

SOME INCLINATION TO RELATIVE WEAK TYPE ORIENTED GROWTH ESTIMATES OF DIFFERENTIAL POLYNOMIALS

Sanjib Kumar Datta

Department of Mathematics, University of Kalyani,
P.O. Kalyani, Dist-Nadia - 741235, West Bengal, India.
e-mail: sanjib_kr_datta@yahoo.co.in

Tanmay Biswas

Rajbari, Rabindrapalli, R. N. Tagore Road,
P.O. Krishnagar, Dist-Nadia - 741101, West Bengal, India.
e-mail: tanmaybiswas_math@rediffmail.com

Md Azizul Hoque

Gobargara High Madrasah (H.S.), P.O.+P.S.- Hariharpara,
Dist-Murshidabad - 742166, West Bengal, India.
e-mail: mhoque3@gmail.com

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Abstract

In this paper our main goal is to establish some comparative growth relationships concerning relative weak type of entire and meromorphic functions generated differential polynomials.

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1 Introduction

Let f be an entire function defined in the open complex plane \mathbb{C} . The function $M_f(r)$ on $|z| = r$ known as maximum modulus function corresponding to f is defined as follows:

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

When f is meromorphic, $M_f(r)$ can not be defined as f is not analytic. In this situation one may define another function $T_f(r)$ known as Nevanlinna's Characteristic function of f , playing the same role as $M_f(r)$ in the following manner:

$$T_f(r) = N_f(r) + m_f(r) .$$

And given two meromorphic functions f and g the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \rightarrow \infty$ is called the growth of f with respect to g in terms of their Nevanlinna's Characteristic function.

When f is entire function, the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r) .$$

We called the function $N_f(r, a)$ ($\bar{N}_f(r, a)$) as counting function of a -points (distinct a -points) of f . In many occasions $N_f(r, \infty)$ and $\bar{N}_f(r, \infty)$ are denoted by $N_f(r)$ and $\bar{N}_f(r)$ respectively. We put

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} dt + \bar{n}_f(0, a) \log r ,$$

where we denote by $n_f(r, a)$ ($\bar{n}_f(r, a)$) the number of a -points (distinct a -points) of f in $|z| \leq r$ and an ∞ -point is a pole of f and the quantity $\Theta(a; f)$ of a meromorphic function f is defined as follows

$$\Theta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, a; f)}{T(r, f)} .$$

Also we denote by $n_p(r, a; f)$ denotes the number of zeros of $f - a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly p times.

Accordingly, $N_p(r, a; f)$ is defined in terms of $n_p(r, a; f)$ in the usual way and we set for any $a \in \mathbb{C} \cup \{\infty\}$

$$\delta_p(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_p(r, a; f)}{T(r, f)} \quad \{\text{cf. [6]}\} ,$$

On the other hand, $m\left(r, \frac{1}{f-a}\right)$ is denoted by $m_f(r, a)$ and we mean $m_f(r, \infty)$ by $m_f(r)$, which is called the proximity function of f . We also put

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \quad \text{where}$$

$$\log^+ x = \max(\log x, 0) \quad \text{for all } x \geq 0 .$$

Further for any non-constant meromorphic function f , $b = b(z)$ is called small with respect to f if $T_b(r) = S_f(r)$ where $S_f(r) = o\{T_f(r)\}$ i.e.,

$\frac{S_f(r)}{T_f(r)} \rightarrow 0$ as $r \rightarrow \infty$. Moreover for any non-constant meromorphic function f , we call $M_j[f] = A_j(f)^{n_{0j}} (f^{(1)})^{n_{1j}} \dots \dots (f^{(k)})^{n_{kj}}$ where $T_{A_j}(r) = S_f(r)$, to be a differential monomial generated by it where $n_{0j}, n_{1j}, \dots, n_{kj}$ ($k \geq 1$) be non-negative integers such that for each $j, \sum_{i=0}^k n_{ij} \geq 1$. In this connection the numbers $\gamma_{M_j} = \sum_{i=0}^k n_{ij}$ and $\Gamma_{M_j} = \sum_{i=0}^k (i+1) n_{ij}$ are called respectively the degree and weight of $M_j[f]$ $\{[1], [8]\}$. The expression $P[f] = \sum_{j=1}^s M_j[f]$ is called a differential polynomial generated by f . The numbers $\gamma_P = \max_{1 \leq j \leq s} \gamma_{M_j}$ and $\Gamma_P = \max_{1 \leq j \leq s} \Gamma_{M_j}$ are called respectively the degree and weight of $P[f]$ $\{[1], [8]\}$. Also we call the numbers $\underline{\gamma}_P = \min_{1 \leq j \leq s} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma}_P = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f i.e. for which $n_{0j} = 0$ for $j = 1, 2, \dots, s$. We consider only those $P[f], P_0[f]$ singularities of whose individual terms do not cancel each other.

The *order* of a meromorphic function f which is generally used in computational purpose is defined in terms of the growth of f with respect to the exponential function as

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r \rightarrow \infty} \frac{\log T_f(r)}{\log \left(\frac{r}{\pi}\right)} = \limsup_{r \rightarrow \infty} \frac{\log T_f(r)}{\log(r) + O(1)}.$$

Lahiri and Banerjee [7] introduced the definitions of *relative order* and *relative lower order* of a meromorphic function with respect to an entire function to avoid comparing growth just with $\exp z$. To compare the relative growth of two meromorphic functions having same non zero finite *relative lower order* with respect to another entire function, Datta and Biswas [3] introduced the notion of *relative weak type* of meromorphic functions with respect to an entire function. Extending these notions of *relative weak type* as cited in the reference, Datta, Biswas and Hoque [4] gave the definition of *relative weak type of differential polynomials* generated by entire and meromorphic functions.

For entire and meromorphic functions, the notion of their growth indicators such as *order*, *lower order* and *weak type* are classical in complex analysis and during the past decades, several researchers have already been continuing their studies in the area of comparative growth properties of composite entire and meromorphic functions in different directions using the same. But at that time, the concept of *relative order* (respectively *relative lower order*) and consequently *relative weak type* of entire and meromorphic functions with respect to another entire function was mostly unknown to complex analysts and they are not aware of the technical advantages of using the relative growth indicators of the functions. Therefore the growth of composite entire and meromorphic

functions needs to be modified on the basis of their *relative order* (respectively *relative lower order*) and *relative weak type* some of which has been explored in this paper. Actually in this paper we establish some newly developed results based on the growth properties of *relative weak type of differential polynomials* generated by entire and meromorphic functions.

2 Notation and preliminary remarks

We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [5] and [9]. Henceforth, we do not explain those in details. Now we just recall some definitions which will be needed in the sequel.

Definition 1 The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r \rightarrow \infty} \frac{\log T_f(r)}{\log r} .$$

To determine the relative growth of two meromorphic functions having same non zero finite lower order, Datta and Jha [2] introduced the definition of *weak type* of a meromorphic function of finite positive lower order in the following way:

Definition 2 [2] The weak type τ_f of a meromorphic function f of finite positive lower order λ_f is defined by

$$\tau_f = \liminf_{r \rightarrow \infty} \frac{T_f(r)}{r^{\lambda_f}} .$$

Similarly, one can define the growth indicator $\bar{\tau}_f$ of a meromorphic function f of finite positive lower order λ_f as

$$\bar{\tau}_f = \limsup_{r \rightarrow \infty} \frac{T_f(r)}{r^{\lambda_f}} .$$

Given a non-constant entire function f defined in the open complex plane \mathbb{C} , its Nevanlinna's Characteristic function is strictly increasing and continuous. Hence there exists its inverse function $T_g^{-1} : (T_g(0), \infty) \rightarrow (0, \infty)$ with $\lim_{s \rightarrow \infty} T_g^{-1}(s) = \infty$.

Lahiri and Banerjee [7] introduced the definition of *relative order* of a meromorphic function f with respect to an entire function g , denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf \{ \mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r \} \\ &= \limsup_{r \rightarrow \infty} \frac{\log T_g^{-1} T_f(r)}{\log r} . \end{aligned}$$

The definition coincides with the classical one [7] if $g(z) = \exp z$. Similarly, one can define the *relative lower order* of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda_g(f) = \liminf_{r \rightarrow \infty} \frac{\log T_g^{-1} T_f(r)}{\log r} .$$

In the case of relative order, it therefore seems reasonable to define suitably the relative weak type of a meromorphic function with respect to an entire function to determine the relative growth of two meromorphic functions having same non zero finite relative lower order with respect to an entire function. Datta and Biswas [3] gave such definitions of relative weak type of a meromorphic function f with respect to an entire function g which are as follows:

Definition 3 [3] *The relative weak type $\tau_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ is defined by*

$$\tau_g(f) = \liminf_{r \rightarrow \infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}} .$$

In a like manner, one can define the growth indicator $\bar{\tau}_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ as

$$\bar{\tau}_g(f) = \limsup_{r \rightarrow \infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}} .$$

3 Some Examples

In this section we present some examples in connection with definitions given in the previous section.

Example 1 (Order (lower order)) *Given any natural number n , let $f(z) = \exp z^n$. Then $M_f(r) = \exp r^n$. Therefore putting $R = 2$ in the inequality $T_f(r) \leq \log M_f(r) \leq \frac{R+r}{R-r} T_f(R)$ {cf. [5]} we get that $T_f(r) \leq r^n$ and $T_f(r) \geq \frac{1}{3} \left(\frac{r}{2}\right)^n$. Hence*

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log T_f(r)}{\log r} = n \text{ and } \lambda_f = \liminf_{r \rightarrow \infty} \frac{\log T_f(r)}{\log r} = n .$$

Further if we take $g = \exp^{[2]} z$, then $T_g(r) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}$ ($r \rightarrow \infty$). Therefore

$$\rho_f = \lambda_f = \infty .$$

Example 2 (Weak type) *Let $f = \exp z$. Then $T_f(r) = \frac{r}{\pi}$. and $\rho_f = 1$. So*

$$\tau_f = \liminf_{r \rightarrow \infty} \frac{T_f(r)}{r^{\lambda_f}} = \frac{\frac{r}{\pi}}{r} = \frac{1}{\pi} \quad \text{and} \quad \bar{\tau}_f = \limsup_{r \rightarrow \infty} \frac{T_f(r)}{r^{\lambda_f}} = \frac{\frac{r}{\pi}}{r} = \frac{1}{\pi} .$$

Further, if we consider $g = \frac{2}{1-\exp(2z)}$, then one can easily verify that

$$\tau_g = \bar{\tau}_g = \frac{2}{\pi} .$$

Example 3 (Relative order (relative lower order)) Suppose $f = g = \exp^{[2]} z$ then $T_f(r) = T_g(r) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}$ ($r \rightarrow \infty$). Now we obtain that

$$\begin{aligned} T_g(r) &\leq \log M_g(r) \leq 3T_g(2r) \quad \{\text{cf. [5]}\} \\ \text{i.e., } T_g(r) &\leq \exp r \leq 3T_g(2r) . \end{aligned}$$

Therefore

$$\begin{aligned} T_g^{-1}T_f(r) &\geq \log \left(\frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \right) \\ \text{i.e., } \liminf_{r \rightarrow \infty} \frac{\log T_g^{-1}T_f(r)}{\log r} &\geq 1 \end{aligned}$$

and

$$\begin{aligned} T_g^{-1}T_f(r) &\leq 2 \log \left(\frac{3 \exp r}{(2\pi^3 r)^{\frac{1}{2}}} \right) \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log T_g^{-1}T_f(r)}{\log r} &\leq 1 . \end{aligned}$$

Hence

$$\rho_g(f) = \lambda_g(f) = 1 .$$

Example 4 (Relative weak type) Suppose $f = g = \exp z$. Then $T_f(r) = T_g(r) = T_{\exp z}(r) = \frac{r}{\pi}$ and $T_g^{-1}T_f(r) = T_g^{-1}\left(\frac{r}{\pi}\right) = r$. So

$$\lambda_g(f) = \liminf_{r \rightarrow \infty} \frac{\log T_g^{-1}T_f(r)}{\log r} = 1.$$

Therefore

$$\tau_g(f) = \liminf_{r \rightarrow \infty} \frac{T_g^{-1}T_f(r)}{r\lambda_g(f)} = 1 \text{ and } \bar{\tau}_g(f) = \limsup_{r \rightarrow \infty} \frac{T_g^{-1}T_f(r)}{r\lambda_g(f)} = 1.$$

4 Lemmas

In this section we present a lemma which will be needed in the sequel.

Lemma 1 [4] If f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and g be an entire function of regular growth having non zero finite order and

$\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$. Then the relative lower order of $P_0[f]$ with respect to $P_0[g]$ are same as those of f with respect to g where $P_0[f]$ and $P_0[g]$ are homogeneous.

Lemma 2 [4] Let f be a meromorphic function either of finite order or of non-zero lower order with $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and g be an entire function of regular growth having non zero finite type and $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$. Then $\tau_{P_0[g]}(P_0[f])$

and $\bar{\tau}_{P_0[g]}(P_0[f])$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g i.e., $\tau_{P_0[g]}(P_0[f]) = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}} \cdot \tau_g(f)$ and $\bar{\tau}_{P_0[g]}(P_0[f]) = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}} \cdot \bar{\tau}_g(f)$ when $\lambda_g(f)$ is positive finite and $P_0[f]$ and $P_0[g]$ are homogeneous.

5 Theorems

In this section we present the main results of the paper.

Theorem 1 Suppose f be a meromorphic function such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$. Also let h be an entire function of regular growth having non zero finite type with $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ and g be any entire function such that $0 < \tau_h(f \circ g) \leq \bar{\tau}_h(f \circ g) < \infty$, $0 < \tau_h(f) \leq \bar{\tau}_h(f) < \infty$ and $\lambda_h(f \circ g) = \lambda_h(f)$. Then

$$\begin{aligned} \frac{\tau_h(f \circ g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \bar{\tau}_h(f)} &\leq \liminf_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \\ &\leq \frac{\tau_h(f \circ g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \tau_h(f)} \\ &\leq \limsup_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \frac{\bar{\tau}_h(f \circ g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \tau_h(f)}. \end{aligned}$$

Proof. From the definition of $\tau_h(f)$, $\tau_h(f \circ g)$ and in view of Lemma 1, Lemma 2 we have for arbitrary positive ε and for all sufficiently large values of r that

$$T_h^{-1} T_{f \circ g}(r) \geq (\tau_h(f \circ g) - \varepsilon)(r)^{\lambda_h(f \circ g)} \quad (1)$$

and

$$T_{P_0[h]}^{-1} T_{P_0[f]}(r) \leq (\bar{\tau}_{P_0[h]}(P_0[f]) - \varepsilon)(r)^{\lambda_{P_0[h]}(P_0[f])}$$

$$\begin{aligned}
& i.e., T_{P_0[h]}^{-1} T_{P_0[f]}(r) \\
& \leq \left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \bar{\tau}_h(f) + \varepsilon \right) (r)^{\lambda_h(f)} .
\end{aligned} \tag{2}$$

Now from (1), (2) and in view of the condition $\lambda_h(f \circ g) = \lambda_h(f)$, it follows for all large values of r that

$$\frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \geq \frac{(\tau_h(f \circ g) - \varepsilon)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \bar{\tau}_h(f) + \varepsilon \right)} .$$

As $\varepsilon (> 0)$ is arbitrary, we obtain from above that

$$\liminf_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \geq \frac{\tau_h(f \circ g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \bar{\tau}_h(f)} . \tag{3}$$

Again for a sequence of values of r tending to infinity,

$$T_h^{-1} T_{f \circ g}(r) \leq (\tau_h(f \circ g) + \varepsilon) (r)^{\lambda_h(f \circ g)} \tag{4}$$

and for all sufficiently large values of r ,

$$T_{P_0[h]}^{-1} T_{P_0[f]}(r) \geq (\tau_{P_0[h]}(P_0[f]) - \varepsilon) (r)^{\lambda_{P_0[h]}(P_0[f])}$$

$$\begin{aligned}
& i.e., T_{P_0[h]}^{-1} T_{P_0[f]}(r) \\
& \geq \left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \tau_h(f) - \varepsilon \right) (r)^{\lambda_h(f)} .
\end{aligned} \tag{5}$$

Combining (4) and (5) and in view of the condition $\lambda_h(f \circ g) = \lambda_h(f)$, we get for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \frac{(\tau_h(f \circ g) + \varepsilon)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \tau_h(f) - \varepsilon \right)} .$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\liminf_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \frac{\tau_h(f \circ g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \tau_h(f)} . \tag{6}$$

Also for a sequence of values of r tending to infinity that

$$T_{P_0[h]}^{-1} T_{P_0[f]}(r) \leq (\tau_{P_0[h]}(P_0[f]) + \varepsilon) (r)^{\lambda_{P_0[h]}(P_0[f])}$$

$$\begin{aligned}
& i.e., \quad T_{P_0[h]}^{-1} T_{P_0[f]}(r) \\
& \leq \left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \tau_h(f) + \varepsilon \right) (r)^{\lambda_h(f)} .
\end{aligned} \tag{7}$$

Now from (1), (7) and in view of the condition $\lambda_h(f \circ g) = \lambda_h(f)$, we obtain for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1} T_{f \circ g}(r)}{T_{P[h]}^{-1} T_{P[f]}(r)} \geq \frac{(\tau_h(f \circ g) - \varepsilon)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \tau_h(f) + \varepsilon \right)} .$$

As $\varepsilon (> 0)$ is arbitrary, we get from above that

$$\limsup_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \geq \frac{\tau_h(f \circ g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \tau_h(f)} . \tag{8}$$

Also for all sufficiently large values of r ,

$$T_h^{-1} T_{f \circ g}(r) \leq (\bar{\tau}_h(f \circ g) + \varepsilon) (r)^{\lambda_h(f \circ g)} . \tag{9}$$

As the condition $\lambda_h(f \circ g) = \lambda_h(f)$ holds, it follows from (5) and (9) for all sufficiently large values of r that

$$\frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \frac{(\bar{\tau}_h(f \circ g) + \varepsilon)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \tau_h(f) - \varepsilon \right)} .$$

Since $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\limsup_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \frac{\bar{\tau}_h(f \circ g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \tau_h(f)} . \tag{10}$$

Thus the theorem follows from (3), (6), (8) and (10).

The following theorem can be proved in the line of Theorem 1 and so its proof is omitted.

Theorem 2 Suppose g be an entire function either of finite order or of non-zero lower order such that $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$.

Also let h be an entire function of regular growth having non zero finite type with $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ and f be any meromorphic function such that $0 < \tau_h(f \circ g) \leq \bar{\tau}_h(f \circ g) < \infty$, $0 < \tau_h(g) \leq$

$\bar{\tau}_h(g) < \infty$ and $\lambda_h(f \circ g) = \lambda_h(g)$. Then

$$\begin{aligned} \frac{\tau_h(f \circ g)}{\left(\frac{\gamma_{P_0[g]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \bar{\tau}_h(g)} &\leq \liminf_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} \\ &\leq \frac{\tau_h(f \circ g)}{\left(\frac{\gamma_{P_0[g]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \tau_h(g)} \\ &\leq \limsup_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} \leq \frac{\bar{\tau}_h(f \circ g)}{\left(\frac{\gamma_{P_0[g]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \tau_h(g)}. \end{aligned}$$

Theorem 3 Suppose f be a meromorphic function such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$. Also let h be an entire function of regular growth having non zero finite type with $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ and g be any entire function such that $0 < \bar{\tau}_h(f \circ g) < \infty$, $0 < \bar{\tau}_h(f) < \infty$ and $\lambda_h(f \circ g) = \lambda_h(f)$. Then

$$\liminf_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \frac{\bar{\tau}_h(f \circ g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \bar{\tau}_h(f)} \leq \limsup_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)}.$$

Proof. From the definition of $\bar{\tau}_{P_0[h]}(P_0[f])$ and in view of Lemma 1 and Lemma 2, we get for a sequence of values of r tending to infinity that

$$\begin{aligned} T_{P_0[h]}^{-1} T_{P_0[f]}(r) &\geq (\bar{\tau}_{P_0[h]}(P_0[f]) - \varepsilon)(r)^{\lambda_{P_0[h]}(P_0[f])} \\ &\text{i.e., } T_{P_0[h]}^{-1} T_{P_0[f]}(r) \\ &> \left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \bar{\tau}_h(f) - \varepsilon \right) (r)^{\lambda_h(f)}. \end{aligned} \quad (11)$$

Now from (9), (11) and in view of the condition $\lambda_h(f \circ g) = \lambda_h(f)$, it follows for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \frac{(\bar{\tau}_h(f \circ g) + \varepsilon)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \bar{\tau}_h(f) - \varepsilon \right)}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\liminf_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \frac{\bar{\tau}_h(f \circ g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \bar{\tau}_h(f)}. \quad (12)$$

Again for a sequence of values of r tending to infinity ,

$$T_h^{-1}T_{f \circ g}(r) \geq (\bar{\tau}_h(f \circ g) - \varepsilon)(r)^{\lambda_h(f \circ g)} . \quad (13)$$

So combining (2) and (13) and in view of the condition $\lambda_h(f \circ g) = \lambda_h(f)$, we get for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f \circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \geq \frac{(\bar{\tau}_h(f \circ g) - \varepsilon)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \bar{\tau}_h(f) + \varepsilon\right)} .$$

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$\limsup_{r \rightarrow \infty} \frac{T_h^{-1}T_{f \circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \geq \frac{\bar{\tau}_h(f \circ g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \bar{\tau}_h(f)} . \quad (14)$$

Thus the theorem follows from (12) and (14).

The following theorem can be carried out in the line of Theorem 3 and therefore we omit its proof.

Theorem 4 Suppose g be an entire function either of finite order or of non-zero lower order such that $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$.

Also let h be an entire function of regular growth having non zero finite type with $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ and f be any meromorphic function such that $0 < \bar{\tau}_h(f \circ g) < \infty$, $0 < \bar{\tau}_h(g) < \infty$ and $\lambda_h(f \circ g) = \lambda_h(g)$. Then

$$\liminf_{r \rightarrow \infty} \frac{T_h^{-1}T_{f \circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[g]}(r)} \leq \frac{\bar{\tau}_h(f \circ g)}{\left(\frac{\gamma_{P_0[g]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \bar{\tau}_h(g)} \leq \limsup_{r \rightarrow \infty} \frac{T_h^{-1}T_{f \circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[g]}(r)} .$$

The following theorem is a natural consequence of Theorem 1 and Theorem 3:

Theorem 5 Suppose f be a meromorphic function such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$. Also let h be an entire function of regular growth having non zero finite type with $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$

or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ and g be any entire function such that $0 < \tau_h(f \circ g) \leq \bar{\tau}_h(f \circ g) < \infty$, $0 < \tau_h(f) \leq \bar{\tau}_h(f) < \infty$ and $\lambda_h(f \circ g) = \lambda_h(f)$. Then

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{T_h^{-1}T_{f \circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} &\leq \min \left\{ A \cdot \frac{\tau_h(f \circ g)}{\tau_h(f)}, A \cdot \frac{\bar{\tau}_h(f \circ g)}{\bar{\tau}_h(f)} \right\} \\ &\leq \max \left\{ A \cdot \frac{\tau_h(f \circ g)}{\tau_h(f)}, A \cdot \frac{\bar{\tau}_h(f \circ g)}{\bar{\tau}_h(f)} \right\} \leq \limsup_{r \rightarrow \infty} \frac{T_h^{-1}T_{f \circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \end{aligned}$$

where $A = \frac{1}{\left(\frac{\gamma_{F_0}[f]}{\gamma_{F_0}[h]}\right)^{\frac{1}{\rho_h}}}$.

The proof is omitted.

Analogously one may state the following theorem without its proof.

Theorem 6 Suppose g be an entire function of finite order or of non-zero lower order and $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$. Also let h be an entire function of regular growth having non zero finite type with $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ and f be any meromorphic function such that $0 < \tau_h(f \circ g) \leq \bar{\tau}_h(f \circ g) < \infty$, $0 < \tau_h(g) \leq \bar{\tau}_h(g) < \infty$ and $\lambda_h(f \circ g) = \lambda_h(g)$. Then

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} &\leq \min \left\{ D \cdot \frac{\tau_h(f \circ g)}{\tau_h(g)}, D \cdot \frac{\bar{\tau}_h(f \circ g)}{\bar{\tau}_h(g)} \right\} \\ &\leq \max \left\{ D \cdot \frac{\tau_h(f \circ g)}{\tau_h(g)}, D \cdot \frac{\bar{\tau}_h(f \circ g)}{\bar{\tau}_h(g)} \right\} \leq \limsup_{r \rightarrow \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} \end{aligned}$$

where $D = \frac{1}{\left(\frac{\gamma_{P_0}[g]}{\gamma_{P_0}[h]}\right)^{\frac{1}{\rho_h}}}$.

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