

On Pál Type Weighted $(0; 0, 1, \dots, r-2, r)$ -Interpolation On Mixed Tchebycheff Abscissas

Neha Mathur

*Department of Mathematics,
Career Convent College, Lucknow, INDIA.
email: neha_mathur13@yahoo.com*

Abstract

In this paper, we have considered the interpolation problem when function values are prescribed on the zeros of $(n - 1)$ th Tchebycheff polynomial of second kind and the weighted $(0, 1, \dots, r - 2, r)$ data is prescribed on the zeros of n th Tchebycheff polynomial of first kind. It has been shown that such an interpolation exists when n is even, the explicit representation of which has been obtained. The converse problem has also been dealt with.

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1 Introduction

Let $\{x_{2i,2n+1}\}_{i=1}^n$ and $\{x_{2i+1,2n+1}\}_{i=1}^{n-1}$ be two distinct point systems in the interval $[-1, 1]$, which are inter scaled such that

$$(1.1) \quad -1 = x_{2n+1,2n+1} < x_{2n,2n+1} < \dots < x_{1,2n+1} = 1.$$

In this paper, we consider the points $\{x_{2i,2n+1}\}_{i=1}^n$ and $\{x_{2i+1,2n+1}\}_{i=1}^{n-1}$ as the distinct zeros of $T_n(x)$ the n^{th} Tchebycheff polynomial of first kind and $U_{n-1}(x)$ the n^{th} Tchebycheff polynomial of second kind respectively. Further let $\{\alpha_{i,2n+1}\}_{i=1}^n$, $\{\alpha_{2i+1,2n+1}^m\}_{i=1}^{n-1}$, $m = 1, 2, \dots, r - 2, r$ be arbitrary given numbers. We seek to find a polynomial $S_n(x)$ of minimum possible degree satisfying the conditions:

$$(1.2) \quad S_n(x_{2i,2n+1}) = \alpha_{2i,2n+1}; \quad i = 1, 2, \dots, n$$

$$(1.3) \quad S_n(x_{1,2n+1}) = \alpha_{1,2n+1}; \quad S_n(x_{2n+1,2n+1}) = \alpha_{2n+1,2n+1}$$

$$(1.4) \quad S_n^{(m)}(x_{2i+1,2n+1}) = \alpha_{2i+1,2n+1}^m; \quad i = 1, 2, \dots, n-1; \quad m = 0, 1, \dots, r-2$$

$$(1.5) \quad (wS_n)^{(r)}(x_{2i+1,2n+1}) = \alpha_{2i+1,2n+1}^r; \quad i = 1, 2, \dots, n-1$$

where $w(x) \in C^r(-1, 1)$, $r \geq 2$ is a weight function defined by $w(x) = (1 - x^2)^{(3r-3)/4}$.

In 1975, L. G. Pál [11] introduced an interpolation process on an interscaled set of points

$$(1.6) \quad -\infty < x_n < y_{n-1} < \dots < x_2 < y_1 < x_1 < \infty.$$

where $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^{n-1}$ are the distinct real zeros of (say)

$$W_n(x) = \prod_{l=1}^n (x - x_l).$$

and of $W'_n(x)$ respectively. Pál proved that for given arbitrary numbers $\{\alpha_i^*\}_{i=1}^n$ and $\{\beta_i^*\}_{i=1}^{n-1}$ there exists a unique polynomial of degree $\leq 2n - 1$ satisfying the conditions:

$$(1.7) \quad R_n(x_i) = \alpha_i^*, \quad i = 1, 2, \dots, n,$$

$$(1.8) \quad R'_n(y_i) = \beta_i^*, \quad i = 1, 2, \dots, n-1$$

and an initial condition

$$R_n(a) = 0$$

where a is a given point, different from the nodal points (1.6). After which many mathematicians have taken up this problem on different sets of nodes. For more details one is referred to [2], [3], [5] - [9], [12] - [15], [17] - [18] etc.

In this paper we have shown that for n odd, there exist a unique polynomial $S_n(x)$ of degree $\leq 2n$ satisfying the conditions (1.2)-(1.5). The explicit representation of $S_n(x)$ is obtained. The converse of the above problem has also been dealt with.

In section 2, we give preliminaries. Existence, uniqueness and the explicit representation of the interpolatory polynomials have been dealt with in Section 3. Section 4 is devoted to the converse problem.

2 Preliminaries

We characterize the points

$$(2.1) \quad x_{2i} = \cos \left(i - \frac{1}{2} \right) \frac{\pi}{n}, \quad i = 1(1)n$$

as the zeros of $T_n(x) = \cos n\theta, x = \cos\theta (-1 < x < 1)$, n^{th} Tchebycheff polynomial of first kind and

$$(2.2) \quad x_{2i+1} = \cos \left(\frac{i\pi}{n} \right), \quad i = 1(1)n-1$$

as the zeros of $U_{n-1}(x)$ the $(n-1)^{th}$ Tchebycheff polynomial of second kind. Obviously $x = 0$ either in $\{x_{2i}\}_{i=1}^n$ or in $\{x_{2i+1}\}_{i=1}^{n-1}$ according as n is even or odd. Also $x_i = -x_{2n+2-i}, i = 1(1)n$. The differential equation satisfied by $T_n(x)$ [16] is

$$(2.3) \quad (1 - x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0$$

and that $U_{n-1}(x)$ is

$$(2.4) \quad (1-x^2)U''_{n-1}(x) - 3xU'_{n-1}(x) + (n^2 - 1)U_{n-1}(x) = 0.$$

We have that for $i = 1, 2, \dots, n$

$$(2.5) \quad \ell_{2i}(x) = \frac{T_n(x)}{(x - x_{2i}T'_n(x_{2i}))}$$

and for $i = 1, 2, \dots, n-1$

$$(2.6) \quad \ell_{2i+1}(x) = \frac{U_{n-1}(x)}{(x - x_{2i+1}U'_{n-1}(x_{2i+1}))}$$

Also,

$$(2.7) \quad (U_{n-1}^{r-1})^{(q)}(x_{2i+1}) = \begin{cases} 0, & q < r-1 \\ (r-1)!U'_{n-1}(x_{2i+1})^{r-1}, & q = r-1 \\ \frac{(r-1)(r)!}{2}U'_{n-1}(x_{2i+1})^{r-1}U''_{n-1}(x_{2i+1}), & q = r \end{cases}$$

$$(2.8) \quad \left[(1-x^2)^{(3r-3)/4}T_n(x)U_{n-1}^{r-1}(x) \right]_{x=x_{2i+1}}^{(r)} = 0.$$

3 Existence, Uniqueness and Explicit Representation of the Interpolatory Polynomials

We shall prove the following:

Theorem 3.1. *Let n be odd and the $(2n+1)$ points in $[-1, 1]$ be given by equation (1.1) together with (2.1) and (2.2), then there exist a unique polynomial $S_n(x)$ of degree $\leq nr+n-r+1$ satisfying the conditions (1.2) - (1.5).*

For n odd, the interpolatory polynomial $S_n(x)$ satisfying the conditions (1.2) - (1.5) can be represented as:

$$(3.1) \quad S_n(x) = \sum_{i=1}^n \alpha_{2i} A_{2i}(x) + \alpha_1 A_1(x) + \alpha_{2n+1} A_{2n+1}(x) \\ + \sum_{i=1}^{n-1} \sum_{t=0}^{r-2} \alpha_{2i+1}^t A_{t,2i+1}(x) + \sum_{i=1}^{n-1} \alpha_{2i+1}^r A_{r,2i+1}(x)$$

where $\{A_{2i}(x)\}_{i=1}^n$, $\{A_{t,2i+1}(x)\}_{i=1}^{n-1}$, $t = 0, 1, \dots, r-2$, $\{A_{r,2i+1}(x)\}_{i=1}^{n-1}$ are the fundamental polynomials each of degree $\leq nr+n-r+1$ which are uniquely determined by the following

conditions:

For $i = 1, 2, \dots, n$

$$(3.2) \quad \begin{cases} A_{2i}(x_{2j}) = \delta_{ij}, & j = 1, 2, \dots, n \\ A_{2i}^{(m)}(x_{2j+1}) = 0, & j = 1, 2, \dots, n-1; m = 0, 1, \dots, r-2 \\ (wA_{2i})^{(r)}(x_{2j+1}) = 0, & j = 1, 2, \dots, n-1 \\ A_{2i}(x_1) = 0; A_{2i}(x_{2n+1}) = 0. \end{cases}$$

For $p = 0, n$

$$(3.3) \quad \begin{cases} A_{2p+1}(x_{2j+1}) = \delta_{pj}, & j = 0, n \\ A_{2p+1}(x_{2j}) = 0, & j = 1, 2, \dots, n \\ A_{2p+1}^{(m)}(x_{2j+1}) = 0, & j = 1, 2, \dots, n-1; m = 0, 1, \dots, r-2 \\ (wA_{2p+1})^{(r)}(x_{2j+1}) = 0, & j = 1, 2, \dots, n-1. \end{cases}$$

For $i = 1, 2, \dots, n-1; m, t = 0, 1, \dots, r-2$

$$(3.4) \quad \begin{cases} A_{t,2i+1}(x_{2j}) = 0, & j = 1, 2, \dots, n \\ A_{t,2i+1}^{(m)}(x_{2j+1}) = \delta_{ji}\delta_{mt}, & j = 1, 2, \dots, n-1 \\ (wA_{t,2i+1})^{(r)}(x_{2j+1}) = 0, & j = 1, 2, \dots, n-1 \\ A_{t,2i+1}(x_1) = 0; A_{t,2i+1}(x_{2n+1}) = 0 \end{cases}$$

and

$$(3.5) \quad \begin{cases} A_{r,2i+1}(x_{2j}) = 0, & j = 1, 2, \dots, n \\ A_{r,2i+1}^{(m)}(x_{2j+1}) = 0, & j = 1, 2, \dots, n-1 \\ (wA_{r,2i+1})^{(r)}(x_{2j+1}) = \delta_{ij}, & j = 1, 2, \dots, n-1 \\ A_{r,2i+1}(x_1) = 0; A_{r,2i+1}(x_{2n+1}) = 0. \end{cases}$$

The explicit forms of fundamental polynomials are given in the following:

Theorem 3.2. For n odd, the fundamental polynomials $\{A_{2i}(x)\}_{i=1}^n$ satisfying the conditions (3.2) can be represented as

$$\begin{aligned} A_{2i}(x) = & c_{1i}(1-x^2)U_{n-1}^{r-1}(x)\ell_{2i}^2(x) \\ & + U_{n-1}^{r-1}(x)T_n(x) \left[E_{2i}(x) + c_{2i} \int_{-1}^x U_{n-1}(x)dx \right] \end{aligned}$$

where $\ell_{2i}(x)$ are defined by (2.5),

$$E_{2i}(x) = \frac{3(r-1)c_{1i}}{2T'_n(x_{2i})} \int_{-1}^x x \left[\frac{l_{2i}(x) + c_{3i}U_{n-1}(x)}{x - x_{2i}} dx \right]$$

$$c_{1i} = \frac{1}{(1-x_{2i}^2)U_{n-1}^{r-1}(x_{2i})}$$

$$c_{2i} = -\frac{n}{2} E_{2i}(1) \text{ and } c_{3i} = -\frac{1}{U_{n-1}(x_{2i})}$$

Theorem 3.3. *The fundamental polynomials $A_{2p+1}(x), p = 0, n$ satisfying the conditions (3.3) can be represented as*

$$\begin{aligned} A_{2p+1}(x) &= \frac{(x_{2p+1} + x) U_{n-1}^r(x) T_n(x)}{2x_{2p+1} U_{n-1}^r(x_{2p+1}) T_n(x_{2p+1})} \\ &\quad - \frac{U_{n-1}^{r-1}(x) T_n(x)}{2x_{2p+1} U_{n-1}^r(x_{2p+1}) T_n(x_{2p+1})} \left[\int_{-1}^x (x_{2p+1} + x) U_{n-1}'(x) dx + c_{4i} \int_{-1}^x U_{n-1}(x) dx \right] \end{aligned}$$

where

$$c_{4i} = -\frac{n}{2} \int_{-1}^1 (1+x) U_{n-1}'(x) dx.$$

Theorem 3.4. *For n odd and $r \geq 2$, the fundamental polynomials $\{A_{r,2i+1}(x)\}_{i=1}^{n-1}$ satisfying the conditions (3.5) can be represented as*

$$(3.6) \quad A_{r,2i+1}(x) = U_{n-1}^{r-1}(x) T_n(x) \left[c_{5i} \int_{-1}^x \ell_{2i+1}(x) dx \right]$$

$$(3.7) \quad + c_{6i} \int_{-1}^x U_{n-1}(x) dx \left[\right]$$

where $\ell_{2i+1}(x)$ are defined by (2.6),

$$c_{5i}(x) = \frac{1}{r! w(x_{2i+1}) T_n(x_{2i+1}) U_{n-1}'(x_{2i+1})^{r-1}}$$

and

$$c_{6i} = -\frac{n}{2} c_{5i} \int_{-1}^1 \ell_{2i+1}(x) dx.$$

Theorem 3.5. *For n odd and $r \geq 2$, the fundamental polynomials $\{A_{t,2i+1}(x)\}$ for $t = 0, 1, \dots, r-2$ and $i = 1, 2, \dots, n-1$ satisfying the conditions (3.4) can be represented as*

$$\begin{aligned} (3.8) \quad A_{t,2i+1}(x) &= c_{7i}(1-x^2) U_{n-1}^t(x) T_n(x) \ell_{2i+1}^{r-t}(x) \\ &\quad + c_{8i} A_{r,2i+1}(x) + U_{n-1}^{r-1}(x) T_n(x) \left[H_{t,2i+1}(x) \right. \\ &\quad \left. + c_{9i} \int_{-1}^x U_{n-1}(x) dx \right] + \sum_{s=t+1}^{r-2} e_{s,2i+1} A_{s,2i+1}(x) \end{aligned}$$

where the summation is zero for $t = r - 2$ and $\ell_{2i+1}(x)$ are defined by (2.6). Also,

$$(3.9) \quad H_{t,2i+1}(x) = -\frac{1}{U'_{n-1}(x_{2i+1})^{r-t-1}} \int_{-1}^x \frac{(1-x^2)}{(x-x_{2i+1})^{r-t-1}} \left[\ell'_{2i+1}(x) \right. \\ \left. - \sum_{k=0}^{r-t-2} g_{k,2i+1}(x-x_{2i+1})^k \ell_{2i+1}(x) \right] dx$$

where for $s = 0, 1, \dots, r-t-2$

$$(3.10) \quad \sum_{k=0}^s g_{k,2i+1} \ell_{2i+1}^{(s-k)}(x_{2i+1}) - \ell_{2i+1}^{(s+1)}(x_{2i+1}) = 0,$$

$$(3.11) \quad c_{7i} = \frac{1}{t!(1-x_{2i+1}^2)T_n(x_{2i+1})} U'_{n-1}(x_{2i+1})^t, \quad c_{9i} = -\frac{n}{2} H_{t,2i+1}(1)$$

$$(3.12) \quad c_{8i} = - \left[c_{7i} \left\{ w(x)(1-x^2) U_{n-1}^t(x) T_n(x) \ell_{2i+1}^{r-t}(x) \right\}_{x=x_{2i+1}}^{(r)} \right. \\ \left. + \frac{1}{(r-t-1)!} \left\{ \sum_{s=0}^{r-t-1} \binom{r-t-1}{s} s! g_{s,2i+1} (\ell_{2i+1}(x))_{x=x_{2i+1}}^{(r-t-s-1)} \right\} \right. \\ \left. - (\ell_{2i+1}(x))_{x=x_{2i+1}}^{(r-t)} \right]$$

and for $s = t+1, t+2, \dots, r-2$

$$(3.13) \quad e_{s,2i+1} = -c_{7i} \left\{ (1-x^2) U_{n-1}^t(x) T_n(x) \right\}_{x=x_{2i+1}}^{(s)}.$$

4 determination of fundamental polynomials (converse problem)

For n odd, the interpolatory polynomial $S_n^*(x)$ satisfying the conditions (1.2) - (1.5) can be represented as:

$$(4.1) \quad S_n^*(x) = \sum_{i=1}^{n-1} \beta_{2i+1} B_{2i}(x) + \beta_1 B_1(x) + \beta_{2n+1} B_{2n+1}(x) \\ + \sum_{i=1}^n \sum_{t=0}^{r-2} \beta_{2i}^t B_{t,2i}(x) + \sum_{i=1}^n \beta_{2i}^r B_{r,2i}(x)$$

where $\{B_{2i+1}(x)\}_{i=1}^n$, $\{B_{t,2i}(x)\}_{i=1}^n, t = 0, 1, \dots, r-2$, $\{B_{r,2i}(x)\}_{i=1}^n$ are the fundamental polynomials each of degree $\leq nr + n$ satisfying the conditions:

For $i = 1, 2, \dots, n-1$

$$(4.2) \quad \begin{cases} B_{2i+1}(x_{2j+1}) = \delta_{ij}, & j = 1, 2, \dots, n-1 \\ B_{2i+1}^{(m)}(x_{2j}) = 0, & j = 1, 2, \dots, n; m = 0, 1, \dots, r-2 \\ (wB_{2i+1})^{(r)}(x_{2j}) = 0, & j = 1, 2, \dots, n \\ B_{2i+1}(x_1) = 0; B_{2i+1}(x_{2n+1}) = 0. \end{cases}$$

For $p = 0, n$

$$(4.3) \quad \begin{cases} B_{2p+1}(x_{2s+1}) = \delta ps, s = 0, n \\ B_{2p+1}(x_{2j}) = 0, & j = 1, 2, \dots, n-1 \\ B_{2p+1}^{(m)}(x_{2j}) = 0, & j = 1, 2, \dots, n; m = 0, 1, \dots, r-2 \\ (wB_{2p+1})^{(r)}(x_{2j}) = 0, & j = 1, 2, \dots, n. \end{cases}$$

For $i = 1, 2, \dots, n; m, t = 0, 1, \dots, r-2$

$$(4.4) \quad \begin{cases} B_{t,2i}(x_{2j+1}) = 0, & j = 1, 2, \dots, n-1 \\ B_{t,2i}^{(m)}(x_{2j}) = \delta_{ji}\delta_{mt}, & j = 1, 2, \dots, n \\ (wB_{t,2i})^{(r)}(x_{2j}) = 0, & j = 1, 2, \dots, n \\ B_{t,2i}(x_1) = 0; B_{t,2i}(x_{2n+1}) = 0 \end{cases}$$

and

$$(4.5) \quad \begin{cases} B_{r,2i}(x_{2j+1}) = 0, & j = 1, 2, \dots, n-1 \\ B_{r,2i}^{(m)}(x_{2j}) = 0, & j = 1, 2, \dots, n \\ (wB_{r,2i})^{(r)}(x_{2j}) = \delta_{ij}, & j = 1, 2, \dots, n \\ B_{r,2i}(x_1) = 0; B_{r,2i}(x_{2n+1}) = 0. \end{cases}$$

The explicit forms of these fundamental polynomials are given in the following:

Theorem 4.1. For n odd, the fundamental polynomials $\{B_{2i+1}(x)\}_{i=1}^n$ satisfying the conditions (4.2) can be represented as

$$\begin{aligned} B_{2i+1}(x) &= c_{10i}(1-x^2)T_n^r(x)\ell_{2i+1}(x) \\ &\quad + T_n^{r-1}(x)U_{n-1}(x) \left[E_{2i+1}^*(x) + c_{11i} \int_{-1}^x T_n(x)dx \right] \end{aligned}$$

where $\ell_{2i+1}(x)$ are defined by (2.6),

$$E_{2i+1}^*(x) = nc_{10i} \int_{-1}^x (1-x^2)\ell_{2i+1}(x)dx$$

$$c_{10i} = \frac{1}{(1-x_{2i+1}^2)T_n^r(x_{2i+1})} \text{ and } c_{11i} = -\frac{n^2-1}{2}E_{2i+1}^*(1).$$

Theorem 4.2. *The fundamental polynomials $B_{2p+1}(x)$ satisfying the conditions (4.3) can be represented as*

$$\begin{aligned} B_{2p+1}(x) &= \frac{(x + x_{2p+1})T_n^r(x)U_{n-1}(x)}{2x_{2p+1}T_n^r(x_{2p+1})U_{n-1}(x_{2p+1})} \\ &\quad - \frac{U_{n-1}(x)T_n^{r-1}(x)}{2T_n^r(x_{2p+1})U_{n-1}(x_{2p+1})} \left[\int_{-1}^x (x + x_{2p+1})T_n'(x)dx + c_{12i} \int_{-1}^x T_n(x)dx \right] \end{aligned}$$

where

$$c_{12i} = -\frac{n^2 - 1}{2} \int_{-1}^1 (1 + x)T_n'(x)dx.$$

Theorem 4.3. *For n odd and $r \geq 2$, the fundamental polynomials $\{B_{r,2i}(x)\}_{i=1}^n$ satisfying the conditions (4.5) can be represented as*

$$B_{r,2i}(x) = U_{n-1}(x)T_n^{r-1}(x) \left[c_{14i} \int_{-1}^x \ell_{2i}(x)dx + c_{15i} \int_{-1}^x T_n(x)dx \right]$$

where $\ell_{2i}(x)$ are defined by (2.5),

$$c_{14i}(x) = \frac{1}{r!w(x_{2i})T_n'(x_{2i})^{r-1}U_{n-1}(x_{2i})} \quad \text{and} \quad c_{15i} = -\frac{n^2 - 1}{2} c_{14i} \int_{-1}^1 \ell_{2i}(x)dx.$$

Theorem 4.4. *For n odd and $r \geq 2$, the fundamental polynomials $\{B_{t,2i}(x)\}$ for $t = 0, 1, \dots, r-2$ and $i = 1, 2, \dots, n$ satisfying the conditions (4.4) can be represented as*

$$\begin{aligned} B_{t,2i}(x) &= c_{16i}(1-x^2)U_{n-1}(x)T_n^t(x)\ell_{2i}^{r-t}(x) \\ &\quad + c_{17i}B_{r,2i}(x) + U_{n-1}(x)T_n^{r-1}(x) \left[H_{t,2i}^*(x) \right. \\ &\quad \left. + c_{18i} \int_{-1}^x T_n(x)dx \right] + \sum_{s=t+1}^{r-2} e_{s,2i}^* B_{s,2i}(x) \end{aligned}$$

where the summation is zero for $t = r-2$, $\ell_{2i}(x)$ are defined by (2.5),

$$\begin{aligned} H_{t,2i}^*(x) &= -\frac{1}{T_n'(x_{2i})^{r-t-1}} \int_{-1}^x \frac{(1-x^2)}{(x-x_{2i})^{r-t-1}} \left[\ell_{2i}'(x) \right. \\ &\quad \left. - \sum_{k=0}^{r-t-2} g_{k,2i}^*(x-x_{2i})^k \ell_{2i}(x) \right] dx \end{aligned}$$

where for $s = 0, 1, \dots, r-t-2$

$$\sum_{k=0}^s g_{k,2i}^* \ell_{2i}^{(s-k)}(x_{2i}) - \ell_{2i}^{(s+1)}(x_{2i}) = 0,$$

$$c_{16i} = \frac{1}{t!(1-x_{2i}^2)T'_n(x_{2i})^t U_{n-1}(x_{2i})}, \quad c_{18i} = -\frac{n^2-1}{2} H_{t,2i}^*(1),$$

$$\begin{aligned} c_{17i} = & - \left[c_{16i} \left\{ w(x)(1-x^2)U_{n-1}(x)T_n^t(x)\ell_{2i}^{r-t}(x) \right\}_{x=x_{2i}}^{(r)} \right. \\ & + \frac{1}{(r-t-1)!} \left\{ \sum_{s=0}^{r-t-1} \binom{r-t-1}{s} s! g_{s,2i}^* (\ell_{2i}(x))_{x=x_{2i}}^{(r-t-s-1)} \right\} \\ & \left. - (\ell_{2i}(x))_{x=x_{2i}}^{(r-t)} \right] \end{aligned}$$

and for $s = t+1, t+2, \dots, r-2$

$$(4.6) \quad e_{s,2i}^* = -c_{16i} \left\{ (1-x^2)U_{n-1}(x)T_n^t(x) \right\}_{x=x_{2i}}^{(s)}.$$

5 Proof of Theorems

We will prove Theorem 3.4 only as the proof of other Theorems is similar to that of this theorem, so we omit details.

Proof of Theorem 3.4. By (3.8), for $t = r-2$ we obtain $A_{r-2,2i+1}(x)$ since $A_{r,2i+1}(x)$ is already known by (3.6) and the last summation vanishes. Similarly, taking $t = r-3$ we can determine $A_{r-3,2i+1}(x)$ in terms of $A_{r,2i+1}(x)$ and $A_{r-2,2i+1}(x)$ by the relation (3.8). Continuing this process, we get $A_{r-4,2i+1}(x)$, $A_{r-5,2i+1}(x)$ and so on in terms of subsequent $A_{q,2i+1}(x)$'s. Since $H_{t,2i+1}(x)$, given by (3.9) is a polynomial of degree $\leq n$, we have $\left(\sum_{k=0}^{r-t-2} g_{k,2i+1}(x-x_{2i+1})^k \ell_{2i+1}(x) - \ell'_{2i+1}(x) \right)_{x=x_{2i+1}}^{(s)} = 0$ for $s = 0, 1, \dots, r-t-2$ determining the values of $g_{j,2i+1}$'s for $j = 0, 1, \dots, r-t-2$, which on simplification turns out to be an equation (3.10). Thus, $A_{t,2i+1}(x)$, $t = 0, 1, \dots, r-t-2$ given in (3.8) is a polynomial of degree $nr+n-r+1$. Obviously, $A_{t,2i+1}(x_{2j}) = 0$ for $j = 1, 2, \dots, n$; $i = 1, 2, \dots, n-1$, $A_{t,2i+1}(x_1) = A_{t,2i+1}(x_{2n+1}) = 0$ gives $c_{8i} = -\frac{n}{2} H_{t,2i+1}(1)$. By Leibnitz's theorem, for $x = x_{2j+1}$, $j = 1, 2, \dots, n-1$ by (2.5) and (2.6) we have

$$A_{t,2i+1}^{(m)}(x_{2j+1}) = \begin{cases} 0, & t > m \\ \delta_{ij}, & t = m \end{cases}$$

when $t < m$ and $j \neq i$, then $A_{t,2i+1}^{(m)}(x_{2j+1}) = 0$ and for $j = i$, we have $A_{t,2i+1}^{(m)}(x_{2i+1}) = 0$ provided

$$e_{s,2i+1} + c_{6i} \left\{ (1-x^2)U_{n-1}^t(x)T_n(x)\ell_{2i+1}^{r-t}(x) \right\}_{x=x_{2i+1}}^{(s)} = 0$$

which gives $e_{s,2i+1}$ as given in (3.13). Also by (3.8), (2.8), (2.5) and (2.6), we have for $j \neq i$,

$$\begin{aligned} (wA_{t,2i+1})^{(r)}(x_{2j+1}) &= c_{6i}\{w(x)(1-x^2)U_{n-1}^t(x)T_n(x)\ell_{2i+1}^{r-t}(x)\}_{x_{2j+1}}^{(r)} \\ &+ r!w(x_{2j+1})U'_{n-1}(x_{2j+1})^{r-1}T_n(x_{2j+1})H'_{t,2i+1}(x_{2j+1}) \\ &= c_{6i}r!w(x_{2j+1})(1-x_{2j+1}^2)U'_{n-1}(x_{2j+1})^tT_n(x_{2j+1}) \\ &\quad \left[\ell'_{2i+1}(x_{2j+1})^{r-t} - \frac{U'_{n-1}(x_{2j+1})^{r-t}}{(x_{2j+1}-x_{2i+1})^{r-t}U'_{n-1}(x_{2i+1})^{r-t}} \right] = 0 \end{aligned}$$

since

$$\ell'_{2i+1}(x_{2j+1}) = \frac{U'_{n-1}(x_{2j+1})}{(x_{2j+1}-x_{2i+1})U'_{n-1}(x_{2i+1})}$$

In the case for $j = i$, we have

$$\begin{aligned} &\lim_{x \rightarrow x_{2i+1}} \frac{\sum_{k=0}^{r-t-2} g_{k,2i+1}(x-x_{2i+1})^k \ell_{2i+1}(x) - \ell'_{2i+1}(x)}{(x-x_{2i+1})^{r-t-1}} \\ &= \frac{1}{(r-t-1)!} \left\{ \sum_{s=0}^{r-t-1} g_{s,2i+1} \binom{r-t-1}{s} s! (\ell_{2i+1}(x))_{x=x_{2i+1}}^{(r-t-s-1)} \right\} - \ell_{2i+1}^{(r-t)}(x_{2i+1}) \end{aligned}$$

Thus

$$\begin{aligned} (wA_{t,2i+1})^{(r)}(x_{2i+1}) &= c_{6i}\{w(x)(1-x^2)U_{n-1}^t(x)T_n(x)\ell_{2i+1}^{r-t}(x)\}_{x_{2i+1}}^{(r)} \\ &+ c_{7i} + \frac{1}{(r-t-1)!} \left\{ \sum_{s=0}^{r-t-1} g_{s,2i+1} \binom{r-t-1}{s} s! (\ell_{2i+1}(x))_{x=x_{2i+1}}^{(r-t-s-1)} \right\} - \ell_{2i+1}^{(r-t)}(x_{2i+1}) = 0, \end{aligned}$$

which gives c_{7i} as given in (3.11).

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