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Θ-IRREGULAR WEYL-HEISENBERG FRAMES FOR $L^2(\mathbb{R})$

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ABSTRACT. Găvruta [Frames for operators, *Appl. Compu. Harmon. Appl.*, 32, (2012), 139–144] introduced and studied K-frames for separable Hilbert spaces. In this paper we generalize the notion of K-frames to irregular Weyl-Heisenberg frames and call it Θ-irregular Weyl-Heisenberg frames, where Θ is a bounded linear operator on $L^2(\mathbb{R})$. A characterization of a system in $L^2(\mathbb{R})$ to be a Θ-irregular Weyl-Heisenberg frame is given. Necessary and sufficient conditions for the existence of Θ-irregular Weyl-Heisenberg frames in $L^2(\mathbb{R})$ are obtained.

1. INTRODUCTION

The theory of frames for Hilbert spaces were formally introduced by Duffin and Schaeffer [3]. A sequence $\{f_k\}$ in a separable complex Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$ is a *frame* (or Hilbert frame) for $\mathcal{H}$ if there exists positive constants A and B such that

$$A\|f\|^2 \leq \|\{\langle f, f_k\rangle\}\|_{\ell^2}^2 \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

If upper inequality in (1.1) holds, then we say that $\{f_k\}$ is a Bessel sequence for $\mathcal{H}$. The positive constants A and B are called *lower* and *upper frame bounds* of the frame $\{f_n\}$, respectively. They are not unique. The positive constants

$$\mathrm{A}_0 = \inf\{B : B \text{ satisfy } (1.1)\}$$
$$\mathrm{B}_0 = \sup\{A : A \text{ satisfy } (1.1)\}$$

are called *optimal* or *best bounds* of the frame. The basic theory of frames can be found in [1, 13].

Recently, Găvruta in [7] introduced and studied K-frames in Hilbert spaces to study atomic systems with respect to a bounded linear operator K on Hilbert spaces. It is observed in [7] that K-frames are more general than standard frames in the

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sense that the lower frame bound only holds for the elements in the range of K, where K is a bounded linear operator on the underlying Hilbert space. Găvruta in [7] characterize K-frames in Hilbert spaces by using bounded linear operators. The theory of K-frames were further studied by Xiao et al. in [12].

Definition 1.1. [7, pp. 142] Let $\mathcal{H}$ be a Hilbert space and let K be a bounded linear operator on $\mathcal{H}$. A sequence $\{f_k\} \subset \mathcal{H}$ is called a K-frame for $\mathcal{H}$ if there exist the constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}.$$

Casazza in [2] introduced and studied irregular Weyl-Heisenberg frames in $L^2(\mathbb{R})$. For a survey of irregular Weyl-Heisenberg frames in $L^2(\mathbb{R})$ one may refer to [11]. We generalize the notion of K-frames to irregular Gabor frames, and we call it Θ-irregular Weyl-Heisenberg frames (or Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frames) in $L^2(\mathbb{R})$, where Θ is a bounded linear operator on $L^2(\mathbb{R})$. It is observed that the standard Hilbert frames and K-frames for Hilbert spaces are particular cases of Θ-irregular Weyl-Heisenberg frames (see Definition 3.1).

2. Preliminaries

In this section, we recall notations and definitions which will be required in this paper. For $1 \leq p < \infty$, let $L^p(\mathbb{R})$ denote the Banach space of complex-valued Lebesgue integrable functions f on $\mathbb{R}$ satisfying

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(t)|^p dt\right)^{\frac{1}{p}} < \infty.$$

For $p = 2$, an inner product on $L^p(\mathbb{R})$ is given by

$$\langle f, g\rangle = \int_R f\overline{g}dt,$$

where $\overline{g}$ denote the complex conjugate of g.

We define the unitary operators $T_a, E_b; a, b \in \mathbb{R}$ on $L^p(\mathbb{R})$ by :
Translation $\leftrightarrow$ $T_a f(t) = f(t-a)$.
Modulation $\leftrightarrow$ $E_b f(t) = e^{2\pi i b t} f(t)$.

One can easily verify that for $g \in L^2(\mathbb{R})$, $E_{mb}T_{na}g(t) = e^{2\pi i m b t} g(t - na)$. If $a, b > 0$ and $(g, a, b) = \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then we call (g, a, b) a *Gabor frame* or a *Weyl- Heisenberg frame* for $L^2(\mathbb{R})$. A nice introduction of Weyl-Heisenberg frames can be found in Gröchenig [8], Feichtinger and Strohmer [6] and in the paper of Heil and Walnut [9]. It is very difficult to classify the g, a, b so that (g, a, b) is a Weyl- Heisenberg frame for $L^2(\mathbb{R})$. A deep result of Rieffel in [10] shows that if (g, a, b) is a Weyl- Heisenberg frame for $L^2(\mathbb{R})$, provided $ab < 1$.

Definition 2.1. [2] Let $(x_m, y_n) \in \mathbb{R}^2$ and let $g \in L^2(\mathbb{R})$. A system $\{E_{x_m}T_{y_n}g(t)\}_{m,n\in\mathbb{Z}}$ is called an *irregular Weyl-Heisenberg frame* for $L^2(\mathbb{R})$, if $\{E_{x_m}T_{y_n}g(t)\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$.

Definition 2.2. A family of real numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ has *uniform density* $D \equiv D(\{\lambda_n\})$ if there is an $L > 0$ such that for all $n \in \mathbb{Z}$, we have $|\lambda_n - \frac{n}{D}| \leq L$.

Definition 2.3. A family of real numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ is *separated* if there is a $\delta > 0$ such that $|\lambda_n - \lambda_m| \geq \delta$, whenever $m \neq n$ and it is *relatively separated* if it is a finite union of separated sequence.

Remark 2.4. Duffin and Schaeffer in [3] showed that if $\{\lambda_n\}$ is separated and has uniform density $D > (a-b)$, then the complex exponentials $\{e^{2\pi\lambda_n t}\}_{n\in\mathbb{Z}}$ form a frame for $L^2[a,b]$.

Theorem 2.5. [4] *Let $T_1 \in \mathcal{B}(H_1, H), T_2 \in \mathcal{B}(H_2, H)$ be two bounded linear operators, where H, H_1, H_2 are Hilbert spaces. The following statement are equivalent.*

(i) $R(T_1) \subset R(T_2)$, where $R(T_i)$ denote the range of T_i.
(ii) $T_1T_1^* \leq \lambda^2 T_2T_2^*$ for some $\lambda \geq 0$
(iii) There exists a bounded linear operator $S \in B(H_1, H)$ such that $T_1 = T_2S$.

3. Main Results

We start with the definition of Θ-irregular Weyl-Heisenberg frames in $L^2(\mathbb{R})$.

Definition 3.1. Let $g \in L^2(\mathbb{R})$ and let Θ be a bounded linear operator on $L^2(\mathbb{R})$. A systen $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is said to be a Θ-*irregular Weyl-Heisenberg frame* (or Θ-*irregular* $\mathcal{W}$-$\mathcal{H}$ *frame*) for $L^2(\mathbb{R})$ if there exist finite positive constants $0 < \alpha_0 \leq \beta_0 < \infty$ such that

$$\alpha_0\|\Theta^* f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, E_{x_m}T_{y_n}g\rangle|^2 \leq \beta_0\|\Theta f\|^2, \text{ for all } f \in L^2(\mathbb{R}) \tag{3.1}$$

The positive constants α_0 and β_0 are called *lower* and *upper frame bounds* of the Θ-irregular Weyl-Heisenberg frame $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$, respectively . If upper ineqality in (3.1) is satisfied, then $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is called the Θ-*Bessel sequence for* $L^2(\mathbb{R})$ with Bessel bound β_0.

Remark 3.2. A Θ-irregular Weyl-Heisenberg frame is a K-frame. More precisely, if $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a Θ-irregular Weyl-Heisenberg frame for a suitable measure space $L^2(\Omega)$ (for example discrete signal space) with a choice of bound a_0, b_0, then $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a K-frame for the underlying space with bounds a_0 and b_0 $\|\Theta\|^2$.

Suppose that $\mathcal{F} \equiv \{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame for $L^2(\mathbb{R})$. The operator $T : l^2 \bigoplus l^2 \longrightarrow L^2(\mathbb{R})$ given by

$$T\{c_{mn}\}_{m,n\in\mathbb{Z}} = \sum_{m,n\in\mathbb{Z}} c_{mn}E_{x_m}T_{y_n}g,$$

is called the *pre-frame operator* or *synthesis operator* associated with $\mathcal{F}$ and the adjoint operator $T^* : L^2(\mathbb{R}) \longrightarrow l^2 \bigoplus l^2$ is given by

$$T^*f = \{\langle f, E_{x_m}T_{y_n}g\rangle\}_{m,n\in\mathbb{Z}}$$

is called the *analysis operator* associated with $\mathcal{F}$. Composing T *and* T^*, we obtain the *frame operator* $\mathcal{S} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ given by

$$\mathcal{S}f = TT^*f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{x_m}T_{y_n}g\rangle E_{x_m}T_{y_n}g. \tag{3.2}$$

Since $\mathcal{F}$ is a Θ-Bessel sequence for $L^2(\mathbb{R})$, the series in (3.2) converges unconditionally for all $f \in L^2(\mathbb{R})$. Note that, in general, frame operator of $\mathcal{F}$ is not invertible on $L^2(\mathbb{R})$, but it is invertible on a subspace Range$(\Theta) \subset L^2(\mathbb{R})$.

Example 3.3. Let $\chi_{[0,1]}$ denote the characteristic function for the interval $[0,1]$, and let $x_m = m$, $y_n = n$, where $m, n \in \mathbb{Z}$ and $g = \chi_{[0,1]}$. Choose $\Theta = T_\xi$, a translation operator. Then, $\mathcal{F} \equiv \{e^{2\pi i m(\bullet - \xi)}\chi_{[0,1]}(\bullet - \xi - n)\}_{m,n\in\mathbb{Z}}$ is a Θ-irregular $\mathcal{W}$-$\mathcal{H}$-frame for $L^2(\mathbb{R})$ with bounds $\alpha_0 = \beta_0 = 1$.
Indeed

$$\begin{aligned}
\sum_{m,n\in\mathbb{Z}} |\langle f, e^{2\pi i m(\bullet-\xi)}\chi_{[0,1]}(\bullet - \xi - n)\rangle|^2 &= \|T_\xi^* f\|^2 \\
&= \|T_{-\xi} f\|^2 \\
&= \|T_\xi f\|^2, \text{ for all } f \in L^2(\mathbb{R}).
\end{aligned}$$

Example 3.4. Consider the Gaussian function $h = e^{-x^2}$. Choose $x_m = m$, $y_n = n$ for all $m, n \in \mathbb{N}$. Then, $\{e^{-2\pi mn} E_{x_m} T_{y_n} h\}_{m,n\in\mathbb{Z}} = \{T_{y_n} E_{x_m} h\}_{m,n\in\mathbb{Z}}$ which is not a Θ-irregular $\mathcal{W}$-$\mathcal{H}$-frame for $L^2(\mathbb{R})$.

The following lemma gives a characterization of Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame in term of operator inequality. This is similar to Hilbert frames for Hilbert spaces [13]. For completeness we include its proof.

Lemma 3.5. *Let $g \in L^2(\mathbb{R})$. Then, $\{E_{x_m} T_{y_n} g\}_{m,n\in\mathbb{Z}}$ is a Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame with bounds A,B if and only if $A\Theta\Theta^* \leq \mathcal{S} \leq B\Theta^*\Theta$, where $\mathcal{S}$ is frame operator for $\{E_{x_m} T_{y_n} g\}_{m,n\in\mathbb{Z}}$.*

Proof. We compute

$$\begin{aligned}
&A\,\|\Theta^* f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, E_{x_m} T_{y_n} g\rangle|^2 \leq B\|\Theta f\|^2, \text{ for all } f \in L^2(\mathbb{R}) \\
&\Leftrightarrow A\|\Theta^* f\|^2 \leq \langle \mathcal{S}f, f\rangle \leq B\|\Theta f\|^2, \text{ for all } f \in L^2(\mathbb{R}) \\
&\Leftrightarrow A\langle \Theta^* f, \Theta^* f\rangle \leq \langle \mathcal{S}f, f\rangle \leq B\langle \Theta f, \Theta f\rangle, \text{ for all } f \in L^2(\mathbb{R}) \\
&\Leftrightarrow A\langle \Theta\Theta^* f, f\rangle \leq \langle \mathcal{S}f, f\rangle \leq B\langle \Theta^*\Theta f, f\rangle, \text{ for all } f \in L^2(\mathbb{R}) \\
&\Leftrightarrow A\Theta\Theta^* \leq \mathcal{S} \leq B\Theta^*\Theta.
\end{aligned}$$

The lemma is proved. □

The following theorem provides necessary and sufficient conditions for a given system to be Θ-irregular Weyl-Heisenberg frame for $L^2(\mathbb{R})$.

Theorem 3.6. *Let $g \in L^2(\mathbb{R})$ and let Θ be a bounded linear coercive operator on $L^2(\mathbb{R})$. Then, $\{E_{x_m} T_{y_n} g\}_{m,n\in\mathbb{Z}}$ is a Θ-irregular Weyl-Heisenberg frame for $L^2(\mathbb{R})$ if and only if there exist a bounded linear operator $L : l^2 \bigoplus l^2 \longrightarrow L^2(\mathbb{R})$ such that*

$$L(e_{mn}) = E_{x_m} T_{y_n} g \text{ and } Range(\Theta) \subset Range(L), \tag{3.3}$$

where $\{e_{mn}\}$ is an orthonormal basis for $l^2 \bigoplus l^2$.

Proof. Suppose first that $\{E_{x_m} T_{y_n} g\}_{m,n\in\mathbb{Z}}$ is a Θ-irregular Weyl-Heisenberg frame for $L^2(\mathbb{R})$. Then, we can found positive constant $\mathrm{a}_0, \mathrm{b}_0$ such that

$$\mathrm{a}_0\|\Theta^* f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, E_{x_m} T_{y_n} g\rangle|^2 \leq \mathrm{b}_0\|\Theta f\|^2, \text{ for all } f \in L^2(\mathbb{R}). \tag{3.4}$$

Define $\mathcal{S} : L^2(\mathbb{R}) \longrightarrow \ell^2 \bigoplus \ell^2$ by

$$\mathcal{S}(f) = \sum_{m,n\in\mathbb{Z}} \langle f, E_{x_m} T_{y_n} g\rangle e_{mn}$$

Clearly, $\mathcal{S}$ is a well defined bounded linear operator on $L^2(\mathbb{R})$.
Now

$$\begin{aligned}
\langle \mathcal{S}^* e_{mn}, h\rangle &= \langle e_{mn}, \mathcal{S}h\rangle \\
&= \langle e_{mn} \sum_{m,n\in\mathbb{Z}} \langle h, E_{x_m} T_{y_n} g\rangle e_{mn}\rangle \\
&= \sum_{m,n\in\mathbb{Z}} \overline{\langle h, E_{x_m} T_{y_n} g\rangle} \langle e_{mn}, e_{mn}\rangle \\
&= \overline{\langle h, E_{x_m} T_{y_n} g\rangle} \\
&= \langle E_{x_m} T_{y_n} g, h\rangle, \text{ for all } h \in L^2(\mathbb{R}).
\end{aligned}$$

Therefore, $\mathcal{S}^* e_{mn} = E_{x_m} T_{y_n} g$.
Now by using (3.4), we have

$$\mathrm{a}_0 \|\Theta^* f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, \mathcal{S}^* e_{mn}\rangle|^2 = \|\mathcal{S} f\|^2, \text{ for all } f \in L^2(\mathbb{R}).$$

Thus, $\mathrm{a}_0 \Theta\Theta^* \leq \mathcal{S}^*\mathcal{S}$, where $L = \mathcal{S}^*$. Also, by Theorem 2.5, we have Range$(\Theta) \subset$ Range(L).
Conversely, assume that $E_{x_m} T_{y_n} g = L e_{mn}$ where $L \in \mathcal{B}(l^2 \bigoplus l^2, L^2(\mathbb{R}))$ and Range$(\Theta) \subset$Range(L). First we find conjugate of L. For this we compute

$$\begin{aligned}
\langle L^* f, h\rangle &= \left\langle L^* f, \sum_{m,n\in\mathbb{Z}} a_{mn} e_{mn} \right\rangle \\
&= \sum_{m,n\in\mathbb{Z}} \overline{a_{mn}} \langle f, L e_{mn}\rangle \\
&= \sum_{m,n\in\mathbb{Z}} \overline{a_{mn}} \langle f, E_{x_m} T_{y_n} g\rangle \\
&= \sum_{m,n\in\mathbb{Z}} \overline{\langle h, e_{mn}\rangle} \langle f, E_{x_m} T_{y_n} g\rangle \\
&= \sum_{m,n\in\mathbb{Z}} \langle e_{mn}, h\rangle \langle f, E_{x_m} T_{y_n} g\rangle \\
&= \left\langle \sum_{m,n\in\mathbb{Z}} \langle f, E_{x_m} T_{y_n} g\rangle e_{mn}, h \right\rangle, \ f, h \in L^2(\mathbb{R}).
\end{aligned}$$

Therefore

$$L^* f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{x_m} T_{y_n} g\rangle e_{mn}, f \in L^2(\mathbb{R}). \tag{3.5}$$

Now Θ is coercive, there exists a positive constant δ such that

$$\|\Theta(f) | \geq \delta \|f\|, \text{ for all } f \in L^2(\mathbb{R}). \tag{3.6}$$

Therefore, by using (3.3) and (3.6), we have

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} |\langle f, E_{x_m}T_{y_n}g\rangle|^2 &= \sum_{m,n\in\mathbb{Z}} |\langle f, Le_{mn}\rangle|^2 \\ &= \sum_{m,n\in\mathbb{Z}} |\langle L^*f, e_{mn}\rangle|^2 \\ &= \|L^*f\|^2 \\ &\leq \|L^*\|^2\delta^{-2}\|\Theta f\|^2, \text{for all } f \in L^2(\mathbb{R}) \end{aligned} \tag{3.7}$$

By Theorem 2.5 and using fact that Range$(\Theta) \subset$ Range(L)., we can find a positive constant A such that $A\Theta\Theta^* \leq LL^*$. Therefore, by (3.6) and lower frame inequality in (3.4), we have

$$\begin{aligned}A\|\Theta^*f\|^2 &\leq \|L^*f\|^2 \\ &= \sum_{m,n\in\mathbb{Z}} |\langle f, E_{x_m}T_{y_n}g\rangle|^2, \text{for all } f \in L^2(\mathbb{R}). \end{aligned} \tag{3.8}$$

By using (3.7) and (3.8), we conclude that $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame for $L^2(\mathbb{R})$. □

Remark 3.7. In the forward part of Theorem 3.6, the coercivity of Θ is not required.

The following theorem provides sufficient conditions for the existence of a Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame for $L^2(\mathbb{R})$.

Theorem 3.8. *Let Θ be a bounded linear coercive operator on $L^2(\mathbb{R})$. Suppose that $\{x_m\}_{m\in\mathbb{Z}}$ is a set of uniform density in $\mathbb{R}$ and $g \in L^2(\mathbb{R})$ is bounded with support $[-a, a]$. Let $\{y_n\}_{n\in\mathbb{Z}}$ be a separated sequence in $\mathbb{R}$ with*

$$A\|\Theta\|^2 \leq \sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 \leq B\|\Theta\|^{-2} \ a.e. \quad (0 < A, B \in \mathbb{R})$$

Then, $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame for $L^2(\mathbb{R})$.

Proof. Since $\{x_m\}_{m\in\mathbb{Z}}$ is a set of uniform density in $\mathbb{R}$, there is an $a > 0$ such that $\{E_{x_m}\}_{m\in\mathbb{Z}}$ is a frame for $L^2[-a, a]$. Let A_1, B_1 be a choice of bounds for $\{E_{x_m}\}_{m\in\mathbb{Z}}$. Then, for all $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} |\langle f, E_{x_m}T_{y_n}g\rangle|^2 &= \sum_{m,n\in\mathbb{Z}} \langle f.T_{y_n}\overline{g}, E_{x_m}\rangle|^2 \\ &\leq B_1\sum_{n\in\mathbb{Z}} \|f.T_{y_n}\overline{g}\|^2 \\ &= B_1\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} |f(t)|^2|g(t-y_n)|^2dt \\ &= B_1\int_{\mathbb{R}} |f(t)|^2\sum_{n\in\mathbb{Z}} |g(t-y_n)|^2dt \\ &\leq B_1B\|\Theta\|^{-2}\int_{\mathbb{R}} |f(t)|^2dt \\ &\leq B_1B\|\Theta\|^{-2}\|f\|^2\end{aligned}$$

$$\leq B_1 B\delta^{-2}\|\Theta\|^{-2}\|\Theta f\|^2, \tag{3.9}$$

where δ is a positive constant which appear in the coercivity of Θ.
For lower frame condition, we compute

$$\begin{aligned}
A_1 A\|\Theta^* f\|^2 &\leq A_1 A\|\Theta^*\|^2\|f\|^2 \\
&= A_1 A\|\Theta\|^2 \int_{\mathbb{R}} |f(t)|^2 dt \\
&= A_1 \int_{\mathbb{R}} |f(t)|^2 \sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 dt \\
&= A_1 \sum_{n\in\mathbb{Z}} \|f.T_{y_n}\overline{g}\|^2 \\
&= \sum_{n\in\mathbb{Z}} A_1 \|f.T_{y_n}\overline{g}\|^2 \\
&\leq \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |\langle f.T_{y_n}\overline{g}, E_{x_m}\rangle|^2 \\
&= \sum_{m,n\in\mathbb{Z}} |\langle f, E_{x_m}T_{y_n}g\rangle|^2, \quad \text{for all } f \in L^2(\mathbb{R}).
\end{aligned} \tag{3.10}$$

By using (3.9) and (3.10), $\{E_{x_m}T_{y_n}g\}_{m.n\in\mathbb{Z}}$ is a Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame for $L^2(\mathbb{R})$ with bounds A_1A and $B_1B\|\Theta\|^{-2}$. This completes the proof. $\square$

The following theorem gives necessary condition for the existence of Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame for $L^2(\mathbb{R})$.

Theorem 3.9. *Let $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ be a Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame for $L^2(\mathbb{R})$ with bounds A and B. Then, there exists a positive number α such that $\{E_{x_m}\}_{m\in\mathbb{Z}}$ is a frame for $L^2[0,\alpha]$. Furthermore, if A_1, B_1 are bounds of $\{E_{x_m}\}_{m\in\mathbb{Z}}$, then*

$$\sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 \leq \frac{B}{A_1}\|\Theta\|^2.$$

Proof. Let $\mathcal{J} = \{(x_m, y_n) : m, n \in \mathbb{Z}\}$. Then, there exist $r, \omega > 0$ such that for all $c, d \in \mathbb{R}$ $([c, c+r] \times [d, d+\omega]) \bigcap \mathcal{J} \neq \phi$. Therefore, $\{x_m\}_{m\in\mathbb{Z}}$ is a system of uniform density in $\mathbb{R}$. Thus, there exists $\alpha > 0$ such that $\{E_{x_m}\}_{m\in\mathbb{Z}}$ is a frame for $L^2[0,\alpha]$. Let A_1, B_1 be a choice of bounds for $\{E_{x_m}\}_{m\in\mathbb{Z}}$. Then, for any interval $I = [b, b+\alpha]$ and bounded function $f \in L^2(I)$, we have

$$\begin{aligned}
\langle f, E_{x_m}T_{y_n}g\rangle &= \int_I f(t)\overline{E_{x_m}T_{y_n}g(t)}dt \\
&= \int_I f(t)\overline{e^{2\pi i x_m t}g(t-y_n)}dt \\
&= \int_I f(t)\overline{g}(t-y_n)\overline{e^{2\pi i x_m t}}dt \\
&= \int_I (f.T_{y_n}\overline{g})(t)\overline{E_{x_m}(t)}dt \\
&= \langle f.T_{y_n}\overline{g}, E_{x_m}\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \|f.T_{y_n}\overline{g}\|^2 &\leq \sum_{m,n\in\mathbb{Z}} |\langle f.T_{y_n}\overline{g}, E_{x_m}\rangle|^2 \\ &= \sum_{m,n\in\mathbb{Z}} |\langle f, E_{x_m}T_{y_n}g\rangle|^2 \\ &\leq B\|\Theta f\|^2 \\ &\leq B\|\Theta\|^2\|f\|^2.\end{aligned}$$

Thus

$$A_1 \int_I f(t)|^2 \sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 dt \leq B\|\Theta\|^2 \int_I |f(t)|^2 \text{ for all } f \in L^2(\mathbb{R}).$$

Hence $\sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 \leq \frac{B}{A_1}\|\Theta\|^2$. □

Now we give the construction of a new Hilbert Θ-frame for $L^2(\mathbb{R})$ from a given Hilbert Θ-frame for $L^2(\mathbb{R})$ with estimate of the excess of a given Hilbert Θ-frame for the underlying space. This can be generalised to Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame for $L^2(\mathbb{R})$.

Proposition 3.10. *Assume that $\{f_i\}$ is a Hilbert Θ-frame for $L^2(\mathbb{R})$ with bounds A, B and that $\{\|f_i\|\}$ is bounded below by $C > 0$. Assume that each g_k appear n_k times in the system $\{f_i\}$. Then, $N := \sup_k n_k \leq \frac{B\|\Theta\|^2}{C^2}$, and $\{g_k\}$ is a Θ-frame with bounds $\frac{A}{N}$, NB.*

Proof. Given $k \in \mathbb{Z}$, the element g_k appears n_k times in $\{f_i\}$. Thus

$$n_k\|g_k\|^4 \leq \sum_{i\in\mathbb{Z}} |\langle g_k, f_i\rangle|^2 \leq B\|\Theta g_k\|^2 \leq B\|\Theta\|^2\|g_k\|^2.$$

Therefore

$$n_k \leq \frac{B\|\Theta\|^2}{\|g_k\|^2} \leq \frac{B\|\Theta\|^2}{C^2}$$

Hence $N = \sup_k n_k \leq \frac{B\|\Theta\|^2}{C^2}$.

Now the family of elements consisting of all g_k, each of them repeated N-times contains $\{f_i\}$. Therefore, for each $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}N\sum_{k\in\mathbb{Z}} |\langle f, g_k\rangle|^2 &\geq \sum_{i\in\mathbb{Z}} \langle f, f_i\rangle|^2 \\ &\geq A\|\Theta^* f\|^2.\end{aligned}$$

Therefore, $\frac{A}{N}\|\Theta^* f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, g_k\rangle|^2$. Similarly we can show that NB is Θ-Bessel constant for $\{g_k\}$. Hence $\{g_k\}$ is a Hilbert Θ-frame for $L^2(\mathbb{R})$ with bounds $\frac{A}{N}$, NB. The proposition is proved. □

To conclude the paper we show that Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame for $L^2(\mathbb{R})$ are invariant under a linear homeomorphism, provided both Θ and its conjugate commutes with the given homeomorphism. A relation between the best bounds of a given

Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame and best bounds of Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame obtained by the action of linear homeomorphism is given in the following theorem.

Theorem 3.11. *Let $\{x_m\}$ and $\{y_n\}$ be relatively separated sequence, let $g \in L^2(\mathbb{R})$ and $\Theta \in \mathcal{B}(L^2(\mathbb{R}))$. Suppose that $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame for $L^2(\mathbb{R})$ with best bounds A_1 and B_1. If $U : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is a linear homeomorphism such that U commutes with both Θ and Θ^*, then $\{U(E_{x_m}T_{y_n}g)\}_{m,n\in\mathbb{Z}}$ is a Θ-frame for $L^2(\mathbb{R})$ and its best bounds A_2, B_2 satisfy the inequalities*

$$A_1\|U\|^{-2} \leq A_2 \leq A_1\|U^{-1}\|^2 \ ; \ B_1\|U\|^{-2} \leq B_2 \leq B_1\|U\|^2. \tag{3.11}$$

Proof. Since B_1 is an upper bound for $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$. Therefore, for all $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} |\langle f, U(E_{x_m}T_{y_n}g)\rangle|^2 &= \sum_{m,n\in\mathbb{Z}} |\langle U^*f, E_{x_m}T_{y_n}g\rangle|^2 \\ &\leq B_1\|\Theta U^*f\|^2 \\ &\leq B_1\|U^*\|^2\|\Theta f\|^2. \end{aligned} \tag{3.12}$$

Also, by using the fact that A_1 is one of the choice for lower bound for $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$, we have

$$\begin{aligned}\|\Theta^*f\|^2 &= \|\Theta^*(UU^{-1})f\|^2 \\ &= \|U\Theta^*(U^{-1}f)\|^2 \\ &\leq \|U\|^2\|\Theta^*(U^{-1}f)\|^2 \\ &\leq \frac{\|U\|^2}{A_1}\sum_{m,n\in\mathbb{Z}} |\langle U^{-1}f, E_{x_m}T_{y_n}g\rangle|^2 \\ &= \frac{\|U\|^2}{A_1}\sum_{m,n\in\mathbb{Z}} |\langle UU^{-1}f, U(E_{x_m}T_{y_n}g)\rangle|^2 \\ &= \frac{\|U\|^2}{A_1}\sum_{m,n\in\mathbb{Z}} |\langle f, U(E_{x_m}T_{y_n}g)\rangle|^2. \end{aligned} \tag{3.13}$$

By using (3.12) and (3.13), we obtain

$$A_1\|U\|^{-2}\|\Theta^*f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, U(E_{x_m}T_{y_n}g)\rangle|^2 \leq B_1\|U^*\|\|\Theta f\|^2, \quad \text{for all } f \in L^2(\mathbb{R}).$$

Hence $\{U(E_{x_m}T_{y_n}g)\}_{m,n\in\mathbb{Z}}$ is a Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame for $L^2(\mathbb{R})$ with one of the choice of a pair $(A_1\|U\|^{-2}, B_1\|U\|^2)$ as its bounds.

Now A_2 and B_2 are best bounds for $\{U(E_{x_m}T_{y_n}g)\}_{m,n\in\mathbb{Z}}$, we have

$$A_1\|U\|^{-2} \leq A_2, B_2 \leq B_1\|U\|^2. \tag{3.14}$$

Also $\{U(E_{x_m}T_{y_n}g)\}_{m,n\in\mathbb{Z}}$ is a Θ-irregular $\mathcal{W}$-$\mathcal{H}$ frame for $L^2(\mathbb{R})$ with (A_2, B_2) as one of the choice of bounds. So, for all $f \in L^2(\mathbb{R})$, we have

$$A_2\|\Theta^*f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, U(E_{x_m}T_{y_n}g)\rangle|^2 \leq B_2\|\Theta f\|^2. \tag{3.15}$$

Now

$$\|\Theta^*f\|^2 = \|U^{-1}U\Theta^*f\|^2$$

$$= \|U^{-1}\Theta^* Uf\|^2$$
$$\leq \|U^{-1}\|^2 \|\Theta^* Uf\|^2, \text{for all } f \in L^2(\mathbb{R}). \tag{3.16}$$

By using (3.15) and (3.16), we have

$$A_2\|U^{-1}\|^{-2}\|\Theta^* f\|^2 \leq A_2\|\Theta^* Uf\|^2$$
$$\leq \sum_{m,n\in\mathbb{Z}} |\langle Uf, U(E_{x_m}T_{y_n}g)\rangle|^2 (= \sum_{m,n\in\mathbb{Z}} |\langle f, E_{x_m}T_{y_n}g\rangle|^2)$$
$$\leq B_2\|\Theta Uf\|^2$$
$$\leq B_2\|U\|^2\|\Theta f\|^2, \text{ for all } f \in L^2(\mathbb{R}). \tag{3.17}$$

Now A_1 and B_1 are the best bounds for $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$. Therefore, by using (3.17), we have

$$A_2\|U^{-1}\|^{-2} \leq A_1, B_1 \leq B_2\|U\|^2. \tag{3.18}$$

The inequalities in (3.11) is follows from (3.14) and (3.18). □

Corollary 3.12. *If $U : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is a unitary operator, then the best bounds of $\{U(E_{x_m}T_{y_n}g)\}_{m,n\in\mathbb{Z}}$ coincides with the best bounds of $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$.*

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