# Weighted representations of transformation groups induced by weighted composition operators

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**Abstract**

In this article we have presented a special class of the weighted composition operators induced by action $\pi$ and co-cycles on $G$ giving rise to a representation of group $G$, and apply them for some special groups to have applications in study of flows on topological space $X$ and semigroups of operators.

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## 1 Introduction and Preliminaries:

Let $(X, G, \pi)$ be a transformation group, where $X$ is a locally compact topological space, $G$ is a topological group and $\pi$ is an action of $G$ on $X$ i.e $\pi : X \times G \to X$ is a continuous map such that $\pi(x, e) = x$ and $\pi(x, gh) = \pi(\pi(x, g), h)$, for all $x \in X$ and $g, h \in G$, where $e$ is the identity element of $G$. Let $E$ be a Banach space and let $C_b(X, E)$ denote the Banach space of all bounded $E$-valued continuous functions on $X$ under sup-norm and pointwise linear operations. If $\varphi : X \to X$ is a continuous map, then the mapping $f \to f \circ \varphi$ is a bounded (linear) operator on $C_b(X, E)$ and we denote it by $C_\varphi$, which is called the composition operator induced by $\varphi$. In case $\varphi$ is a homeomorphism, $C_\varphi$ is an isometric isomorphism. If $X$ is a compact space, then every onto isometry is a weighted multiplier of $C_\varphi$. This is well known Banach-Stone theorem. If $\theta : X \to \mathbb{C}$ (or $B(E)$) is a continuous map, then $f \to \theta \cdot f$ is a linear transformation on $C_b(X, E)$, where $B(E)$ is Banach algebra of all bounded linear operators on $E$. In case $\theta$ is bounded, $f \to \theta \cdot f$ is a linear operator on $C_b(X, E)$ and we denote it by $M_\theta$, which is known as the multiplication operator induced by the map $\theta$. The operator $M_\theta C_\varphi$ is known as the weighted composition operator on $C_b(X,E)$ induced by $\theta$ and $\varphi$. Thus $(M_\theta C_\varphi)(x) = \theta(x)\cdot f(\varphi(x))$, for $x \in X$ and $f \in C_b(X,E)$. We denote the operator $M_\theta C_\varphi$ by $W_\varphi^\theta$. These operators have been the subject matter of study for more than four decades and have many applications in representation theory, wavelet theory, dynamical systems and semi-group of operators. For details we refer to [2, 4, 9] . In this article we have presented a special class of the weighted composition operators induced by action $\pi$ and co-cycles on $G$ giving rise to a representation of group $G$, and apply them for some special groups to have applications in study of flows on $X$ and semigroups of operators.

## 1.1 Composition operators induced by a transformation group

Let $(X,G,\pi)$ be a transformation group. Let $g \in G$ .Then define $\pi_g : X \to X$ by $\pi_g(x) = \pi(x,g)$. Since

$$\begin{aligned}
\pi_g \circ \pi_{g^{-1}}(x) &= \pi_g(\pi_{g^{-1}}(x)) \\
&= \pi_g(\pi(x,g^{-1})) \\
&= \pi(\pi(x,g^{-1}),g)) \\
&= \pi(x,g^{-1}g) \\
&= \pi(x,e) \\
&= x,
\end{aligned}$$

the map $\pi_g$ belongs to $H(X,X)$, the group of all homeomorphisms of $X$. Since

$$\begin{aligned}
\pi_{gh}(x) &= \pi(x,gh) \\
&= \pi(\pi(x,g),h) \\
&= \pi_h(\pi_g(x)),
\end{aligned}$$

we have $\pi_{gh} = \pi_h \circ \pi_g$. The mapping $\pi_g$ gives rise to the composition operator $C_{\pi_g}$ on $C_b(X,E)$ given by $C_{\pi_g} f = f \circ \pi_g$. The composition operator $C_{\pi_g}$ is denoted by $C_g$. Let $\psi : G \to H(X,X)$ be the map defined as $\psi(g) = \pi_g$ and let $\psi^{'} : G \to B(C_b(X,E))$ be the map defined as $\psi^{'}(g) = C_g$. It is clear that $\psi(gh) = \pi_{gh} = \pi_h \circ \pi_g$. Thus $\psi$ is an anti-homomorphism and $\psi(G)$ is a non-abelian subgroup of $H(X,X)$ with $\pi_g^{-1} = \pi_{g^{-1}}$. $Ker\psi$ is closed in $G$ if $X$ is a Hausdorff space and $G$ acts effectively iff $Ker\psi = \{e\}$. Since $\psi^{'}(gh) = C_{gh} = C_g C_h, \psi^{'}$ is a homomorphism from $G$ to group $\{C_g : g \in G\}$ of all composition operators induced by elements of $G$. By $C_G$, we denote the group of all composition operators on $C_b(X,E)$ induced by elements of $G$. We shall record some results in the following theorem.

**Theorem 1.1.** *Let $(X,G,\pi)$ be a transformation group and $E$ be a Banach space. Then*

1. *$C_b(X,E)$ is a Banach space and if $E$ is a Banach algebra, then $C_b(X,E)$ is also a Banach algebra.*

2. *For every $g \in G, C_g$ is a bounded linear operator on $C_b(X,E)$ with $\|C_g\| = 1$.*

3. *For $g \in G, C_g$ is an isometric isomorphism on $C_b(X,E)$ with $C_g^{-1} = C_{g^{-1}}$ and the set $C_G = \{C_g : g \in G\}$ is a group under operation of operator multiplication.*

4. *The mapping $\psi^{'} : G \to C_G$ defined as $\psi^{'}(g) = C_g$ is a homomorphism and $\psi^{'}$ is a representation of $G$ in terms of composition operators induced by elements of $G$.*

5. *The mapping $\psi^{''} : G \to H(X,X)$ defined by $\psi^{''}(g) = \pi_{g^{-1}}$ is a homomorphism.*

**Outline of the proof**:

1. For proof we refer to [9].

2. Since $\|C_g f\| = \sup \|f(\pi_g(x))\| = \|f\|$ . We have $\frac{\|C_g f\|}{\|f\|} = 1$, for every $f \in C_b(X,E)$. Hence $\|C_g\| = 1$ .

3. Also $C_g^{-1} = C_{g^{-1}}$ since $C_{g^{-1}} C_g f = C_{g^{-1}}(f \circ \pi_g) = f \circ \pi_g \circ \pi_{g^{-1}} = f \circ \pi_e = f$.

4. $[\psi^{'}(e)f](x) = (C_e f)(x) = (f \circ \pi_e)(x) = f(\pi(x,e)) = f(x)$, for every $x \in X$ Thus $\psi^{'}(e) = 1$.
   Let $g, h \in G$ and let $f \in C_b(X,E)$. Then $\psi^{'}(gh)f = C_{gh}f = f \circ \pi_{gh} = f \circ (\pi_h \circ \pi_g) = \psi^{'}(g)\psi^{'}(h)(f)$, for all $f \in C_b(X,E)$.
   Thus $\psi^{'}(gh) = \psi^{'}(g)\psi^{'}(h)$. This shows that $\psi^{'}$ is a homomorphism and hence a representation of $G$.

5.

$$\begin{aligned}
\psi^{''}(gh)f &= \pi_{(gh)^{-1}} f \\
&= \pi_{h^{-1}g^{-1}} f \\
&= (\pi_{g^{-1}} \circ \pi_{h^{-1}})(f) \\
&= \pi_{g^{-1}}(\pi_{h^{-1}} f) \\
&= \pi_{g^{-1}}(\psi^{''}(h) f) \\
&= \psi^{''}(g)\psi^{''}(h)f, for every f \in C_b(X,E).
\end{aligned}$$

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Thus $\psi^{''}(gh) = \psi^{''}(g)\psi^{''}(h)$. This shows that $\psi^{''}$ is a homomorphism. If $G$ is a semi-group with identity, a topology on it such that operation on $G$ is continuous and $\pi : X \times G \to X$ is an action of $G$ on $X$, then $(X, G, \pi)$ is called semi-transformation group. If $g \in G$, then certainly $C_g$ is a bounded operator on $C_b(X,E)$ but it may not be invertible. But $g \to C_g$ is a representation of semigroup $G$. If $G = \mathbb{R}$ with usual topology, then $(\mathbb{R}, X, \pi)$ is called continuous dynamical system and if $G = \mathbb{R}^+$, then $(\mathbb{R}^+, X, \pi)$ is known as semi-dynamical system. If $G = \mathbb{Z}$ or $Z^+$, then we have discrete dynamical systems. These dynamical systems make contact with several areas of Mathematics and Mathematical physics.[1, 4].

The translation operators on $C_b(\mathbb{R}, E)$ or $C_b(\mathbb{R}^+, E)$ give a representation of the additive group $\mathbb{R}$.
In general, any topological group $G$ has representation on $C_b(G, E)$ as it acts on itself through operation on $G$. In particular every Lie-group has a representation on space of analytic functions on $G$ [13]. If $G = \mathbb{Z}$, then it has a representation on Hilbert space $l^2(\mathbb{Z})$ through shift operators.

## 2 Some weighted composition operators associated with a transformation group

Let $(X, G, \pi)$ be a transformation group. Let $\nu : X \times G \to \mathbb{C}$ (or $B(E)$) be a continuous map . For $g \in G$, let the map $\pi_g : X \to X$ be defined as $\pi_g(x) = \pi(x, g)$, for every $x \in X$ and $\nu_g : X \to \mathbb{C}$ (or $B(E)$) be defined as $\nu_g(x) = \nu(x, g)$ for $x \in X$. For $h, g \in G$, define the map $\nu_h C_g : C_b(X, E) \to C_b(X, E)$ as $\nu_h C_g f = \nu_h . f \circ \pi_g$. Then $\nu_h C_g$ is the weighted composition operator on $C_b(X, E)$ induced by the pair $(h, g)$. In case $\nu_h$ is a bounded function on $X$, $\nu_h C_g$ is a continuous operator on $C_b(X, E)$. Define the map $\psi : G \times G \to L(C_b(X, E))$ by $\psi(h, g) = \nu_h C_g$. In case $h = e$, $\psi(e, g) = C_g$ (composition operator induced by $\pi_g$). In the following theorem, we record some properties of weighted composition operators.

**Theorem 2.1.** *Let $(X, G, \pi)$ be a transformation group and $\nu : X \times G \to \mathbb{C}$ be a continuous map. Then*

1. *$\nu_h C_g$ is bounded iff $\nu_h$ is bounded.*

2. *$\nu_h C_g$ is invertible iff $\nu_h(x) \neq 0$ for every $x \in X$.*

**Outline of the proof**: For the proof of (1), (2), we refer to [13].

Consider the map $\hat{T} : G \to L(C_b(X, E))$ defined by $\hat{T}(g) = \nu_g C_g$. This map is not necessarily a homomorphism.

**Definition 2.1.** By a scalar-valued co-cycle over $G$, we mean a continuous map $w : X \times G \to \mathbb{C}$ such that $w(x, gh) = w(\pi_h(x), g)w(x, h)$ for $g, h \in G, x \in X$. For $g \in G$, we define $w_g : X \to \mathbb{C}$ by $w_g x = w(x, g)$. It is clear that $w_g \in C(X)$, the vector space of all continuous functions on $X$. Let $W = \{w_g : g \in G\}$, $W$ is called the family of weights induced by co-cycle $w$. For $g, h \in G$, define the map $W_g^h$ on $C_b(X, E)$ as $W_g^h f = w_g . f \circ \pi_h$. Clearly it is a linear transformation and if $w_g$ is a bounded function, then $W_g^h$ is a continuous operator on $C_b(X, E)$. Define $S : G \to B(C_b(X, E))$ by $w_g C_g$. Then $\{S(g) : g \in G\}$ is a semigroup of weighted composition operators induced by the pair $(w, \pi)$.

Let $f \in C_b(X, \mathbb{C})$ suchthat $f(x) \neq 0$, for every $x \in X$. Let $G = \mathbb{R}$. Then define a map

$w_t : X \to \mathbb{C}$ by $w_t(x) = \frac{f(x)}{f(\pi_t x)}$. Then $w_t(x)$ is a cocycle, since $w_0(x) = \frac{f(x)}{f(\pi_0 x)} = 1$ and

$$\begin{aligned} w_s(x)w_t(\pi_s(x)) &= \frac{f(x)}{f(\pi_s x)}.\frac{f(\pi_s(x))}{f(\pi_t(\pi_s(x))} \\ &= \frac{f(x)}{f(\pi_t \circ \pi_s(x))} \\ &= \frac{f(x)}{f(\pi_{s+t}(x)} \\ &= w_{s+t}(x). \end{aligned}$$

. For more examples we refer to [4, 5]

**Definition 2.2.** Let $(X, G, \pi)$ be transformation group. By an operator-valued co-cycle over $G$(or $\pi$), we mean a map $\nu : X \times G \to B(E)$ suchthat

1. $\nu$ is strongly continuous i.e continuous with respect to strong topology on $B(E)$.
2. $\nu(x, gh) = \nu(x, g)\nu(\pi(x, g), h)$.
3. $\nu(x, e) = I$, for every $x \in X$.

In case $G = \mathbb{R}$ with addition and if there exist $M > 0$ and $w > 0$ suchthat $\|\nu(x, t)\| \leq Me^{wt}$ for $t \in \mathbb{R}$. and $x \in X$, then co-cycle $\nu$ is called exponentially bounded. If $\nu$ is a co-cycle, then define map $\hat{\pi}_g : X \times E \to X \times E$ by $\hat{\pi}_g(x, y) = (\pi_g(x), \nu_g(x)y)$. Then $(X \times E, G, \hat{\pi})$ is a transformation group.

1. Let $G = \mathbb{R}$ and $\pi(s, t) = s + t$ be action on $\mathbb{R}$. Let $\{U(s, t) : s \geq t, s, t \in \mathbb{R}\}$ be strongly continuous exponentially bounded evolution family on $E$ . Define $\nu(s, t) = U(s+t, s)$. Then $\nu$ is an operator-valued co-cycle. This cocycle is closely related to Abstract Cauchy Problem [11].
2. If $A \in B(E)$ and define $\nu : \mathbb{R} \times \mathbb{R} \to B(E)$ as $\nu(t, s) = e^{tA}$. Then $\nu$ is a co-cycle.
3. Let $G = \mathbb{R}$ and $X = \mathbb{R}$ and $A : \mathbb{R} \to B(E)$ be a norm-continuous function. Let $\frac{dx}{dt} = A(t)x$ be a differential equation. Suppose the evolution family $\{u(s, t) : t, s \in \mathbb{R}\}$ is defined so that $t \to x(t)$ is a solution of differential equation given above. Let $\pi(s, t) = s + t$ be an action on $\mathbb{R}$. Define $\nu(s, t) = u(s + t, s)$. Then $\nu$ is a co-cycle over $\pi$ and $A(t) = \frac{d}{dt}\nu(t, s)$ at $s = 0$.
4. Let $X$ be a compact metric space and $\pi : X \times G \to X$ be an action (flow). Let $A : X \to B(E)$ be a (norm) continuous function. Consider the differential equation $\frac{dy}{dt} = A(\pi_t(x)y), s \in X$ and $t \in \mathbb{R}$.If $\nu_t(x)$ is the solution operator for above differential equation i.e. $y(t) = \nu_t(x)y(0)$, then the co-cycle $\nu_t(x)$ is norm continuous in $x$. This co-cycles is smooth with $A(x) = \frac{d}{dt}\nu_t(x)$ at $t = 0, x \in X$.

For more examples of operator-valued co-cycles and their applications in solutions of Abstract Cauchy Problems, we refer to [4, 11].
In the following theorem, we present a representation of transformation group $G$ in terms of the bounded weighted composition operators induced by a co-cycles and action $G$.

**Theorem 2.2.** *Let $(X, G, \pi)$ be a transformation group and let $\nu$ be a cocycle over $G$. Then $\nu_g C_g$ is a weighted composition transformation on $C_b(X, E)$ and the mapping $\hat{T}$ : $G \to B(C_b(X, E))$ given by $\hat{T}(g) = \nu_g C_g$ is a (faithful) weighted representation of $G$, where $B(C_b(X, E))$ is the algebra of all bounded operators on $C_b(X, E)$.*

**Outline of the proof**: We have already seen that $\nu_h C_g$ is a weighted composition transformation on $C_b(X, E)$. We shall show that the mapping $\hat{T}$ is a representation.

1. Since $[\hat{T}(g)f](x) = [\nu_g C_g f](x)$ for every $f \in C_b(X, E), g \in G$ and $x \in U$, we have $[\hat{T}(e)f](x) = \nu_e(x)(C_e f)(x) = \nu(x, e)f(\pi(x, e)) = f(x)$. Hence $\hat{T}(e) = 1$.

2. Let $g$ and $h$ be in $G$. Then for $f \in C_b(X, E)$,

$$\begin{aligned} \hat{T}(gh)(f) &= \nu_{gh} C_{gh} f \\ &= \nu_g . \nu_h \circ \pi_g \cdot f \circ \pi_h \circ \pi_g \\ &= \nu_g . (\nu_h . f \circ \pi_h) \circ \pi_g \\ &= \nu_g . (\hat{T}(h) f) \circ \pi_g \\ &= \hat{T}(g) \hat{T}(h) f. \end{aligned}$$

Thus $\hat{T}(gh) = \hat{T}(g)\hat{T}(h)$. This shows that $\hat{T}$ is a homomorphism and hence a multiplier representation of $G$ induced by co-cycle $\nu$

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