(\beta)$ be such that $L^{\beta}_{\phi(z^2)} \neq 0$ and $T^{\beta}_{\psi} \neq 0$.

Suppose $z\psi(z) \in H^\infty(\beta)$. Then $\phi(z^2)\psi(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, with c_n given by $c_{2n} = \sum_{k=-\infty}^{n} a_k b_{2n-2k}$ and $c_{2n-1} = \sum_{k=-\infty}^{n} a_k b_{2n-1-2k}$. A routine computation shows that for $i, j \geq 0$

$$\langle L^\beta_{\phi(z^2)} T^\beta_\psi e_j, e_i \rangle = \begin{cases} \frac{\beta_i}{\beta_j} \sum_{k=0}^{\infty} a_{-k-i-l} b_{2k} & \text{if } j = 2l \\ \frac{\beta_i}{\beta_j} \sum_{k=0}^{\infty} a_{-k-i-l} b_{2k-1} & \text{if } j = (2l+1) \end{cases}$$

and

$$\begin{aligned} \langle L^\beta_{\phi(z^2)\psi} e_j, e_i \rangle &= \begin{cases} \frac{\beta_i}{\beta_j} \sum_{k=-\infty}^{-i-l} a_k b_{-2i-2l-2k} & \text{if } j = 2l \\ \frac{\beta_i}{\beta_j} \sum_{k=-\infty}^{-i-l} a_k b_{-2i+2l-2k-1} & \text{if } j = (2l+1) \end{cases} \\ &= \begin{cases} \frac{\beta_i}{\beta_j} \sum_{k=0}^{\infty} a_{-i-l-k} b_{2k} & \text{if } j = 2l \\ \frac{\beta_i}{\beta_j} \sum_{k=0}^{\infty} a_{-i-l-k} b_{2k-1} & \text{if } j = (2l+1) \end{cases}. \end{aligned}$$

This proves $L^\beta_{\phi(z^2)} T^\beta_\psi = L^\beta_{\phi(z^2)\psi}$.

For converse, let $L^\beta_{\phi(z^2)} T^\beta_\psi = L^\beta_\xi$ for some $\xi \in L^\infty(\beta)$. This gives $L^\beta_{\phi(z^2)} T^\beta_\psi V P^\beta = L^\beta_\xi V P^\beta$, which on using the fact $L^\beta_\phi = W H^\beta_\phi$ and the observations listed before Proposition 2.1 reduces to $H^\beta_\phi T^\beta_{W\psi} = H^\beta_{W\xi}$. As $L^\beta_{\phi(z^2)} \neq 0$ so $H^\beta_\phi \neq 0$ and from [2, Theorem 3.6] we deduce that $W\psi \in H^\infty(\beta)$. Also, we have $L^\beta_{\phi(z^2)} T^\beta_\psi T^\beta_z V P^\beta = L^\beta_\xi T^\beta_z V P^\beta$ and on applying the similar arguments we get $W(z\psi) \in H^\infty(\beta)$. Finally, it is elementary to see that the conditions $W\psi, W(z\psi) \in H^\infty(\beta)$ implies that $z\psi \in H^\infty(\beta)$. □

Our next result, proof of which is almost along the lines of the proof of Theorem 3.6 of [5], is stated as a lemma without proof.

Lemma 2.3. *Let H^β_ϕ and T^β_ψ be non-zero operators on $H^2(\beta)$ for $\phi, \psi \in L^\infty(\beta)$. Then the product $T^\beta_\psi H^\beta_\phi$ is a weighted Hankel operator if and only if $\widetilde{\psi} \in H^\infty(\beta)$.*

Now we discuss the situation under which the product $T^\beta_\psi L^\beta_\phi$ is the compression of a weighted slant Hankel operator.

Theorem 2.4. *Let $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ be bounded and L^β_ϕ, T^β_ψ be non-zero operators for $\phi, \psi \in L^\infty(\beta)$. Then the product $T^\beta_\psi L^\beta_\phi$ is a compression of a weighted slant Hankel operator if and only if $\widetilde{\psi} \in H^\infty(\beta)$.*

Proof. Suppose first that $\widetilde{\psi} \in H^\infty(\beta)$. Using the observation that $L^\beta_\phi = W H^\beta_\phi$, the definition of T^β_ϕ and observation (1) listed before Proposition 2.1, one has

$$\begin{aligned} T^\beta_\psi L^\beta_\phi &= T^\beta_\psi W H^\beta_\phi = P^\beta M^\beta_\psi W H^\beta_\phi \\ &= P^\beta W M^\beta_{\psi(z^2)} H^\beta_\phi = W T^\beta_{\widetilde{\psi}(z^2)} H^\beta_\phi. \end{aligned}$$

Since $\widetilde{\psi} \in H^\infty(\beta)$, we have $\widetilde{\psi}(z^2) \in H^\infty(\beta)$ (see [2,Corollary 2.10]). Thus, it follows from Theorem 4.2 of [2] that $T^\beta_{\widetilde{\psi}(z^2)} H^\beta_\phi = H^\beta_\phi T^\beta_{\widetilde{\psi}(z^2)}$. This gives

$$T^\beta_\psi L^\beta_\phi = W H^\beta_\phi T^\beta_{\widetilde{\psi}(z^2)} = L^\beta_\phi T^\beta_{\widetilde{\psi}(z^2)} = L^\beta_\xi$$

with $\xi(z) = \phi(z)\widetilde{\psi}(z^2)$, which proves that $T^\beta_\psi L^\beta_\phi$ is the compression of a weighted slant Hankel operator.

For converse, let $\phi, \psi \in L^\infty(\beta)$ be such that $L^\beta_\phi \neq 0, T^\beta_\psi \neq 0$ and $T^\beta_\psi L^\beta_\phi = L^\beta_\xi$ for some $\xi \in L^\infty(\beta)$. This gives $T^\beta_\psi L^\beta_\phi V P^\beta = L^\beta_\xi V P^\beta$ and by arguing as in the proof of Theorem 2.2, we obtain $T^\beta_\psi H^\beta_{W\phi} = H^\beta_{W\xi}$. By Lemma 2.3, we conclude that $\widetilde{\psi} \in H^\infty(\beta)$ provided $H^\beta_{W\phi} \neq 0$. Again, we have $T^\beta_\psi L^\beta_\phi T^\beta_z V P^\beta = L^\beta_\xi T^\beta_z V P^\beta$, which implies that $T^\beta_\psi H^\beta_{W(z\phi)} = H^\beta_{W(z\xi)}$. Now by repeating the arguments made above, we get $\widetilde{\psi} \in H^\infty(\beta)$ provided $H^\beta_{W(z\phi)} \neq 0$. But $L^\beta_\phi \neq 0$, therefore either $H^\beta_{W\phi} \neq 0$ or $H^\beta_{W(z\phi)} \neq 0$. Thus $\widetilde{\psi} \in H^\infty(\beta)$. □

Theorem 2.5. *Let $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ be bounded. Then WL^β_ϕ is compression of a weighted slant Hankel operator if and only if $z\phi \in H^\infty(\beta)$.*

Proof. Suppose that for $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$ in $L^\infty(\beta)$, WL^β_ϕ is compression of a weighted slant Hankel operator. This gives that, for each $i, j \geq 0$

$$\langle T^\beta_{z^{-2}} W L^\beta_\phi e_j, e_i\rangle = \langle W L^\beta_\phi T^\beta_{z^2} e_j, e_i\rangle.$$

This yields that $a_{-n} = 0$ for $n \geq 2$ and thus $z\phi \in H^\infty(\beta)$. For converse, let $z\phi \in H^\infty(\beta)$, then simple computations show that for each $n \geq 0$, $WL^\beta_\phi e_n = L^\beta_\phi e_n$. Hence the result. □

We use this theorem to conclude the following.

Corollary 2.6. *Let $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ be bounded. Then $L^\beta_\phi W_{\ H^2(\beta)}$ is the compression of a weighted slant Hankel operator if and only if $\phi \in H^\infty(\beta)$.*

Proof. From observation (1) listed before Proposition 2.1, one has

$$\begin{aligned} L^\beta_\phi W|_{H^2(\beta)} &= P^\beta J^\beta W M^\beta_\psi W|_{H^2(\beta)} \\ &= P^\beta J^\beta W W M^\beta_{\psi(z^2)}|_{H^2(\beta)} = W L^\beta_{\psi(z^2)}. \end{aligned}$$

Now result follows using Theorem 2.5 and the fact that $z\phi(z^2) \in H^\infty(\beta)$. □

In [8], it is shown that the compression of a slant Hankel operator on $H^2(\mathbb{T})$(Hardy space) cannot be an isometry. In the same direction we prove the following.

Theorem 2.7. *The compression of a weighted slant Hankel operator on $H^2(\beta)$ cannot be an isometry.*

Proof. If possible, L^β_ϕ on $H^2(\beta)$, $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, is a non-zero compression and is an isometry. Then for each $j \geq 0$, $\|H^\beta_\phi e_j\| = 1$ i.e.

$$\frac{1}{\beta_j^2} \sum_{n=0}^{\infty} |a_{-2n-j}|^2 \beta_{-n}^2 = 1. \qquad (2.8.1)$$

Equation (2.8.1) used for $j = 0, 2$ gives that $a_0 = 0$. Now assume that for a natural number m, $a_0 = a_{-2} = a_{-4} = \cdots = a_{-2m} = 0$. Then equation (2.8.1) for $j = 0$ and $j = 2(m+2)$ gives

$$\begin{aligned}\sum_{n=m+1}^{\infty} |a_{-2n}|^2 \beta_n^2 &= \frac{1}{\beta_{2(m+2)}^2} \sum_{n=0}^{\infty} |a_{-2n-2(m+2)}|^2 \beta_n^2 \\ &\le \sum_{n=m+2}^{\infty} |a_{-2n}|^2 {\beta_n}^2.\end{aligned}$$

This implies that, $a_{-2(m+1)} = 0$. Therefore, by the principle of mathematical induction, $a_{2n} = 0$ for each $n \le 0$. Again, on applying (2.8.1) for $j = 1, 3$, we get $a_1 = 0$. Now by the same arguments as above we can see that $a_{-2n-1} = 0$ for each $n \le 0$. Consequently, $\phi \in zH^{\infty}(\beta)$ and hence $L_\phi^\beta = 0$. This is a contradiction. Hence the result. □

Now we discuss the compactness of the compression of a weighted slant Hankel operator. Consider the semi-dual sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ of positive numbers, where $\beta_0 = 1$, $r \le \frac{\beta_n}{\beta_{n+1}} \le 1$ for $n \ge 0$ and $r \le \frac{\beta_n}{\beta_{n-1}} \le 1$ for $n \le 0$, for some $r > 0$. Define semi-dual sequences $\gamma = \{\gamma_n\}_{n\in\mathbb{Z}}$ and $\beta' = \{\beta'_n\}_{n\in\mathbb{Z}}$ of positive numbers as

$$\gamma_n = \sqrt{\sum_{k=0}^{n} \frac{\beta_k^2}{\beta_{n-k}^2}}$$

and

$$\beta'_n = \sqrt{\sum_{k=0}^{[\frac{n}{2}]} \frac{\beta_k^2}{\beta_{n-2k}^2}}$$

for $n \ge 0$, where $[\cdot]$ is the greatest integer function. If $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$ then we define $\widetilde{\phi_+}(z) = \sum\limits_{n=0}^{\infty} a_{-n} z^n$. We use these notations to obtain a generalization for the compression of weighted slant Hankel operator to be Hilbert-Schmidt.

Proposition 2.8. *If $\phi \in L^{\infty}(\beta)$ is such that $\widetilde{\phi_+} \in H^2(\beta')$ then L_ϕ^β is a Hilbert-Schmidt operator.*

Proof. It can be deduced from [4] that H_ϕ^β is a Hilbert-Schmidt operator, which provides the result. □

Proposition 2.9. *Let $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. Then $L_{\phi(z^2)}^\beta$ is a compact operator if and only if H_ϕ^β is a compact operator.*

Proof. Proof follows using the facts $L_{\phi(z^2)}^\beta V = H_\phi^\beta$ and $H_\phi^\beta W = L_{\phi(z^2)}^\beta$. □

Theorem 2.10. *L_ϕ^β on $H^2(\beta)$ induced by $\phi \in L^\infty(\beta)$ is a Hilbert-Schmidt operator if and only if $\widetilde{\phi_+} \in H^2(\beta')$.*

Proof. The proof follows as

$$\begin{aligned}
\sum_{j=0}^{\infty}\|L_{\phi}^{\beta}e_j\|^2 &= \sum_{j=0}^{\infty}\|\frac{1}{\beta_j}\sum_{k=0}^{\infty}a_{-2k-j}\beta_k e_k\|^2\\
&= \sum_{j=0}^{\infty}\frac{1}{\beta_j^2}(\sum_{k=0}^{\infty}|a_{-2k-j}|^2\beta_k^2)\\
&= \sum_{n=0}^{\infty}(\sum_{k=0}^{[\frac{n}{2}]}\frac{\beta_k^2}{\beta_{n-2k}^2})|a_{-n}|^2\\
&= \sum_{n=0}^{\infty}\beta'^2_n|a_{-n}|^2.
\end{aligned}$$

□

In the familiar case of the compression of a slant Hankel operator L_ϕ on $H^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle, L_ϕ is Hilbert-Schmidt if and only if $\widetilde{\phi_+}$ lies in the weighted Hardy space $H^2(\beta')$, where $\beta' = (\beta'_n)_{n\geq 0}$ and $\beta'_n = \sqrt{[\frac{n}{2}]+1}$ for each $n \geq 0$.

Remark 2.11. Each L_ϕ^β satisfies $T_{z^{-1}}^{\beta}L_\phi^\beta = L_\phi^\beta T_{z^2}^\beta$. Hence we have the following.

(1) L_ϕ^β cannot be invertible.
(2) The kernel of L_ϕ^β, if non-zero, is always infinite dimensional.
(3) The kernel of the adjoint of L_ϕ^β, if non zero, is also infinite dimensional.

In the next result we find that no L_ϕ^β on $H^2(\beta)$ can be a Fredholm operator.

Theorem 2.12. *L_ϕ^β on $H^2(\beta)$ can never be a Fredholm operator.*

Proof. In the light of Remark 2.11, we prove the result when the kernels of L_ϕ^β and its adjoint are zero. Now, on contrary, if we assume that the operator L_ϕ^β is Fredholm then it gives that L_ϕ^β is invertible, which contradicts the fact (1) of Remark 2.11. Hence the result follows. □

3. Spectra of L_ϕ^β

Now we discuss some spectral properties of the compression of weigh-ted slant Hankel operators. The symbols $\sigma(T), \sigma_P(T)$, $\Pi(T)$, $\Pi_{00}(T)$, $\omega(T)$ and $\sigma_e(T)$ respectively refer to the spectrum, the point spectrum, the approximate point spectrum, the set of isolated points of the spectrum of T that are eigenvalues of finite multiplicity, the Weyl spectrum and the essential spectrum of an operator T on a Hilbert space. We refer [1] for the details and meaning of the symbols. An operator T is said to satisfy the Weyl's theorem if $\sigma(T)\setminus\omega(T) = \Pi_{00}(T)$ (see [1]). We use the symbols $\mathrm{N}(T)$ and $\mathrm{R}(T)$ to denote the kernel and the range of the operator T respectively. Symbol $\dim(\mathrm{N}(T))$ is used for the dimension of $\mathrm{N}(T)$ and T^* stands for the adjoint of the operator T.

Proposition 3.1. *If L_ϕ^β and L_ψ^β on $H^2(\beta)$ are non-zero and their product $L_\phi^\beta L_\psi^\beta$ is the compression of a weighted slant Hankel operator on $H^2(\beta)$ then $0 \in \sigma_P(L_\phi^\beta) \bigcap \sigma_P((L_\psi^\beta)^*)$.*

Proof. Suppose $L^{\beta}_{\phi}L^{\beta}_{\psi} = L^{\beta}_{\xi}$ for some $\xi \in L^{\infty}(\beta)$. If possible $\sigma_P(L^{\beta}_{\phi})$ does not contain 0. Now

$$\begin{aligned} L^{\beta}_{\phi}(T^{\beta}_{z^2} - T^{\beta}_{z^{-1}})L^{\beta}_{\psi} &= L^{\beta}_{\phi}T^{\beta}_{z^2}L^{\beta}_{\psi} - L^{\beta}_{\phi}T^{\beta}_{z^{-1}}L^{\beta}_{\psi} \\ &= T^{\beta}_{z^{-1}}L^{\beta}_{\phi}L^{\beta}_{\psi} - L^{\beta}_{\phi}L^{\beta}_{\psi}T^{\beta}_{z^2} \\ &= T^{\beta}_{z^{-1}}L^{\beta}_{\xi} - L^{\beta}_{\xi}T^{\beta}_{z^2} = 0. \end{aligned}$$

As L^{β}_{ϕ} is one-one so this implies $(T^{\beta}_{z^2} - T^{\beta}_{z^{-1}})L^{\beta}_{\psi} = 0$. If $\psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n$, then for each $n \geq 0$, $0 = (T^{\beta}_{z^2} - T^{\beta}_{z^{-1}})L^{\beta}_{\psi}e_n$. This provides that $b_{-n} = 0$ for each $n \geq 0$ so that $L^{\beta}_{\psi} = 0$. This is a contradiction and hence $0 \in \sigma_P(L^{\beta}_{\phi})$.

Similarly, if $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and we assume that 0 is not in $\sigma_P(L^{\beta *}_{\psi})$ then on using the fact $(L^{\beta}_{\psi})^*((T^{\beta}_{z^2})^* - (T^{\beta}_{z^{-1}})^*)(L^{\beta}_{\phi})^* = 0$, we have $L^{\beta}_{\phi}(T^{\beta}_{z^2} - T^{\beta}_{z^{-1}}) = 0$. This yields a contradiction against the fact that L^{β}_{ϕ} is a non-zero operator. Evidentally, the result follows. □

Corollary 3.2. *If the product $L_{\phi}L_{\psi}$ of non-zero operators L_{ϕ} and L_{ψ} on $H^2(\mathbb{T})$ is the compression of a slant Hankel operator on $H^2(\mathbb{T})$ then $0 \in \sigma_P(L_{\phi}) \bigcap \sigma_P(L_{\psi})$.*

Proof. If $(\beta_n)_{n\in\mathbb{Z}}$ is bounded then dim(N(H^{β}_{ϕ}))= dim(N(($H^{\beta}_{\phi})^*$)) (see [4]). Also $0 \in \sigma_P(L^*_{\psi})$ implies $0 \in \sigma_P(H^*_{\psi})$. Thus $0 \in \sigma_P(L_{\psi})$. □

We note that for each $f(z) = \sum_{j=0}^{\infty} a_j z^j \in H^2(\beta)$, $\|T^n_{z^{-1}}(f)\|^2_{\beta} = \sum_{j=0}^{\infty} |a_{j+n}|^2 \beta_j^2 \to 0$ as $n \to \infty$. This helps us to show in the next result that $0 \in \sigma_e(L^{\beta}_{\phi})$. However, we know that for each $\phi \in L^{\infty}(\beta)$, $0 \in \sigma(L^{\beta}_{\phi})$.

Proposition 3.3. *For $\phi \in L^{\infty}(\beta)$, $0 \in \sigma_e(L^{\beta}_{\phi})$.*

Proof. If we suppose that $\sigma_e(L^{\beta}_{\phi})$ does not contain 0 then we find operators A and K with K compact and $AL^{\beta}_{\phi} = I + K$. Since $T^{\beta}_{z^{-1}}L^{\beta}_{\phi} = L^{\beta}_{\phi}T_{z^2}$ and $Ke_n \to 0$ as $n \to \infty$ (K being compact), so we have

$$\begin{aligned} 1 = \|(AL^{\beta}_{\phi} - K)e_{2n}\| &\leq \|AL^{\beta}_{\phi}e_{2n}\| + \|Ke_{2n}\| \\ &= \frac{1}{\beta_{2n}}\|AL^{\beta}_{\phi}T^{2n}_{z}1\| + \|Ke_{2n}\| \\ &\leq \frac{1}{\beta_{2n}}\|A\|\|T^{n}_{z^{-1}}L^{\beta}_{\phi}1\| + \|Ke_{2n}\| \to 0 \end{aligned}$$

as $n \to \infty$. This is an absurd. This completes the proof. □

Theorem 3.4. *Let $(\frac{\beta_{2n}}{\beta_n})_{n\in\mathbb{Z}}$ be bounded. Then for $\phi \in L^{\infty}(\beta)$, $\{0\} \bigcup \sigma_p(L^{\beta}_{\phi}) = \sigma_p(L^{\beta}_{\phi(z^2)})$.*

Proof. Suppose that $0 \neq \lambda \in \sigma_p(L_\phi^\beta)$. Then there exists a $0 \neq f \in H^2(\beta)$ such that $L_\phi^\beta f = \lambda f$, that is, $\lambda f = WH_\phi^\beta f$. It yields that $H_\phi^\beta f \neq 0$ and

$$\begin{aligned} L^\beta_{\phi(z^2)}(H_\phi^\beta f) &= P^\beta J^\beta W M^\beta_{\phi(z^2)} H_\phi^\beta f \\ &= P^\beta J^\beta M_\phi^\beta W H_\phi^\beta f = H_\phi^\beta L_\phi^\beta f \\ &= H_\phi^\beta(\lambda f) = \lambda(H_\phi^\beta f). \end{aligned}$$

Therefore $\lambda \in \sigma_p(L^\beta_{\phi(z^2)})$. Conversely, let $0 \neq \mu \in \sigma_p(L_\phi^\beta)$. Then we find a $g(\neq 0)$ in $H^2(\beta)$ satisfying $L^\beta_{\phi(z^2)} g = \mu g$, which gives $Wg \neq 0$ and $L_\phi^\beta(Wg) = \mu(Wg)$. Therefore $\mu \in \sigma_p(L^\beta_{\phi(z^2)})$. Since 0 always belongs to $\sigma_p(L^\beta_{\phi(z^2)})$, hence the result. □

By using the fact that $\{0\} \cup \sigma(AB) = \{0\} \cup \sigma(BA)$, we can prove the following.

Theorem 3.5. *Let $(\frac{\beta_{2n}}{\beta_n})_{n\in\mathbb{Z}}$ be bounded. Then for $\phi \in L^\infty(\beta)$, $\sigma(L_\phi^\beta) = \sigma(L^\beta_{\phi(z^2)})$.*

Proof. The proof follows as

$$\begin{aligned} \sigma(L^\beta_{\phi(z^2)}) &= \sigma(L^\beta_{\phi(z^2)}) \cup \{0\} \\ &= \sigma(H_\phi^\beta W P^\beta) \cup \{0\} \\ &= \sigma(P^\beta W H_\phi^\beta) \cup \{0\} \\ &= \sigma(L_\phi^\beta). \end{aligned}$$

□

Now we discuss the Weyl's theorem for the compression of slant Hankel operators and get the following result.

Theorem 3.6. *Weyl's theorem holds for each compact L_ϕ^β on $H^2(\beta)$.*

Proof. Suppose that L_ϕ^β is compact. Then $\sigma(L_\phi^\beta) = \sigma_P(L_\phi^\beta) \cup \{0\}$ and $\omega(L_\phi^\beta) = \{0\}$. Now we divide the proof in two cases.
Case (i): Let 0 is not in $\sigma_P(L_\phi^\beta)$. Then $\Pi_{00}(L_\phi^\beta) = \sigma_P(L_\phi^\beta)$ and hence

$$\begin{aligned} \Pi_{00}(L_\phi^\beta) &= \sigma_P(L_\phi^\beta) \\ &= \sigma(L_\phi^\beta) \setminus \{0\} \\ &= \sigma(L_\phi^\beta) \setminus \omega(L_\phi^\beta). \end{aligned}$$

Case (ii): Let $0 \in \sigma_P(L_\phi)$ then 0 is an eigenvalue of L_ϕ^β of infinite multiplicity and hence $\Pi_{00}(L_\phi^\beta) = \sigma_P(L_\phi^\beta) \setminus \{0\} = \sigma(L_\phi^\beta) \setminus \omega(L_\phi^\beta)$.
Hence the result. □

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