OPTIMAL HARVESTING POLICY OF A PREY–PREDATOR COMMUNITY WITH HOLLING TYPE–IV FUNCTIONAL RESPONSE

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Abstract

The present paper deals with the problem of optimal harvesting of a prey–predator community in which prey follows the law of logistic growth and predator functional response is represented by a Holling type–IV function. In this type of functional response, the predator's per capita rate of predation decreases due to the phenomenon of group defence among the prey. Criteria for the stability of system are derived and bifurcation analysis is done. The optimal harvesting policy is discussed by considering taxation as control instrument and using Pontryagin's maximum principle.

Key Words: Type–IV functional response, optimal harvesting, prey–predator, stability and bifurcation analysis.
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1. Introduction

It is clearly necessary to develop an ecologically acceptable strategy for harvesting of any renewable resource, be it animals, fishes, plants, or whatever. We also usually want to maximize yield with the minimum effort. The inclusion of economic factors in population models of renewable resources is increasing and these introduce important constraints. Several extensive studies have been done by many researchers [2, 6–10] to discuss the economic and biological management of renewable resources.

There has been interesting research work on the dynamics of a prey–predator system also. Bhattacharya and Begum [2] proposed three types of models of two species system. In one of them, they discussed Lotka–Volterra model of one prey and one predator and obtained the feasible bionomic equilibrium points. Mesterton–Gibbons [13] investigated an optimal policy for maximizing the present value from the combined harvesting of two ecologically independent species which would coexist as predator and prey in the absence of harvesting. This optimal policy gave an estimate of the true loss of resource value due to a catastrophic fall in stock level. Chattopadhyay et.al. [5] proposed and analyzed a resource based model in three species fishery consisting of two predators and one prey with competition amongst predators. Recently, Zhang et.al. [16] studied the optimal harvesting policy of a stage–structured predator–prey model and obtained necessary and sufficient condition for the permanence of two species and the extinction of one species and two species.

In most prey–predator models considered in above literature, the predator response to prey density is assumed to be monotonic increasing, the inherent assumption being that the more prey in the environment, the better off the predator, however, this need not always be true.

The term ‘group defence’ is used to describe the phenomenon where predation is decreased or even prevented altogether due to increased ability of the prey to better defend or disguise themselves when their numbers are large enough. An example of this phenomenon is described by Tener [14]. Lone musk ox can be successfully attacked by wolves. Small herds of musk ox (2–6 animals) are attacked but with rare success. No successful attacks have been observed in larger herds. Freedman and Wolkowicz [11] proposed and analyzed predator–prey system with group defence.

Zhu et.al. [18] in their paper considered a predator–prey system with non–monotonic functional response $P(x)=\dfrac{mx}{ax^2+bx+1}$ and did bifurcation analysis. Bhattacharya et.al. [13] studied the stability and bifurcation analysis of a prey–predator ecosystem where predation is represented by a Holling type–IV function. They also performed a comparative study of local stability in the presence and absence of diffusion.

With the phenomenon of group defence as mentioned in above literature in mind, it is therefore, of interest to study optimal harvesting of prey–predator community in which the predator response function is not necessarily a monotone increasing function of prey density but rather is only monotone increasing until some critical density and then becomes monotone decreasing.

So, in this paper, we analyze a prey–predator model with dynamic harvesting effort and a specific inhibition response function the Holling type–IV response function.

2. Model

We consider the model of Freedman and Wolkowicz [11] along with dynamic harvesting effort, governed by the system of equations:

$$
\begin{aligned}
\frac{dx}{dt} &= rx\left[1-\frac{x}{K}\right]-P(x)y-q_1Ex \\
\frac{dy}{dt} &= \pi P(x)y-yd-q_2Ey \\
\frac{dE}{dt} &= \alpha_0E[(p_1-\tau)q_1x+(p_2-\tau)q_2y-c] \\
&x(0)\ \geq 0,\quad y(0)\geq 0,\quad E(0)\geq 0
\end{aligned}
\tag{2.1}
$$

Here x and y represent the densities of prey and predator respectively and E is the combined effort to harvest prey and predator population at time. In the absence of predator and harvesting effort prey population grow logistically with intrinsic growth rate r and carrying capacity K. The natural death rate of the predator is denoted by d and the predator response function is denoted by P(x). The predator converts consumed prey into new predators with efficiency 'π' where $0 < \pi < 1$. We have taken P(x) a type–IV functional response of the form

$$P(x) = \frac{mx}{\frac{x^2}{a} + x + b} \tag{2.2}$$

where 'm', 'a', 'b' are all positive constants. This type of functional response was first introduced by Haldane [12] and it was then used by Andrews [11] and Yano [15] as a substrate uptake function. This function increases to a maximum and then decreases, approaching zero as x approaches infinity. Thus, P(x) models the situation where the prey can better defend or disguise themselves when their population becomes large enough. The parameters m and b can be interpreted as the maximum per capita predation rate and the half saturation constant in the absence of any inhibitary effect. For large 'a', P(x) reduces to a type II functional response.

In the model (2.1), it is assumed that both prey and predator are harvested in direct proportion to their densities and applied effort with constant catchability coefficients q_1 and q_2 respectively. τ is the tax per unit biomass of landed species, imposed by regulatory agency in order to maintain the desired level. Here, the tax for both populations is assumed to be same. p_1 and p_2 are the constant price per unit biomass of prey and predator respectively, c is the constant harvesting cost per unit effort applied and α_0 is positive constant called stiffness parameter measuring the strength of reaction of effort to the perceived rent.

Next there is a theorem that proves the boundedness and positivity of the system.

Theorem 2.1: The solutions of (2.1) are bounded and non–negative.

Proof:

Let us consider $\mu = \alpha_0(p - \tau)\, x(t) + \alpha_0(p - \tau)\, y(t) + E(t)$, $p = \max(p_1, p_2)$

Choosing $\in < \min\left(c, \frac{d}{\alpha_0(p-\tau)}\right)$,we get

$$\dot{\mu} + \in \mu < \frac{\alpha_0(p-\tau)K}{4r}(r + \in)^2 = \mu_0 \quad \text{(a constant)}$$

On integrating and taking limit $t \to \infty$, we have

$$\mu = \alpha_0(p - \tau)[x + y] + E < \frac{\mu_0}{\in}.$$

For positivity, we have

$$\left.\frac{dx}{dt}\right|_{x=0} = 0 \quad , \quad \left.\frac{dy}{dt}\right|_{y=0} = 0 \quad , \quad \left.\frac{dE}{dt}\right|_{E=0} = 0$$

$$\frac{dx}{dt} \geq -\left\{\frac{my}{x^2/a + x + b}\right\}x \quad > -\left[\frac{m\mu_0}{\alpha_0(p-\tau)\in b}\right]x = -c_0 x$$

where

$$c_0 = \frac{m\mu_0}{\alpha_0(p-\tau)\in b}$$

On integrating $x(t) > x(0)\exp(-c_0 t) \geq 0$. Similarly,

$$\frac{dy}{dt} \geq -dy \quad \Rightarrow \quad y(t) \geq y(0)e^{-dt} \geq 0.$$

$$\frac{dE}{dt} \geq -c\alpha_0 E \quad \Rightarrow \quad E(t) \geq E(0)e^{-c\alpha_0 t} \geq 0.$$

Hence, all solutions of the system (2.1) are positive with initial conditions.

3. Analysis of Equilibria

First of all, we will discuss the case of no harvesting. In this case, we have at most four equilibria namely, $P_0(0, 0)$, $P_1(K, 0)$, $P_2(x_1, y_1)$, $P_3(x_2, y_2)$. Existence of P_0 and P_1 are obvious. P_2 and P_3 are given by solution of equations

$$rx\left[1-\frac{x}{K}\right] - P(x)y = 0$$

$$\Rightarrow y = \frac{rx}{P(x)}\left[1-\frac{x}{K}\right] = w(x) \tag{3.1(a)}$$

$$\pi P(x) - d = 0.$$

(3.1(b))

x_1 and x_2, $(x_1 < x_2)$ are positive roots of quadratic equation (3.1(b)) that can be written as

$$x_1 = \frac{a}{2d}\left[(m\pi - d) - \sqrt{(m\pi - d)^2 - \frac{4bd^2}{a}}\right]$$

$$x_2 = \frac{a}{2d}\left[(m\pi - d) + \sqrt{(m\pi - d)^2 - \frac{4bd^2}{a}}\right]$$

The condition for existence of P_2 and P_3 are

$$d < m\pi \qquad \text{and} \qquad a > \frac{4bd^2}{(m\pi - d)^2}$$

Also for P_2 , $x_1 < K$ and for P_3 , $x_2 < K$.

In order to investigate local behavior of system about equilibria, we will consider variational matrix

$$M = \begin{bmatrix} r\left(1 - \frac{2x}{K}\right) - P'(x)y & -P(x) \\ \pi P'(x) & \pi P(x) - d \end{bmatrix}$$

It can be seen here that

(a) P_0 is saddle point

(b) P_1 is locally asymptotically stable if $d > \pi P(K)$, other wise it is also saddle point.

Variational matrix corresponding to interior equilibria $P_2(x_1, y_1)$ can be written as

$$M_2 = \begin{bmatrix} -\frac{rx_1}{K} + \frac{P(x_1)w(x_1)}{x_1} - P'(x_1)w(x_1) & -P(x_1) \\ \pi P'(x_1) & 0 \end{bmatrix}$$

The characteristic equation corresponding to this matrix will be

$$\lambda^2+\lambda\left[\frac{-dw(x_1)}{\pi x_1}+\frac{rx_1}{K}+P'(x_1)w(x_1)\right]+dP'(x_1)=0$$

Here, $P'(x_1)>0$, so P_2 is stable or unstable according as

$$\frac{-dw(x_1)}{\pi x_1}+\frac{rx_1}{K}+P'(x_1)w(x_1)>\text{or}<0.$$

Therefore, $\frac{-dw(x_1)}{\pi x_1}+\frac{rx_1}{K}+P'(x_1)w(x_1)=0$ is a bifurcation point which on simplification gives

$$K=\frac{\frac{3x_1^2}{a}+2x_1+b}{\frac{2x_1}{a}+1}=\overline{K} \tag{3.2}$$

So, a hopf bifurcation occurs at the parameter value given by (3.2).

For $P_3(x_2, y_2)$, characteristic equation is

$$\lambda^2+\lambda\left[\frac{-dw(x_2)}{\pi x_2}+\frac{rx_2}{K}+P'(x_2)w(x_2)\right]+dP'(x_2)=0 \tag{3.3}$$

In this case, $P'(x_2)<0$, so by Descartes' rule of sign equation (3.3) has two real roots which are of opposite sign. Consequently, P_3 is unstable saddle point.

Effect of Harvesting:

In this we will consider complete model (2.1). It can be seen that in this case, there are seven equilibria $S_0(0, 0, 0)$, $S_1(K, 0, 0)$, $S_2(x_1, y_1, 0)$, $S_3(x_2, y_2, 0)$, $S_4(\overline{x},0,\overline{E})$, $S_5(x_1^*, y_1^*, E_1^*)$, $S_6(x_2^*, y_2^*, E_2^*)$.

The equilibrium points corresponding to E = 0 are discussed in previous part, now we will discuss the equilibrium points with $E\neq 0$.

$S_4(\overline{x},0,\overline{E})$ is given by

$$r\left[1-\frac{x}{K}\right]-q_1E=0 \quad\Rightarrow\quad \overline{E}=\frac{r}{q_1}\left(1-\frac{\overline{x}}{K}\right)$$

$$(p_1-\tau)q_1x-c=0 \quad\Rightarrow\quad \overline{x}=\frac{c}{(p_1-\tau)q_1}$$

$$\frac{c}{(p_1-\tau)q_1}<K$$

For existence of S_4, $\bar{x} < K$ and

$$\Rightarrow \tau < p_1 - \frac{c}{Kq_1} \tag{3.4}$$

This equation gives the upper bound for the tax in the absence of predator.
Now, $S_5(x_1^*, y_1^*, E_1^*)$ and $S_6(x_2^*, y_2^*, E_2^*)$ are given by

$$rx\left(1-\frac{x}{K}\right) - P(x)y - q_1Ex = 0$$

$$\pi P(x) - d - q_2E = 0 \tag{3.5}$$

$$(p_1 - \tau)q_1x + (p_2 - \tau)q_2y - c = 0.$$

$$y = \frac{x\left[r\left(1-\frac{x}{K}\right) - q_1E\right]}{P(x)} \tag{3.6(a)}$$

$$E = \frac{1}{q_2}[\pi P(x) - d] \tag{3.6(b)}$$

$$y = \frac{c - (p_1 - \tau)q_1x}{(p_2 - \tau)q_2} \tag{3.6(c)}$$

$$y = \frac{x\left[r\left(1-\frac{x}{K}\right) - \frac{q_1}{q_2}\{\pi P(x) - d\}\right]}{P(x)} \tag{3.6(d)}$$

On equating (3.6(c)) and (3.6(d)), x_1^* and x_2^* are positive roots of equation

$$x\left[r\left(1-\frac{x}{K}\right) - \frac{q_1}{q_2}\{\pi P(x) - d\}\right] = P(x)\left[\frac{c - (p_1 - \tau)q_1x}{(p_2 - \tau)q_2}\right]$$

$$\frac{r}{Ka}x^3+\left\{\frac{r}{K}-\frac{1}{a}\left(r+\frac{q_1}{q_2}d\right)\right\}x^2+x\left\{\frac{rb}{K}-\left(r+\frac{q_1}{q_2}d\right)+\frac{q_1}{q_2}m\left(\pi-\frac{(p_1-\tau)q_1}{(p_2-\tau)q_2}\right)\right\}$$

$$+\frac{mc}{(p_2-\tau)q_2}-b\left(r+\frac{q_1}{q_2}d\right)=0$$

By Descartes rule of sign, there is at least one positive equilibria if

$$\frac{mc}{(p_2-\tau)q_2}<b\left(r+\frac{q_1}{q_2}d\right) \tag{3.7(a)}$$

Along with this, for two positive roots of equation, we must have

$$\frac{1}{a}\left(r+\frac{q_1}{q_2}d\right)>\frac{r}{K} \tag{3.7(b)}$$

So for the existence of S_5 and S_6, we must have

$$\left(r+\frac{q_1}{q_2}d\right)>\max\left\{\frac{mc}{(p_2-\tau)q_2b},\frac{ar}{K}\right\} \tag{3.8}$$

Variational matrix with harvesting effort can be written as

$$\overline{M}=\begin{bmatrix} r\left(1-\frac{2x}{K}\right)-p'(x)y-q_1E & -P(x) & -q_1x \\ \pi P'(x) & \pi P(x)-d-q_2E & -q_2y \\ \alpha_0E(p_1-\tau)q_1 & \alpha_0E(p_2-\tau)q_2 & \alpha_0[(p_1-\tau)q_1x \\ & & (p_2-\tau)q_2y-c] \end{bmatrix}$$

For $S_4(\bar{x},0,\bar{E})$, system is locally asymptotically stable if

$$d+q_2\bar{E}>\pi P(\bar{x}) \qquad \text{and} \qquad P'(\bar{x})>0.$$

For S_5 and S_6, let us assume that $x_1^* < x_2^*$, so $P'(x_1^*) > 0$ and $P'(x_2^*) < 0$. Instead of taking x_1^* and x_2^* separately, we will discuss both at the same time by considering them as x^*. Now, the characteristic equation corresponding to interior equilibria can be written as

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \tag{3.9}$$

where

$$A = \frac{rx^*}{K} + P'(x^*)y^* - \frac{P(x^*)y^*}{x^*}$$

$$B = q_1q_2\alpha_0(p_2 - \tau)E^*y^* + \pi P(x^*)P'(x^*) + q_1^2\alpha_0(p_1 - \tau)E^*x^*$$

$$C = q_1q_2\alpha_0(p_2 - \tau)E^*y^*\left[\frac{rx^*}{K} + P'(x^*)y - \frac{P(x^*)y^*}{x^*}\right]$$
$$- q_1^2\alpha_0(p_1 - \tau)E^*y^*P(x^*) + \pi\alpha_0q_1q_2(p_2 - \tau)\, P'(x^*)x^*.$$

So, if $A > 0$, $B > 0$, $C > 0$ and $AB - C > 0$, then by Routh – Hurwitz criteria, all roots of characteristic equation will have negative real parts and the system will be locally asymptotically stable about the interior equilibria.

Also, at bifurcation point, $A = 0$ which on simplification gives

$$K = \frac{r(3x^{*2}/a + x^* + b)}{(r - q_1E^*)\left[\frac{2x^*}{a} + 1\right]} = K'$$

Hence, under the influence of harvesting effort, the bifurcation value $K = \bar{K}$ has been shifted to new value given by $K = K'$ and this value depends on the equilibrium level of effort.

4. Optimal Harvesting Policy

In this section, we will discuss harvesting policy that will maximize the net revenue to the society.

The net economic revenue is given by

$$\pi = [p_1q_1x + p_2q_2y - c]\, E \tag{4.1}$$

Our objective is to solve the problem:

$$\max \int_0^\infty e^{-\delta t}(p_1q_1x + p_2q_2y - c)E \; dt \tag{4.2}$$

where δ is the instantaneous annual rate of discount.

In order to solve the problem associated Hamiltonian function is given as

$$H = e^{-\delta t}(p_1q_1x + p_2q_2y - c)E + \lambda_1(t)\left[rx\left(1-\frac{x}{K}\right) - P(x)y - q_1Ex\right]$$

$$+\lambda_2(t)[\pi P(x)y - dy - q_2Ey] + \lambda_3(t)[\alpha_0 E\{(p_1-\tau)q_1x + (p_2-\tau)q_2y - c\}] \quad (4.3)$$

For H to be maximum on control set $\tau_{min} \le \tau \le \tau_{max}$ we must have

$$\frac{\partial H}{\partial \tau} = 0 \Rightarrow \lambda_3(t) = 0 \quad (4.4)$$

As per the maximum principle rule, we must have

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} \quad , \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} \quad , \quad \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial E}$$

Above equations can be rewritten as

$$\frac{d\lambda_1}{dt} = -e^{-\delta t}p_1q_1E - \lambda_1(t)\left[r\left(1-\frac{2x}{K}\right) - P'(x)y - q_1E\right] - \lambda_2(t)[\pi P'(x)]$$

$$\frac{d\lambda_2}{dt} = -e^{-\delta t}p_2q_2E - \lambda_1(t)[-P(x)] - \lambda_2(t)[\pi P(x) - d - q_2E]$$

$$\frac{d\lambda_3}{dt} = -e^{-\delta t}(p_1q_1x + p_2q_2y - c) + \lambda_1(t)q_1x + \lambda_2(t)q_2y$$

From (4.4)

$$\lambda_1q_1x + \lambda_2q_2y = e^{-\delta t}(p_1q_1x + p_2q_2y - c) \quad (4.5)$$

$$\lambda_2(t) = \frac{-e^{-\delta t}(p_1q_1x + p_2q_2y - c) - \lambda_1q_1x}{q_2y} \quad (4.6)$$

Corresponding to interior equilibria, we have

$$\frac{d\lambda_1}{dt} = -e^{-\delta t}p_1q_1E^* - \lambda_1(t)\left[\frac{rx^*}{K} + P'(x^*)y - \frac{P(x^*)y^*}{x^*}\right] - \pi P'(x)\lambda_2(t) \quad (4.7(a))$$

$$\frac{d\lambda_2}{dt} = -e^{-\delta t} p_1 q_1 E^* + P(x^*)\lambda_1(t) \tag{4.7(b)}$$

On solving these equations we get

$$\lambda_1(t) = \frac{A_2}{A_1+\delta} e^{-\delta t}, \qquad \lambda_2(t) = \frac{A_3}{\delta} e^{-\delta t} \tag{4.8}$$

$$A_1 = \frac{rx^*}{K} + P'(x^*)y^* - \frac{P(x^*)y^*}{x^*} + \frac{\pi P'(x^*)q_1 x^*}{q_2 y^*}$$

$$A_2 = p_1 q_1 E^* + \frac{\pi P'(x)}{q_2 y^*}(p_1 q_1 x^* + p_2 q_2 y^* - c)$$

$$A_3 = p_1 q_1 E^* - \frac{A_2}{A_1+\delta}$$

Note: Condition for $\lambda_2(t) > 0$, $p_1 q_1 E^* - \frac{A_2}{A_1+\delta} > 0$. From (4.5) and (4.8), we get

$$\frac{A_2}{A_1+\delta} q_1 x^* + \frac{A_3}{\delta} q_2 y^* = p_1 q_1 x^* + p_2 q_2 y^* - c \tag{4.9}$$

For an optimal effort, we have

$$E^* = \frac{1}{q_1}\left[r\left(1 - \frac{x^*}{K}\right) - \frac{P(x^*)y^*}{x^*} \right] = \frac{1}{q_2}[\pi P(x^*) - d] \tag{4.10}$$

Equation (4.10) together with (4.9) gives the optimal equilibrium levels of population $x^* = x_\delta$, $y^* = y_\delta$ then the optimal levels of effort and tax are given by

$$E^* = E_\delta = \frac{1}{q_2}[\pi P(x_\delta) - d] \tag{4.11(a)}$$

$$\tau = \tau_\delta = \frac{p_1 q_1 x_\delta + p_2 q_2 y_\delta - c}{q_1 x_\delta + q_2 y_\delta} \tag{4.11(b)}$$

Remarks: (i) From (4.9), we note that $\lambda_i(t)e^{\delta t}$ (i = 1, 2) is independent of time in an optimal equilibrium. Hence, they satisfy the transversality condition at ∞ i.e. they remain bounded as $t \to \infty$.

(ii) From (4.9), it can be noted that

$$p_1 q_1 x^* + p_2 q_2 y^* - c = \frac{A_2}{A_1 + \delta} q_1 x^* + \frac{A_3}{\delta} q_2 y^* \to 0 \text{ as } \delta \to \infty$$

So net economic revenue is zero when δ approaches to infinity. This shows that in case of infinite discount rate, the net economic rent is zero and hence both prey and predator populations remain unexploited. Thus, we can conclude that whenever there is a danger of extinction of population, the regulatory agency may increase the tax and or discount rate to discourage the harvesting of prey–predator community.

5. Numerical Example:

In this section, we present a numerical example for hypothetical values of parameter as,

$r = 2$, $m = 1$, $b = 4$, $q_1 = 0.1$,
$K = 6$, $a = 1$, $\pi = 0.5$, $q_2 = 0.1$,
$d = 0.06$, $p_1 = 8$, $p_2 = 10$, $\alpha_0 = 1$,
$\delta = 0.1$.

For these values, we obtain two optimal equilibrium levels of prey–predator population and harvesting effort as

$x_{1\delta} = 3.2158$, $y_{1\delta} = 14.6883$, $E_{1\delta} = 0.3158$.
$x_{2\delta} = 5.6941$, $y_{2\delta} = 1.4556$, $E_{2\delta} = 0.07599$.

Also, the values of bifurcation parameter K' is obtained as $K_1' = 5.6678$ and $K_2' = 9.1285$ and hence $K_1' < K = 6 < K_2'$. So, the optimal equilibrium level corresponding to more prey population gives the stable interior equilibrium point of the system. Then the corresponding optimal level of tax is given bys_δ = 0.0152.

5. Discussion

In the present paper, we have considered and analyzed a non–linear mathematical model to study the optimal harvesting of a prey–predator community in which the predation is represented by a Holling type–IV function. The objective of the paper is both economical and ecological. The economic goal is to maximize the net benefit to the society and ecologically we wish to keep the prey and predator away from extinction to keep the ecological balance. The instrument used to derive the system towards such a state is tax. In the paper, we have shown the existence of equilibrium points and a comparative study of stability in the absence and presence of harvesting effort has been performed. It has been observed that the bifurcation point of the system has been shifted to new values of 'K', the carrying capacity of prey in the absence of predator and harvesting effort.

Using the Pontryagin's Maximum principle, the optimal harvesting policy has been discussed and optimal equilibrium levels of prey–predator population, the effort and the tax have been obtained. It has been found that if the discount rate increases, then the economic rent decreases and even it may tend to zero as discount rate tends to infinity.

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