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A STUDY OF GENERALIZED EXTENDED HANKEL TRANSFORMATIONS

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Abstract

In this paper, we will study generalized extended hankel transformations $B_{1,\mu,m}$ and $B_{2,\mu,m}$ on the spaces.

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1 Introduction

Rooney in [7] has studied the boundedness and range of the transformation

$$(H_{\mu,m}f)(x) = \int_0^\infty \sqrt{xy} J_{\mu,m}(xy) f(y) dy \tag{1.1}$$

where,

$$J_{\mu,m}(x) = \sum_{k=m}^{\infty} \frac{(-1)^k (x/2)^{\mu+2k}}{\Gamma(k+1)\Gamma(\mu+k+1)} \tag{1.2}$$

on $L_{\mu,p}$ space, whereas [1], studied (1.1) on the space H_μ and it's generalized function spaces $H_\mu^{'}$ introduced by [9]. Mendez Perez and Sanchez Quintana in [6], have studied the transformations,

$$(B_{1,\mu}\psi(x))(y) = \int_0^\infty x^{2\mu+1} \mathcal{J}_\mu(xy)\psi(x)dx \tag{1.3}$$

$$(B_{2,\mu}\psi(x))(y) = y^{2\mu+1} \int_0^\infty \mathcal{J}_\mu(xy)\psi(x)dx \tag{1.4}$$

where $\mathcal{J}_\mu(z) = z^{-\mu} J_\mu(z)$

for the testing function space H and H_μ and their dual spaces. By using the cut Bessel function (1.2) we can extend the transformation (1.3) & (1.4) as

$$(B_{1,\mu,m}\psi(x))(y) = y^{-\mu} \int_0^\infty x^{\mu+1} J_{\mu,m}(xy)\psi(x)dx \tag{1.5}$$

$$(B_{2,\mu,m}\psi(x))(y) = y^{\mu+1} \int_0^\infty x^{-\mu} J_{\mu,m}(xy)\psi(x)dx \tag{1.6}$$

respectively.

Our aim is now to study the transformation (1.5) & (1.6) on the spaces H and H_μ and it's dual $H_\mu{}'$. For our convenience, we recall briefly the necessary definitions and important results. Let μ be arbitrary real number, H_μ is the space of all infinitely differentiable complex valued functions $\psi(x)$ defined on I, for which

$$\rho^\mu_{\mu,k} = \sup_{x \in I} \mid x^m x^{-1} D^k x^{-2\mu-1}\psi(x) \mid \tag{1.7}$$

exists for each pair of non negative integers m & k with topology generated by the multinorm $\rho^\mu_{m,k}$. H_μ is a Frechet space. Now suppose that $\psi(x)$ admits the expansion

$$\psi(x) = x^{2\mu+1}\left[b_0 + b_1 x^2 + \ldots + b_k x^{2k} + o(x^{2k})\right] \tag{1.8}$$

in some vicinity of the origin. Obviously function $\psi(x)$ and $x \in I$ belongs to the space H_μ if and only if $\psi(x)$ is infinitely differentiable, has the form (1.8) at the origin and $D^k\psi(x)$ is of rapid discent as $x \to \infty$ for each $k = 0, 1, 2...$ $H_\mu{}'$ denote the dual space of H_μ and it's members are generalized functions of slow growth. The Altenburg space H turns to be particular case of H_μ when $\mu = -1/2$ that is $H = H_{-\frac{1}{2}}$. The following differential operators will be studied for the transformation (1.5) & (1.6).

$$P_\mu\psi(x) = x^{-2\mu-1} D x^{2\mu+2}\psi(x) \tag{1.9}$$

$$T\psi(x) = x^{-1} D\psi(x) \tag{1.10}$$

$$P^*_\mu\psi(x) = -x^{2\mu+2} D x^{-2\mu-1}\psi(x) \tag{1.11}$$

$$T^*\psi(x) = -D x^{-1}\psi(x) \tag{1.12}$$

2 A Study of $B_{1,\mu,m}$ & $B_{2,\mu,m}$ on H and H_μ spaces

In this section we will first study the operators (1.9) and (1.10) for the transformations $B_{1,\mu,m}$ and $B_{2,\mu,m}$.

Theorem 2.1. *For $\mu + 2m \geq -\frac{1}{2}$ and $\mu \in H$*

$$B_{1,\mu+1,m}T\psi = -B_{1,\mu+1,m}\psi \tag{2.1}$$

$$TB_{1,\mu,m}\psi = -B_{1,\mu+1,m-1}\psi \tag{2.2}$$

$$B_{1,\mu,m}(P_\mu T\psi) = -y^2 B_{1,\mu,m-1}\psi \tag{2.3}$$

$$P_\mu TB_{1,\mu,m}\psi = B_{1,\mu,m-1}\left(-x^2\psi\right) \tag{2.4}$$

$$B_{1,\mu,m}(P_\mu\psi) = y^2 B_{1,\mu+1,m-1}\psi \tag{2.5}$$

$$P_\mu B_{1,\mu+1,m}\psi = B_{1,\mu,m-1}\psi \tag{2.6}$$

Proof. We can write L.H.S. of (1.1) as

$$B_{1,\mu+1,m}(T\psi(x)) = y^{\mu+1}\int_0^\infty x^{\mu+1}J_{\mu+1,m-1}(xy)\psi(x)dx$$

which on integrating by parts and using

$$\frac{d}{dx}[x^\nu J_{\nu,m}(x)] = x^\nu J_{\nu-1,m}(x) \tag{2.7}$$

{see [4] [p.186]} gives the required result.

We can write L.H.S. of (2.2) as

$$TB_{1,\mu,m}\psi = y^{-\mu}D\left[y^{-\mu}\int_0^\infty x^{\mu+1}J_{\mu,m}(xy)\psi(x)dx\right] \tag{2.8}$$

On using

$$\frac{d}{dx}[x^{-\nu}J_{\nu,m}(x)] = x^{-\nu}J_{\nu+1,m-1}(x) \tag{2.9}$$

{see [4] [p.186]}

$$\begin{aligned} TB_{1,\mu,m}\psi(x) &= y^{-(\mu+1)}\int_0^\infty x^{\mu+2}J_{\mu+1,m-1}(xy)\psi(x)dx \\ &= -B_{1,\mu+1,m-1}\psi \end{aligned} \tag{2.10}$$

We can write L.H.S. of (1.3) as

$$B_{1,\mu,m}(P_\mu T\psi) = y^{-\mu}\int_0^\infty x^{\mu+1}J_{\mu,m}(xy)x^{-2\mu-1}Dx^{2\mu+2}x^{-1}D\psi(x)dx$$

and

$$B_{1,\mu,m}(P_\mu T\psi) = y^{-\mu+1}\int_0^\infty x^{\mu+1}J_{\mu+1,m-1}(xy)D\psi(x)dx$$

Again integrating by parts & using (2.9) we get the required result.
We can write L.H.S. of (2.4) as

$$\begin{aligned} P_\mu T B_{1,\mu,m}\psi &= y^{-2\mu-1}Dy^{2\mu+1}Dy^{-\mu}\int_0^\infty x^{\mu+1}J_{\mu,m}(xy)\psi(x) \\ &= -y^{-(2\mu+1)}Dy^{2\mu+1}\int_0^\infty Dy^{-\mu}x^{\mu+1}J_{\mu,m}(xy)\psi(x)dx \\ \text{using}(2.9) \\ &= -y^{-(2\mu+1)}Dy^{2\mu+1}\int_0^\infty x^{2\mu+2}y^{-(\mu)}J_{\mu+1,m-1}(xy)\psi(x)dx \end{aligned}$$

which on using (2.7) gives the required result.

We can write L.H.S. of (2.5) as

$$B_{1,\mu,m}(P_\mu\psi) = y^{-\mu}\int_0^\infty x^{\mu+1}J_{\mu,m}(xy)x^{-2\mu-1}Dx^{2\mu+2}\psi(x)dx$$

which on integrating by parts & using (1.8) gives

$$y^{-\mu+1}\int_0^\infty x^{\mu+2}J_{\mu+1,m-1}(xy)\psi(x)dx = y^2 B_{1,\mu+1,m-1}\psi$$

L.H.S. of (1.6) can be written as

$$\begin{aligned} P_\mu B_{1,\mu+1,m}\psi &= y^{-2\mu-1}Dy^{2\mu+2}y^{-\mu-1}\int_0^\infty x^{\mu+2}J_{\mu+1,m}(xy)\psi(x)dx \\ &= B_{1,\mu,m-1}(x^2\psi) \end{aligned}$$

□

Theorem 2.2. *If $\psi \in H_\mu$ then*

$$B_{2,\mu+1,m}(P_\mu^*\psi) = y^2 B_{2,\mu,m}\psi \tag{2.11}$$

$$P_\mu^* B_{2,\mu,m}\psi = B_{2,\mu+1,m-1}(x^2\psi) \tag{2.12}$$

$$B_{2,\mu,m}(T^*P_\mu^*\psi) = -y^2 B_{2,\mu,m-1}\psi \tag{2.13}$$

$$T^*P_\mu^* B_{2,\mu,m}\psi = B_{2,\mu,m-1}(-x^2\psi) \tag{2.14}$$

Proof. The proof follows as theorem (2.1). □

Theorem 2.3. *Let $\mu + 2m \geq -\frac{1}{2}$ and if $\psi \in H_{\mu+1}$ then*

$$B_{2,\mu,m}(T^*\psi) = -B_{2,\mu+1,m-1}\psi \tag{2.15}$$

$$T^* B_{2,\mu+1,m}\psi = -B_{1,\mu,m}\psi \tag{2.16}$$

Proof. Similar as above. □

Theorem 2.4. *If $m \geq 0$ and $Re\mu + 2m \geq -\frac{1}{2}$ then $B_{1,\mu,m}$ is an automorphism on H.*

Proof. Repeating (1.6) k times and multiplying by $(y^2)^n$ we get

$$(y^2)^n P_{\mu+k+1}.....P_{\mu+1}.P_{\mu}.B_{1,\mu+k,m+k-1}\psi = (y^2)^n B_{1,\mu+k-1,m+k-2}(x^2)^k\psi$$

which on using (1.5) n times gives

$$\begin{aligned}&(y^2)^n P_{\mu+k+1}.....P_{\mu+1}.P_{\mu}.B_{1,\mu+k,m+k-1}\psi\\ &= B_{1,\mu+k+n,m+k-n-1}(P_{\mu+n-1}....P_{\mu})(x^2)^k\psi\end{aligned} \tag{2.17}$$

since

$$P_{\mu+k-1}...P_{\mu+1}P_{\mu}\psi(x) = x^{-2\mu+2k-2}(x^{-1}D)^k x^{2\mu+2}\psi \tag{2.18}$$

thus (2.17) becomes

$$\begin{aligned}&(x^{2n}x^{-2\mu+2(k-1)}(x^{-1}D)^k x^{2\mu+2}B_{1,\mu+k,m+k-1}\psi =\\ &x^{-\mu-k+n}\int_0^\infty y^{\mu+k-n+1}J_{\mu+k-n,m+k-n-1}(xy)\\ &y^{-2\mu+2n-2}(y^{-1}D)^n y^{2\mu+2}\psi(y)dy\end{aligned} \tag{2.19}$$

$$\begin{aligned}&x^{-2\mu+n+2k-2}(x^{-1}D)^k x^{2\mu+2}B_{1,\mu+k,m+k-1\psi}(x) =\\ &\int_0^\infty y^{2k+n-1}(y^{-1}D)^n y^{2\mu+2}\\ &\psi(y)(xy)^{-\mu-k}B_{1,\mu+k-n,m+k-n-1}(xy)dy < \infty \text{ for } \mu = -1/2\end{aligned} \tag{2.20}$$

which implies that $B_{1,\mu,m}$ is an automorphism on H. □

3 The Generalized Schwartz's Hankel Transformation $B'_{1,\mu,m}$

Let μ be arbitrary real number such that $\mu + 2m \geq -\frac{1}{2}$. The generalized Hankel transformation $B'_{1,\mu,m}$ is defined on H'_{μ} as the adjoint operator $B_{2,\mu,m}$ on H_{μ} that is

$$\langle B_{1,\mu,m}f, \varphi\rangle = \langle f, B_{2,\mu,m}\varphi\rangle \tag{3.1}$$

Theorem 3.1. *The generalized Schwartz's Hankel transformation $B'_{1,\mu,m}$ of order $\mu + 2m \geq -\frac{1}{2}$ is an automorphism on H'_{μ}.*

Proof. proof will be similar (2.4). $\square$

Theorem 3.2. *Let $\mu + 2m \geq -1/2$ for every $f \in H'_{\mu}$, we obtain*

$$B'_{1,\mu+1,m}(Tf) = -B'_{1,\mu,m}f \tag{3.2}$$

$$TB'_{1,\mu,m}(f) = -B'_{1,\mu+1,m}f \tag{3.3}$$

$$B'_{1,\mu,m}(P_{\mu}Tf) = -y^2 B'_{1,\mu,m-1}f \tag{3.4}$$

$$P_{\mu}TB'_{1,\mu,m}f = B'_{1,\mu,m-1}(-x^2 f) \tag{3.5}$$

Proof. L.H.S. of (1.2) may be written as

$$\begin{aligned} \langle B'_{1,\mu+1,m}Tf, \varphi\rangle &= \langle Tf, B_{2,\mu+1,m}\varphi\rangle \\ &= \langle f, T^* B_{2,\mu+1,m}\varphi\rangle \\ &= \langle f, -B_{2,\mu,m}\varphi\rangle \end{aligned}$$

Thus

$$B'_{1,\mu+1,m}Tf = -B'_{1,\mu,m}f \tag{3.6}$$

Now (3.3) to (3.5) can be proved in a similar manner. $\square$

Theorem 3.3. *If $\mu + 2m \geq -1/2$*

$$\langle B'_{1,\mu,m}(P_{\mu}Tf), \varphi\rangle = \langle P_{\mu}Tf, B_{2,\mu,m}\varphi\rangle \tag{3.7}$$

$$\langle P_{\mu}Tf, B_{2,\mu,m}\varphi\rangle = \langle f, T^* P^*_{\mu} B_{2,\mu,m}\varphi\rangle \tag{3.8}$$

$$\langle f, B_{2,\mu,m}(-y^2\varphi)\rangle = \langle -y^2 B'_{1,\mu,m}f, \varphi\rangle \tag{3.9}$$

$$\langle B'_{1,\mu,m}(P_{\mu}f), \varphi\rangle = \langle P_{\mu}f, B_{2,\mu,m}\varphi\rangle \tag{3.10}$$

Proof. Proof will be similar as (2.1). $\square$

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