Some Results on the Product of distributions

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Abstract

In this paper, we propose some generalized results on the product of distributions $x_+^\lambda \ln^p x_+$, $x_+^\mu \ln^q x_+$, $x_+^\nu \ln^r x_+$, $x_-^\lambda \ln^p x_-$, $x_-^\mu \ln^q x_-$, $x_-^\nu \ln^r x_-$ and $\operatorname{sgn} x\, |x|^\lambda \ln^p x_+$ given by Fisher, B.

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1 Introduction

Let the space $\mathcal{D}$ be the space of infinitely differentiable functions with compact support in $[a, b]$ and $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Locally summable functions $x_+^\lambda \ln^p x_+$ and $x_-^\lambda \ln^p x_-$ for $\lambda \geq -1$ and $p = 0, 1, 2, ..$ defined by Fisher are

$$x_+^\lambda \ln^p x_+ = \begin{cases} x^\lambda \ln^p x & x > 0 \\ 0 & x < 0 \end{cases} \tag{1.1}$$

$$x_-^\lambda \ln^p x_- = \begin{cases} |x|^\lambda \ln^p |x| & x > 0 \\ 0 & x < 0 \end{cases} \tag{1.2}$$

Derivatives of x_+^λ and x_-^λ, for $\lambda < -1$ and $\lambda \neq -2, -3, \ldots$, are given by

$$(x_+^\lambda)' = \lambda\, x_+^{\lambda-1} \tag{1.3}$$

$$(x_-^\lambda)' = -\lambda\, x_-^{\lambda-1} \tag{1.4}$$

If r is a positive integer and $-r-1 < \lambda < -r$ then for arbitrary φ in $\mathcal{D}$, we can define the inner product as follows

$$\langle x_+^\lambda, \varphi(x)\rangle = \int_0^\infty x^\lambda \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i\right] dx \tag{1.5}$$

$$\langle x_-^\lambda, \varphi(x)\rangle = \int_{-\infty}^0 |x|^\lambda \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i\right] dx \tag{1.6}$$

For $p = 0, 1, \ldots$ the distributions $x_+^\lambda \ln^p x_+$ and $x_-^\lambda \ln^p x_-$ are defined as,

$$\langle x_+^\lambda \ln^p x_+, \varphi(x)\rangle = \frac{\partial^p}{\partial\lambda^p}\langle x_+^\lambda, \varphi(x)\rangle$$

$$(1.7) \qquad = \int_0^\infty x^\lambda \ln^p x \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i\right] dx$$

$$(1.8) \qquad \langle x_-^\lambda \ln^p x_-, \varphi(x)\rangle = \int_{-\infty}^0 |x|^\lambda \ln^p(|x|) \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i\right] dx$$

If φ is a function whose support is contained in the interval $[-1, +1]$. Then

$$(1.9) \langle x_+^\lambda \ln^p x_+, \varphi(x)\rangle = \int_0^1 x^\lambda \ln^p(x) \left[\psi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i\right] dx + \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!(\lambda+i+1)}$$

Definition 1.1. Let f be the distribution in $\mathcal{D}'$ and let g be an infinitely differentiable function. Then the product $f.g$ is defined by

$$\langle f.g, \varphi\rangle = \langle f, g\varphi\rangle$$

for all test function φ with compact support contained in (a, b).

Definition 1.2. Let f and g be distributions in $\mathcal{D}'$. Let f is the k^{th} derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $f.g(= g.f)$ of f and g is defined on the interval (a, b) and is given by

$$f.g = \Sigma_{i=0}^k \binom{k}{i=0} (-1)^i [F g^{(i)}]^{(k-i)}$$

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties,

(i) $\rho(x) = 0$ for $|x| \geq 1$

(ii) $\rho(x) \geq 0$

(iii) $\rho(x) = \rho(-x)$

(iv) $\int_{-1}^{+1} \rho(x) dx = 1$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. It follows that $\langle \delta_n(x)\rangle$ is regular sequences of infinitely differentiable functions converging to the dirac delta function $\delta(x)$. If f is arbitrary distribution in $\mathcal{D}'$, we define for $n = 1, 2, \ldots$

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t)\rangle$$

It follows that $f_n(x)$ is regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

Definition 1.3. Let f and g be distributions in $\mathcal{D}'$ and let $g_n(x) = (g * \delta_n)(x)$. The non commutative product $f.g$ of f and g exists and is equal to the distribution h on the interval (a, b), if

$$\lim_{n\to\infty} \langle f(x)g_n(x), \varphi(x)\rangle = \langle h(x), \varphi(x)\rangle$$

for all function φ in $\mathcal{D}[a, b]$.

We next, provide the generalization of definitions (1.1), (1.2) and (1.3) as given in [3].

Definition 1.4. Let $g_n(x) = (g * \delta_n)(x)$. The non commutative neutrix product fog of f and g exists and is equal to the distribution h in the interval (a, b) if

$$N - \lim_{n\to\infty} \langle f(x)g_n(x), \varphi(x)\rangle = \langle h(x), \varphi(x)\rangle \quad \forall \quad \varphi \in \mathcal{D}$$

where N is the neutrix, for more details we refer to [7] with domain natural numbers and range real numbers with negligible functions $n^\lambda \ln^{r-1} n$, $\ln^r n$; $\lambda > 0, r = 1, 2, \dots$ and all functions which converge to zero in the normal sense as $n \to \infty$. It is obvious that if the product $f.g$ exists, then the neutrix product fog exists and $f.g = fog$.

The following theorem is stated in [5].

Theorem 1.1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the non commutative neutrix products fog and $fog'(or f'og)$ exists then the product $fog'(or f'og)$ exists and $(fog)' = f'og + fog'$.*

The next theorem is proved in [5]

Theorem 1.2. *The non commutative neutrix products of $x_+^\lambda \ln^p x_+$ and $x_+^\mu \ln^q x_+$ and of $x_-^\lambda \ln^p x_-$ and $x_-^\mu \ln^q x_-$ exist and*

$$(x_+^\lambda \ln^p x_+)o(x_+^\mu \ln^q x_+) = x_+^{\lambda+\mu} \ln^{p+q} x_+ \tag{1.10}$$

$$(x_-^\lambda \ln^p x_-)o(x_-^\mu \ln^q x_-) = x_-^{\lambda+\mu} \ln^{p+q} x_- \tag{1.11}$$

for $\lambda + \mu < -1$ and $\lambda, \mu, \lambda + \mu \neq -1, -2, \dots$ and $p, q = 0, 1, 2, \dots$.

2 Product of distributions of the type $x_+^\lambda \ln^p x_+$, $x_-^\lambda \ln^p x_-$, $\operatorname{sgn} x|x|^\lambda \ln^p x_+$ and $\operatorname{sgn} x|x|^\lambda \ln^p x_-$

In this section we provide the neutrix product of three distributions $x_+^\lambda \ln^p x_+$, $x_+^\mu \ln^q x_+$ and $x_+^\nu \ln^r x_+$. In the same way we can get the product of distributions $x_-^\lambda \ln^p x_-$, $x_-^\mu \ln^q x_-$ and $x_-^\nu \ln^r x_-$.

Theorem 2.1. *The noncommutative neutrix products of $x_+^\lambda \ln^p x_+$, $x_+^\mu \ln^q x_+$, and $x_+^\nu \ln^r x_+$ and of $x_-^\lambda \ln^p x_-$, $x_-^\mu \ln^q x_-$ and $x_-^\nu \ln^r x_-$ exist and*

$$(x_+^\lambda \ln^p x_+)o(x_+^\mu \ln^q x_+)o(x_+^\nu \ln^r x_+) = x_+^{\lambda+\mu+\nu} \ln^{p+q+r} x_+ \tag{2.1}$$

$$(x_-^\lambda \ln^p x_-)o(x_-^\mu \ln^q x_-)o(x_-^\nu \ln^r x_-) = x_-^{\lambda+\mu+\nu} \ln^{p+q+r} x_- \tag{2.2}$$

for $\lambda, \lambda + \mu, \lambda + \mu + \nu < -1$ and $\lambda, \mu, \nu, \lambda+\mu, \lambda+\mu+\nu \neq -1, -2, \dots$ and $p, q, r = 0, 1, 2, \dots$.

Proof. We will first prove the following results

$$(x_+^\lambda)o(x_+^\mu)o(x_+^\nu) = x_+^{\lambda+\mu+\nu} \tag{2.3}$$

$$(x_-^\lambda)o(x_-^\mu)o(x_-^\nu) = x_-^{\lambda+\mu+\nu} \tag{2.4}$$

Fisher has proved that

$$(x_+^\lambda)o(x_+^\mu) = x_+^{\lambda+\mu} \tag{2.5}$$

$$(x_-^\lambda)o(x_-^\mu) = x_-^{\lambda+\mu} \tag{2.6}$$

Using equation(2.5) and equation(2.6) in equation(2.3) and equation(2.4), we get

$$(x_+^{\lambda+\mu})o(x_+^\nu) = x_+^{\lambda+\mu+\nu} \tag{2.7}$$

$$(x_-^{\lambda+\mu})o(x_-^\nu) = x_-^{\lambda+\mu+\nu} \tag{2.8}$$

So our aim is to prove above two equations. For proving equation(2.7) we are taking here $-s-1 < \nu < -s$, for some nonnegative integer s; $\lambda, \mu > -1, \lambda + \mu > -1$ and $\lambda + \mu + \nu \neq -1, -2, \ldots$. Let k be the smallest positive integer greater than $-\lambda - \mu - \nu$. We know that

$$\begin{aligned} x_+^\nu * \delta_n(x) &= \int_{-1/n}^{1/n} (x-t)^\nu \delta_n(t)\, dt \\ &= \int_{-1/n}^{x} (x-t)^\nu \delta_n(t)\, dt + \int_{x}^{1/n} (x-t)^\nu \delta_n(t)\, dt \end{aligned} \tag{2.9}$$

Using

$$(x-t)_+^\nu = \begin{cases} (x-t)^\nu & for \quad x > t \\ 0 & for \quad x < t \end{cases} \tag{2.10}$$

second integral of equation(2.9) vanishes and further using the property of $\delta_n(t)$ we get

$$\begin{aligned} x_+^\nu * \delta_n(x) &= \int_{-1/n}^{x} (x-t)^\nu \delta_n(t)\, dt \\ &= \frac{1}{\nu+1}\left[\delta_n(t)(x-t)^{\nu+1}\right]_{-1/n}^{x} + \frac{1}{\nu+1}\int_{-1/n}^{x} (x-t)^{\nu+1}\delta_n^{(1)}(t)\, dt \\ &= \frac{1}{(\nu+1)(\nu+2)}\int_{-1/n}^{x} (x-t)^{\nu+2}\delta_n^{(2)}(t)\, dt \\ &= \frac{1}{(\nu+1)(\nu+2)(\nu+3)}\int_{-1/n}^{x} (x-t)^{\nu+3}\delta_n^{(3)}(t)\, dt \\ &= \frac{1}{(\nu+1)(\nu+2)(\nu+3)\ldots(\nu+s)}\int_{-1/n}^{x} (x-t)^{\nu+s}\delta_n^{(s)}\, dt \end{aligned}$$

$$= \frac{\Gamma(\nu+1)}{\Gamma(\nu+1)}\left[\frac{1}{(\nu+1)(\nu+2)(\nu+3)\dots(\nu+s)}\right] x_+^{\nu+s} * \delta_n^{(s)} \tag{2.11}$$

$$= \frac{\Gamma(\nu+1)}{\Gamma(\nu+s+1)} x_+^{\nu+s} * \delta_n^{(s)} \tag{2.12}$$

where Γ denotes the gamma function.
We have

$$\begin{aligned}
\int_{-1}^{+1} x^i x_+^{\lambda+\mu}(x_+^\nu)_n dx &= \int_0^1 x^{i+\lambda+\mu}(x_+^\nu)_n dx \\
&= \int_0^{1/n} x^{i+\lambda+\mu}(x_+^\nu)_n dx + \int_{1/n}^1 x^{i+\lambda+\mu}(x_+^\nu)_n dx \\
&= \frac{\Gamma(\nu+1)}{\Gamma(\nu+s+1)} \int_0^{1/n} x^{\lambda+\mu+i} \int_{-1/n}^{x} (x-t)^{\nu+s}\delta_n^{(s)}(t)dt\, dx \\
&\qquad + \int_{1/n}^1 x^{\lambda+\mu+i} \int_{-1/n}^{1/n} (x-t)^\nu \delta_n(t)dtdx \\
&= I_1 + I_2 \qquad (2.13)
\end{aligned}$$

On putting $nt = v$ and $nx = u$ and using $\delta_n(t) = n\rho(nt)$ in I_1, we have

$$\begin{aligned}
I_1 &= \frac{\Gamma(\nu+1)}{\Gamma(\nu+s+1)} \int_0^1 \frac{u^{\lambda+\mu+i}}{n^{\lambda+\mu+i}} \int_{-1}^{u} \frac{(u-v)^{\nu+s}}{n^{\nu+s}} n^{s+1} \rho^{(s)}(v) \frac{dv}{n}\ \frac{du}{n} \\
&= \frac{\Gamma(\nu+1)}{\Gamma(\nu+s+1)} \int_0^1 \frac{u^{\lambda+\mu+i}}{n^{\lambda+\mu+i+\nu+1}} \int_{-1}^{u} (u-v)^{\nu+s}\rho^{(s)}(v)\ dv\ du
\end{aligned}$$

$$I_1 = n^{-\lambda-\mu-\nu-i-1} \frac{\Gamma(\nu+1)}{\Gamma(\nu+s+1)} \int_0^1 u^{\lambda+\mu+i} \int_{-1}^{u} (u-v)^{\nu+s}\rho^{(s)}(v)dvdu \tag{2.14}$$

Hence,

$$N - \lim_{n\to\infty} I_1 = 0 \qquad \text{for } i = 0, 1, 2, \dots k-1 \tag{2.15}$$

On changing the order of integration in I_2

$$\begin{aligned}
I_2 &= \int_{-1/n}^{1/n} \delta_n(t) \int_{1/n}^{1} x^{\lambda+\mu+i}(x-t)^{\nu} dx dt \\
&\quad \text{for } nx = u \text{ and } nt = v \\
&= \int_{-1}^{+1} \rho(v) \int_{1}^{n} n^{-i-\lambda-\mu-\nu-1} u^{\lambda+\mu+i}[u-v]^{\nu} du dv \\
&= n^{-i-\lambda-\mu-\nu-1} \int_{-1}^{1} \rho(v) \int_{1}^{n} u^{\lambda+\mu+\nu+i} \left[1-\frac{v}{u}\right]^{\nu} du dv \\
&= n^{-i-\lambda-\mu-\nu-1} \int_{-1}^{1} \rho(v) \int_{1}^{n} u^{\lambda+\mu+\nu+i} \left[1-\frac{v}{u}\nu+\ldots\right] du dv
\end{aligned}$$

$$\begin{aligned}
N-\lim_{n\to\infty} I_2 &= (i+\lambda+\mu+\nu+1)^{-1} \int_{-1}^{+1} \rho(v) dv \\
&= (i+\lambda+\mu+\nu+1)^{-1} \qquad (2.16)
\end{aligned}$$

for $i = 0, 1, 2, \ldots, k-1$ and using property (iv) of the function $\rho(x)$.
By equation(2.13), equation(2.15) and equation(2.16)

$$N-\lim_{n\to\infty} \int_{-1}^{+1} x^i x_+^{\lambda+\mu} (x_+^{\nu})_n dx = (i+\lambda+\mu+\nu+1)^{-1} \qquad (2.17)$$

for $i = 0, 1, 2, \ldots k-1$

When we take $i = k$ in equ.(2.14), we get

$$\begin{aligned}
I_1 &= n^{-k-\lambda-\mu-\nu} \frac{\Gamma(\nu+1)}{\Gamma(\nu+s+1)} \int_{0}^{1} u^{k+\lambda+\mu} \int_{-1}^{u} (u-v)^{\nu+s} \rho^{(s)}(v) dv du \\
&= \int_{0}^{1/n} x^{k+\lambda+\mu} (x_+^{\nu})_n dx \qquad (2.18)
\end{aligned}$$

If ψ is an arbitrary continuous function then

$$\lim_{n\to\infty} \int_{0}^{1/n} x^{k+\lambda+\mu} (x_+^{\nu})_n \psi(x) dx = 0 \qquad (2.19)$$

since $k+\lambda+\mu+\nu > 0$

Next if $x > 1/n$, we have

$$\begin{aligned}
(x_+^{\nu})_n &= \int_{-1/n}^{1/n} (x-t)^{\nu} \delta_n(t) dt \\
&= \int_{-1}^{1} (x-u/n)^{\nu} \rho(u) du \qquad \text{using } t = u/n
\end{aligned}$$

$$
\begin{aligned}
(x_+^\nu)_n &= x^\nu \int_{-1}^{1} \left[1 - \frac{u}{nx}\right]^\nu \rho(u)dt \\
&= x^\nu \int_{-1}^{1} \left[1 - \frac{\nu u}{nx} + \dots\right] \rho(u)du \\
&= x^\nu + o(x^{\nu-1} n^{-1}) \qquad (2.20)
\end{aligned}
$$

$$
\lim_{n\to\infty} \int_{1/n}^{1} x^{k+\lambda+\mu} (x_+^\nu)_n \psi(x)dx = \int_0^1 x^{k+\lambda+\mu+\nu} \psi(x)dx \qquad (2.21)
$$

Now let φ be an arbitrary function in $\mathcal{D}[-1,1]$. By the mean value theorem we have

$$
\varphi(x) = \sum_{i=0}^{k-1} \frac{\varphi^i(0)}{i!} x^i + \frac{\varphi^k(\xi x)}{k!} x^k \qquad \text{where } 0 < \xi < 1 \qquad (2.22)
$$

then

$$
\begin{aligned}
\left\langle x_+^{\lambda+\mu} (x_+^\nu)_n, \varphi(x) \right\rangle &= \int_{-\infty}^{\infty} x_+^{\lambda+\mu} (x_+^\nu)_n \varphi(x) dx \\
&= \sum_{i=0}^{k-1} \frac{\varphi^i(0)}{i!} \int_{-1}^{+1} x^i x_+^{\lambda+\mu} (x_+^\nu)_n dx + \int_0^{1/n} \frac{x^{k+\lambda+\mu} (x_+^\nu)_n \varphi^k(\xi x)}{k!} dx \\
&\quad + \int_{1/n}^{1} \frac{x^{k+\lambda+\mu} (x_+^\nu)_n \varphi^k(\xi x)}{k!} dx
\end{aligned}
$$

Using equation(2.16), equation(2.19) and equation(2.21)

$$
\begin{aligned}
&N - \lim_{n\to\infty} \left\langle x_+^{\lambda+\mu} (x_+^\nu)_n, \varphi(x) \right\rangle \\
&= \sum_{i=0}^{k-1} \frac{\psi^i(0)}{(i)!} \int_0^1 x^{\lambda+\mu+\nu+i} dx + \int_0^1 x^{\lambda+\mu+\nu} \frac{x^k \varphi^k(\xi x)}{k!} dx \\
&= \int_0^1 x^{\lambda+\mu+\nu} \left[\varphi(x) - \sum_{i=0}^{k-1} \frac{\varphi^i(0)}{i!} x^i\right] dx + \sum_{i=0}^{k-1} \frac{\varphi^{(i)}(0)}{i!(\lambda+\mu+\nu+i+1)} \\
&= \left\langle x_+^{\lambda+\mu+\nu}, \varphi(x) \right\rangle \qquad (2.23)
\end{aligned}
$$

This implies the result

$$
(x_+^{\lambda+\mu}) \; o \; (x_+^\nu) = x_+^{\lambda+\mu+\nu} \qquad (2.24)
$$

Thus equation holds on the interval $[-1,1]$ for $\lambda > -1$, $\mu > -1$ and $\nu < 0$ and $\nu, \lambda+\mu, \lambda+\mu+\nu \neq -1, -2, \dots$ Now differentiating above equation partially with respect to λ, p times we get

$$
(x_+^\lambda \ln^p x_+) \; o \; (x_+^\mu) \; o \; (x_+^\nu) = x_+^{\lambda+\mu+\nu} \ln^p x_+ \qquad (2.25)
$$

Differentiating partially with respect to μ, q times, we get

$$(x_+^\lambda \ln^p x_+) \; o \; (x_+^\mu \ln^q x_+) \; o \; (x_+)^\nu = x_+^{\lambda+\mu+\nu} \ln^{p+q} x_+ \tag{2.26}$$

Again differentiating partially with respect to ν, r times, we have

$$\begin{aligned}(x_+^\lambda \ln^p x_+) \; o \; (x_+^\mu \ln^q x_+) \; o \; (x_+^\nu \ln^r x_+) \\ = x_+^{\lambda+\mu+\nu} (\ln^p x_+) \; o \; (\ln^q x_+) \; o \; (\ln^r x_+)\end{aligned} \tag{2.27}$$

By Fisher[6], we have

$$(\ln^p x_+) \; o \; (\ln^q x_+) = \ln^{p+q} x_+ \tag{2.28}$$

Using this, we get

$$(x_+^\lambda \ln^p x_+) \; o \; (x_+^\mu \ln^q x_+) \; o \; (x_+^\nu \ln^r x_+) = x_+^{\lambda+\mu+\nu} \ln^{p+q+r} x_+ \tag{2.29}$$

□

Theorem 2.2. *The neutrix products of $x_+^\lambda \ln^p x_+$, $x_+^\mu \ln^q x_+$, $x_+^\nu \ln^r x_+$, and of $x_-^\lambda \ln^p x_-$, $x_-^\mu \ln^q x_-$, $x_-^\nu \ln^r x_-$ exist and*

$$(x_+^\lambda \ln^p x_+) \; o \; (x_-^\mu \ln^q x_-) \; o \; (x_+^\nu \ln^r x_+) = 0 \tag{2.30}$$

$$(x_-^\lambda \ln^p x_-) \; o \; (x_+^\mu \ln^q x_+) \; o \; (x_-^\nu \ln^r x_-) = 0 \tag{2.31}$$

for $\lambda+\mu < -1, \quad \lambda+\mu+\nu < -1, \;\; \lambda, \mu, \nu, \lambda+\mu, \lambda+\mu+\nu \neq -1, -2, \dots$ *and* $\;\; p, q, r = 0, 1, 2, \dots$

Proof. Fisher in [6] has given the neutrix product of $x_-^\lambda \ln^p x_-$ and $x_+^\mu \ln^q x_+$ and of $x_+^\lambda \ln^p x_+$ and $x_-^\mu \ln^q x_-$ in [6] as-

$$(x_-^\lambda \ln^p x_-) o (x_+^\mu \ln^q x_+) = 0 \tag{2.32}$$

$$(x_+^\lambda \ln^p x_+) o (x_-^\mu \ln^q x_-) = 0 \tag{2.33}$$

for $\lambda + \mu < -1, \lambda, \mu, \lambda + \mu \neq -1, -2, \dots$ and $p, q = 0, 1, 2, \dots$
Composing these two equation by $x_+^\nu \ln^r x_+$ and $x_-^\nu \ln^r x_-$ from the left, we get the required result. □

Theorem 2.3.

$$\begin{aligned}\Big(\operatorname{sgn} x\, |x|^\lambda \ln^p |x|\Big) \; o \Big(\, |x|^\mu \ln^q |x| \Big) o \Big(\operatorname{sgn} x\, |x|^\nu \ln^r |x| \, \Big) \\ = |x|^{\lambda+\mu+\nu} \ln^{p+q+r} |x|\end{aligned} \tag{2.34}$$

$$\begin{aligned}\Big(|x|^\lambda \ln^p |x|\Big) \; o \Big(\operatorname{sgn} x\, |x|^\mu \ln^q |x| \, \Big) o \Big(\, |x|^\nu \ln^r |x| \, \Big) \\ = \operatorname{sgn} x\, |x|^{\lambda+\mu+\nu} \ln^{p+q+r} |x|\end{aligned} \tag{2.35}$$

for $\lambda+\mu < -1, \quad \lambda+\mu+\nu < -1, \;\; \lambda, \mu, \nu, \lambda+\mu, \lambda+\mu+\nu \neq -1, -2, \dots$ *and* $\;\; p, q, r = 0, 1, 2, \dots$

Proof. Fisher in [2] have shown that

$$\operatorname{sgn} x\,|x|^{\lambda}\ln^{p}|x| = x_{+}^{\lambda}\ln^{p}x_{+} - x_{-}^{\lambda}\ln^{p}x_{-} \tag{2.36}$$

$$|x|^{\lambda}\ln^{p}|x| = x_{+}^{\lambda}\ln^{p}x_{+} + x_{-}^{\lambda}\ln^{p}x_{-} \tag{2.37}$$

In [6], it is proved that

$$\Big(\operatorname{sgn} x\,|x|^{\lambda}\ln^{p}|x|\Big)\ o\ \Big(|x|^{\mu}\ln^{q}|x|\Big) = \operatorname{sgn} x\,|x|^{\lambda+\mu}\ln^{p+q}|x| \tag{2.38}$$

$$\Big(|x|^{\lambda}\ln^{p}|x|\Big)\ o\ \Big(\operatorname{sgn} x\,|x|^{\mu}\ln^{q}|x|\Big) = \operatorname{sgn} x\,|x|^{\lambda+\mu}\ln^{p+q}|x| \tag{2.39}$$

Using equations (2.36), (2.37), (2.38) and (2.39)

$$\begin{aligned}
&\Big(\operatorname{sgn} x\,|x|^{\lambda}\ln^{p}|x|\Big)\ o\ \Big(|x|^{\mu}\ln^{q}|x|\Big)\ o\ \Big(\operatorname{sgn} x\,|x|^{\nu}\ln^{r}|x|\Big)\\
&\quad= \Big(\operatorname{sgn} x\,|x|^{\lambda+\mu}\ln^{p+q}|x|\Big)\ o\ \Big(\operatorname{sgn} x\,|x|^{\nu}\ln^{r}|x|\Big)\\
&\quad= \Big(x_{+}^{\lambda+\mu}\ln^{p+q}x_{+} - x_{-}^{\lambda+\mu}\ln^{p+q}x_{-}\Big)\ o\ \Big(x_{+}^{\nu}\ln^{r}x_{+} - x_{-}^{\nu}\ln^{r}x_{-}\Big)\\
&\quad= x_{+}^{\lambda+\mu+\nu}\ln^{p+q+r}x_{+} + x_{-}^{\lambda+\mu+\nu}\ln^{p+q+r}x_{-}\\
&\quad= |x|^{\lambda+\mu+\mu}\ln^{p+q+r}|x|
\end{aligned}$$

Similarly we can prove equation (2.35). □

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