A brief write up on

1) Dr. S. Minakshisundaram (1913 - 1968)
2) Dr. Vijaykumar Patodi (1945 1976)

M. Sitaramayya [1]
School of Mathematics & Statistics
University of Hyderabad, Hyderabad 500 046

email: malladi_sr@yahoo.com

1. Dr. Subbaramaiah Minakshisundaram (SMS)

Prof. SMS (1913-1968) was a distinguished Professor of Mathematics of 20th century whose contributions to the development and teaching of pure and applied mathematics are highly significant. (His colleagues and students used to fondly call him as Prof. SMS)

Born: 12th October, 1913, Trichur, Kerala

Died: August 1968, Visakhapatnam, A.P.

1. Education details: Studied in C.R.C. High School (1919-29), Perambur, Madras State). Studied Intermediate in Pachiayappas College, Madras 1929-31. Studied B.A. (Hons.) Maths from Loyola College, Madras 1931-34. Got M.A. degree in Maths of Madras University (1935).

2. Research Work: Inspired by the researches of Prof. M.R. Siddiqui of Osmania University, Hyderabad, Prof. SMS was spending a lot of time and energy on the study of PDEquations, Eigen function expansions and BVPs. His teachers were the well known Prof. K. Anand Rau and Prof. R. Vaidyanathaswamy. He worked for a few years since 1936 under Prof. R. Vaidyanathaswamy. His work on Fourier Ansatz and nonlinear Parabolic equations enabled him to secure the doctorate degree (D.Sc.) in Mathematics from the Madras University in 1940. His research work (some papers) was published in Indian Academy of Sciences, Journal of Indian Mathematical Society, Canadian Journal of Mathematics, etc. and his thesis in Madras University Journal.

3. Works and Contributions: First while at Madras University during 1940-42 he has guided a dissertation of K. Chandrasekharan for M.S. (by research) on Bessel Summabillity and applications to heat problems published as 2 papers in Mathematics student. His research work attracted the attention of Stalwarts in the field of PDEquations at that time like Prof. Richard Courant, Courant Institute of Mathematical Sciences, Newyork University. He was awarded the Narasinga Rao medal for best thesis of 1940. He was also awarded the Ramanujan medal. From 1943-46 he worked as Lecturer in the Dept. of

[1] invited talk delivered on November 22, 2015 at the inauguration of Minakshi Sundaram Patodi Lecture series under the auspices of Indian Mathematical Society and Bharata Ganita Parishad at the Department of Mathematics, Lucknow University.

Mathematical Physics in the Andhra University. In 1946 he was invited to be a member of the School of Mathematics of the Institute of Advnced Study, Princeton, USA and he was there from 1946 to 1948. He returned to Andhra University as Reader in 1948 and became Professor and Head in 1951. He was invited to an International Symposium on Differential Equations Stillwater, Oaklahoma, USA and to the ICM held at Cambridge, Mass, USA.

In 1958 he was invited on an educational tour to USA as a member of the general education team of Govt. of India. He was invited to the ICM, Edinburough to deliver a lecture which was well received.

He had several distinguished research students at Andhra University like Prof. M. Rajagopalan etc. and published many research papers in Applied Mathematics and Mathematical Physics. He authored in 1952 the classical book Typical means jointly with Prof. K. Chandrasekharan of TIFR. A part of his earlier work with A. Pleijel on The Eigen function expansions of Laplace operators on Riemannian manifolds (2 papers in Canadian Journal of Mathematics) was so significant that it influenced the refinement of the famous AtiyahSinger index theorem, one of the greatest theorems of 20th century . He was invited in 1967 as a Research Professor at the Math. Institute of Fluid Dynamics and Applied Mathematics of Maryland University, USA but could not join due to bad health. After a prolonged illness he passed awayin August, 1968.

Prof. SMS had worked on Fourier expansions, eigen functions solutions of special partial differential equations in Euclidean set up and also in the Riemannian manifold set up, the Dirichlet series and their associated Zeta functions including Epstein zeta function. He developed the famous indicatrix method and using it the eigen function solution of Laplace and heat equations in [1, 2] (which was later generalized by Patodi and RaySinger at different levels for differential forms and vector bundle valued differential forms respectively) which appeared in proceedings of National Academy of Sciences, Journal of Indian Math. Society and Canadian Journal of mathematics, we cite here only two:

1. S. Minakshisundaram and A. Pleijel: Some properties of the eigen functions of the laplace Operator on Riemannian manifolds, Canad. Jl. Maths 1 (1949), 242-256.

2. −−−−−: Eigen functions on Riemannian manifolds, Jour. Indian. Math. Soc. 17 (1953), 159165.

2. Dr. Vijay Kumar Patodi (1945-1976)

Dr. Vijay Kumar Patodi was an Indian Mathematician who made fundamental and breakthrough contributions in Differential Geometry and Topology. He was a Professor at TIFR, Mumbai.

Born: 12th March, 1945 at Guna, M.P., India

Died: 21st December, 1976, Bombay, India

1. Educational details: Obtained B.Sc. from Vikram University, Ujjain, M.P. (1964). Obtained Masters degree in Mathematics from BHU, Varanasi in 1966. He spent a year at the Centre for Advanced Study at University of Bombay and then joined the School of Mathematics at TIFR, Bombay in 1967. He was awarded Ph.D. degree by Bombay University in 1971 for his thesis on Heat Equations and the index of elliptic operators under Prof. M.S. Narasimhan and S. Ramanan.

2. Research Work and Contributions: Prof. Patodi's research has two stages (a) 196873 (b) 1974-76. In the first stage he focused on the local index theorem for elliptic differential operators. As a preparation Patodi understood the work of SMS Pleijel and applied them in his 1970 paper on Curvature and the fundamental solutions of the heat operator (JIMS34 (1970), p.251-283) (P_3). His first paper (P_1) Curvature and the eigenfunctions of Laplace operator JDG5 (1971), p.233-249 based on his thesis made fundamental breakthrough in the sense that the GaussBonnetChern theorem was realized as a local index theorem for the deRham-Hodge Operator and also the celebrated MckeanSinger Conjecture [6] on fantastic cancellation was proved in this paper.

His second paper from his thesis was Analytic proof of RiemannRochHirzebruch theorem for Kahler manifolds (JDG5(1971), p.251-283 (P_2) which extended the methods of his first paper to much more complicated situation, wherein he proved the local index theorem for Dolbeault operator in a Kahler manifold whose global version is the RRH theorem for algebraic manifolds [5]. He also proved the local index theorem for the signature operator whose global vision gives one the Hirzebruch signature theorem. This work was done by Patodi while in India.

During 1971-73 he was invited to the Inst. for Advanced Study, Princeton where he collaborated with M.F. Atiyah, R.Bott and I.M. Singer. At Princeton Patodi concentrated on the local index theorem for the Dirac operator (the missing one out of his 4 main theorems he aimed as index theorems). This is the joint work with Atiyah and Bott "P,A,B: On the heat equations and the index theorem"Inv. Math. 19 (1973), p.279-330 (28 (1975) 277-280). In this the authors showed that the local index theorem holds for the (i) twisted signature operator (ii) Dirac operator (iii) twisted Dirac operators. (P4)

This paper gave a new proof to the global AS index theorem for general elliptic differential operators using heat equation methods which is a significant and remarkable contribution and it also lead to the local index theorem for manifolds with boundary. (which was later investigated)

Then Patodi in his paper "Holomorphic fixed point formula"BAMS 79 (1973), p.25-28 and in his joint paper with Donnelly "Spectrum and the fixed point set of Isometries II"Topology 16 (1977), p.111 generalized his method to the equivariant situation and proved the Lefschetz fixed point formula of A-BSegal- Singer by the heat equation method and also equivariant signature theorem with respect to isometry of a R-manifold-I) $[P_5, P_{11}]$.

Here it should be noted that as the deRham-Hodge operator, the signature operator and the Dolbeault operator on Kahler manifolds can be expressed as twisted Dirac operators, everything reduces to the local index theorem for twisted Dirac operators of P,A,B, paper (P_4). This finishes Patodi's first stage of research program.

In the second stage of research (1974-1976), Patodi continued his index theory for manifolds with boundary in the joint work with AtiyahSinger in 3 fundamental papers

"Spectral asymmetry and Riem. Geometry I, II, III", Proc. Camb. Phil. Soc. 77 (1975), p.43-69; p.405-432; 79 (1976), p.71-99 wherein the fundamental eta invariant was defined and studied (P_4, P_7). With this his study on local index theorem was completed. ([P_7] This is the best work of all papers of Patodi as well as the best work of all collected papers of Atiyah [1])

Then Patodi's interest moved onto two other important problems: (a) to give an explicit combinatorial formula for the Pontryagin classes of a smooth manifold (b) to give analytic interpretation of the Reidemeister torsion closely following the work of SingerRay [8]. This is the content of his ICM invited talk in 1974 at Vancouver, Canada. [P_8].

On (a) Patodi gave a combinatorial formula (Patodi: A Comb. Formula for Pontryagin classes: Instituto Nationale di Alta Matematica Symp. Mat. 20, p.497505 (P9) (also S.S. Chern and J. Simons: Charaforms and geometric invariants, Ann. Of Maths 99 (1974), p.48-69 wherein for a 4manifold the first Pontryagin class was computed and this problem is open still. (May be Patodi has some program to capture the limiting result on this and more comments on this problem later).

Patodi made a significant contribution for problem (b) in VKP and J. Dodziuk: Riemannian Structures and triangulations of manifolds, Jl. Ind. M.S. 40 (1976) p.152. (P10) In this P&D aimed to prove the Ray Singer conjecture the analytic torsion T_p = the R-torsion τ_P and after proving his fundamental approximation result Patodi made two conjectures and outlined a program which may lead to the RS result as a limiting case but could not complete the program and Dodziuk has not continued this paper. But W. Muller completed Patodi's program by proving his conjectures and refining Patodi's method for finer approximations in his famous paper W. Muller: Analytic torsion and Rtorsion for Rieman manifolds Adv. In Maths 28 (1978), p.233-305 [7].

3. Some Remarks on Patodi's work:

Patodi's work centres around the AtiyahSinger index theory, a major subject in global analysis in sixties. Globally, the AS index theorem states that the analytically defined index of an elliptic differential operator on a manifold (closed or not) can be computed by purely topologically defined quantities. Patodi worked on the local version of the index theorem keeping 4 main theorems in focus and used heat equation techniques closely following SMS approach. His methods are direct unlike Gilkeys indirect methods using Weyls invariant theory.

Patodi published in all 13 research papers. His collected papers were edited and published by Atiyah and Narasimhan in 1996 under World Scientific [P_0]. Many of these papers have become classical literature in Mathematics. They cover almost all aspects of the index theory systematically from classical period where topological methods play a major role to modern era where more geometrical and analytical methods, including those direct heat equation techniques developed by Patodi himself, become more and more important.

After returning to TIFR in 1973 from IAS, Princeton, Patodi became full Professor at the age of 30 and lived only 31 years. There were lot of similarities in the mathematical life of Patodi and Ramanujan. Both created highly original and lasting mathematics in their very short lives. Ramanujan collaborated with Hardy, Patodi with Atiyah, Singer and Bott etc. Both their significant works appeared in Proc. Camb. Phil. Soc. Journal. Both worked with severe health problems throughout their career and that makes their contributions to mathematics even more remarkable.

Between 1980 and 2013 that the Atiyah-Patodi-Singer etainvariant played a major role in subsequent advances in this area. In fact it is now clear there is a relation between etainvairant and RaySinger Analytic torsion through the works of Bismut and Zhang (1992). Mullers theorem (Patodi's program) was proved by further refining the combinatorial approximations methods of Patodi by completing the program in PatodiDodziuk paper $[P_{10}]$. Thus we could say that in some sense Patodi also played a pioneering role in the study of the RaySinger conjecture, now called Cheeger Muller theorem. This CheegerMuller theorem was further generalized by J.M. BismutW.Zhang "An extension of CheegerMuller theorem Asterisque tom 205 (1992) Soc. Math. Fr. [2]"

Strangely enough in the proof of this BismutZhang theorem the local index theoretic technique developed in Patodi's first paper [P1] plays an essential role. Thus it suggests that Patodi's local index theorem for manifolds with boundary and the R-S analytic torsion for manifolds with boundary are intimately related. Infact, Patodi in his P,A,S,-Paper-I on Spectral Asymmery and Riemann geometry, Proc. Camb. Phil. Soc. 1975 ($P_7 - I$, p.45 bottom) observed that a "unification of A-P-S eta-invariant and the Ray-Singer torsion was not known". In this sense all the 13 papers of Patodi form a comprehensive single well knit everlasting mathematical fabric created over just 10 years short period.

On this open problem of Patodi (in quotations above), Witten is attempting to shed some light through his ChernSimons Gauge Theory [9].

It should be noted that the direct method of Patodi and his algebraic identities needed to be explained with the help of some theoretical physicists using Supersymmetry.

On combinatorial Pontryagin classes work around 1971-74 there were 3 papers (a) Patodi's (b) ChernSimons (c) GabrielovGelfandLosiks (GGL). Some comments are in order. Firstly patodis short paper of eight pages contains a direct formula for any Pontryagin class of a Riemannian manifold but it depends on the smooth structure and he indicated some remarks in the end on the limiting case and suspected an obstruction in the process. ($[P_9]$, p.504-505 Remark)

On the other hand ChernSimons paper (written 3 years after announcing the result in Proc. Nat. Acad. Sciences, USA, 1971) (due to the presence of the obstruction suspected by Patodi) gave some differential geometric interpretations of this obstruction leading to conformal invariance of 3-manifold in ambient 4-space and this is formula for the first Pontryagin class only and also depends on smooth structure). Finally, in the GGL paper [4] they computed the first Pontryagin class of a 4-manifold realizing the obstruction in the form of "linearizability" with the merit (though highly technical combinatorics) that their construction was independent of the smooth structure of the underlying manifold. On the whole this problem is still open and needs further investigation.

In another direction the most recent and significant advance of the Patodi's local index theory was the theory of hypoelliptic Laplacians as developed by Bismut and his collaborators (2005-2013). In this whole development of Bismuts hypoellipic Laplacian theory, we find that the local index techniques of Patodi, the Atiyah-Patodi-Singer eta-invariant and the Ray-Singer analytic torsion all occupy their new important places.

Patodi gave several colloquium talks (a) ICM 1974 (b)IMS 1974 Powai (c) 1976 Rome Symposium, etc. It is remarkable that even in such a short period of life Patodi made quite a number of significant and fundamental contributions to mathematics. These contributions have had a deep and wide spread influence in geometry, topology, number theory, representation theory and mathematical physics.

It is amazing that Patodi's direct computation method of index has connections with so many areas from Physics to Probability.

Patodi's strength was his analytic ability to carry out detailed calculations at deeper levels and in complicated situations. That was his forte. In fact as pointed out by Atiyah, Patodi's best papers are his joint work with Atiyah and Singer on spectral asymmetry and Riemanngeometry I, II, III [P_7]. In his collected works [1] Atiyah says "On the Index Theory in many ways the papers with Patodi (P_7) on spectral asymmetry were perhaps the most satisfying ones I was involved with"and further Atiyah says "it was a great collaboration with Patodi exploiting the different talents of the collaborators (Bott, Patodi, Singer) and I am glad it came to a successful conclusion in P_7".

Let me mention "THE THEOREM $P_7 - I$"(Theorem 4.3, p.62) (see Appendix) which was one of the deepest results of Patodi.

This is such a significant one such that for the above "The Theorem"there are now different proofs as in

(a) X. Dai and W. Zhang: Real embeddings and the AtiyahPatodiSinger Index Theroem for Dirac Operators, Asian J. Math. 4 (2000), 775-794.

(b) J.M. Bismut and J. Cheeger: Families index for manifolds with boundaryI. Jl. Fnal. Anal. 89 (1990), 313-363.

Also, there are now several books on "The Theorem"of Patodi. For Example

(c) B. BoossBavnbeek and K. Wojciechowski: Elliptic boundary problems for Dirac Operators:

Mathematics: Theory and Applications. Birkhauser, BostonInc. Boston, MA, 1993

(which gives a fairly complete treatment following the lines of Patodi work.)

(d) R.B. Melrose, The AtiyahPatodiSinger Index Theorem, A.K. Peters 1993.

(which gives a different treatment on the analytic aspects of the problem.)

We close this topic with this remark.

Patodi had neither seen his major works [P_7, P_9, P_{10}, P_{11}] in print nor saw his index theory generalized to (a) families of operators (b) hjypoelliptic Laplacians by Bismut and his collaborators.

We believe that if Patodi still be alive today, he would be happy to see that his fundamental results have occupied firmly the central areas in global analysis.

General References

1. M.F. Atiyah, Collected works, Clarenden Press, Oxford (1988), Oxford.
2. J.M. Bismut, W. Zhang: An extension of CheegerMuller Theorem, Asterisque Tom 205 (1992), Soc. Math. de France.
3. (a) S. Chern and J. Simons: Some Cohomology classes in principal fibre bundles and applications to Riemannian geometry, Proc. Nat. Acad. Sci. USA, 68 (1971), 791-794.
   (b) $---$ and $---$, Characteristic forms and geometric invariants, Ann. Math. 99 (1974), 48-69.
4. A.M. Gabrielov, I.M. Gelfand and M.V. Losik, Combinatorial computations of characteristic classes, Functional Analysis and Applications, Vol. 9(2), 1975, p.12-28.
5. F. Hizebruch, Topological methods in Algebraic Geometry, Grindl. Math. Wiss Band, 131, (1966).
6. H. Mckean and I.M. Singer, Curvature and Eigenvalues of the Laplacian, Jl. Diff. Geom. 1 (1967), 43-69.
7. W. Muller: Analytic torsion and Rtorsion for Riemann manifolds, Adv. In Maths, 28 (1978), 233-305.
8. D.B. Ray and I.M. Singer, Rtorsion and the Laplacian on Riemannmanifolds, Adv. In Math. 7 (1971), 145-210.
9. E. Witten: Quantum Field Theory and the jones Polynomial, Comm. Math. Physics, 121 (1989), 351399 also Vol. 141 (1991), 423-440.

Papers of V.K. Patodi

[P_0] Collected Papers Eds. M.F. Atiyah and M.S. Narasimhan, World Scientific, 1996.

[P_1] Curvature and eigen forms of the Laplace operator, JDG5 (1971), p.233-249.

[P_2] An analytic proof of RRH theorem for Kahler manifolds, JDG5 (1971), p.251-283.

[P_3] Curvature and the fundamental solutions of the heat operator, JIMS, 34 (1970), p.251-283.

[P_4] (with M.F. Atiyah and R. Bott) on the heat equations and the index theorem, Inv. Math. 19 (1973), p.279-330 (also 28 (1975), p.277-280)

[P_5] Holomorphic Lefschetz fixed point formula, BAMS 79 (1973), p.825-828.

[P_6] (with M.F. Atiyah and I.M. Singer), Spectral Asymmetry and Riemannian Geometry, BLMS, 5 (1973), p.229-234.

[P_7] (with M.F. Atiyah and I.M. Siniger): Spectral Asymmetry and Riemannian Geometry I, II, III, Proc. Camb. Phil. Soc. 77 (1975), 43-69, 405-432, 79 (1976) 71-99.

[P_8] Riemannian Structures and triangulations of manifolds, Proc. ICM 1974, Vancouver, p.39-43.

[P_9] A combinatorial formula for Pontryagin classes, Instituto Nationale di Alta Mathematica, Symposia Mathematica 20 (1976), 497-505.

[P_{10}] (with J. Dodziuk) Riemannian Structure and triangulations of manifolds, JIMS 40 (1976), 1-52.

[P_{11}] (with H. Donnelly) Spectrum and the fixed point set of Isometries II, Topology 16 (1977), p.1-11.

Acknowledgements

I thank Prof. Raj Kishore Singh to encourage me in writing down about my beloved (late) friend and Junior at BHU, Varanasi where we were all students learning mathematics with our teacher Prof. V.L.N. Sarma.

Appendix

I. ATIYAH PATODI SINGER INDEX THEOREM

Let M be an oriented spin manifold with smooth boundary ∂M. We assume M is of even dimension $2n$ then ∂M is of dimension $2n-1$ and carries an induced orientation and also an induced spin structure. Let g^{TM} be a Riemann metric on TM. We assume with out loss of generality that g^{TM} is of product structure near the boundary ∂M. That is, there exists a neighborhood $\partial M\times[0,a)\subset M$ of ∂M, with $a>0$ sufficiently small, such that

(1) $g^{TM}\big|_{\partial M\times[0,a)}=\pi^*(g^{TM}\big|_{\partial M})\oplus dt^2$ where $t\in[0,a)$ is the parameter and

$\pi:\partial M\times[0,a]\to\partial M\times\{0\}$ denotes the canonical projection.

Let $S(TM)=S_+(TM)\oplus S_-(TM)$ be the hermitian vector bundle of spinors associated to (TM,g^{TM}). Let (E,g^E) be a hermitian vector bundle over M carrying a hermitian connection $\bigtriangledown^E$. We assume that g^E and $\bigtriangledown^E$ are of product structure on $\partial M\times[0,a)$. That is

(2) $g^E\big|_{\partial M\times[0,a)}=\pi^*(g^{TM}\big|_{\partial M});\ \bigtriangledown^E\big|_{\partial M\times[0,a)}=\pi^*(\bigtriangledown^E\big|_{\partial M})$.

Let $D^E:\Gamma(S(TM)\otimes E)\to\Gamma(S(TM)\otimes E)$ denote the twisted (by E) Dirac operator defined by the above geometric data. Let $D^E_\pm:\Gamma(S_\pm(TM)\otimes E)\to\Gamma(S_\pm(TM)\otimes E)$ denote the obvious restriction of D^E. In view of the product structures given by (1) and (2) above on $\partial M\times[0,a)$ one has

(3) $D^E\equiv c(\frac{\partial}{\partial t})(\frac{\partial}{\partial t}+\pi^*D^E_{\partial M})$, where $c(.)$ is the notation for the clifford action and

$D^E_{\partial M}:\Gamma(S(TM)\otimes E\big|_{\partial M})\to\Gamma(S(TM)\otimes E\big|_{\partial M})$ is the induced Dirac operator on ∂M.

Let $D^E_{\partial M,\pm}:\Gamma\big((S_\pm(TM)\otimes E)\big|_{\partial M}\big)\to\Gamma\big((S_\pm(TM)\otimes E)\big|_{\partial M}\big)$ denote the obvious restriction. Then both $D^E_{\partial M,\pm}$ are elliptic and formally self adjoint. We call them also the induced Dirac operators on the boundary ∂M.

Let $L^2\big((S_\pm(TM)\otimes E)\big|_{\partial M}\big)$ be the L^2-completion of $\Gamma\big((S_\pm(TM)\otimes E)\big|_{\partial M}\big)$.

Let $L^2_{\geq 0}\big((S_+(TM)\otimes E)\big|_{\partial M}\big)\subset L^2\big((S_\pm(TM)\otimes E)\big|_{\partial M}\big)$ and

$L^2_{>0}\big((S_\pm(TM)\otimes E)\big|_{\partial M}\big)\subset L^2\big((S_\pm(TM)\otimes E)\big|_{\partial M}\big)$ be defined by

(5) $\mathrm{ind}\big(D^E_+,P^E_{+,\geq 0}\big)=\dim\Big(\ker\big(D^E_+,P^E_{+,\geq 0}\big)\Big)-\dim\Big(\ker\big(D^E_-,P^E_{-,>0}\big)\Big)$,
where both

(6) $\ker\big(D^E_+,P^E_{+,\geq 0}\big)=\Big\{u\in\Gamma(S_+(TM)\otimes E):D^E_+u=0,P^E_{+,\geq 0}(u|_{\partial M}=0)\Big\}$ and

(7) $\ker\big(D^E_-,P^E_{-,\geq 0}\big)=\Big\{u\in\Gamma(S_-(TM)\otimes E):D^E_-u=0,P^E_{-,>0}(u|_{\partial M}=0)\Big\}$ are of finite dimension.

First note that $P^E_{\pm,\geq 0}$ and $P^E_{\pm,>0}$ do not satisfy the requirements of the local boundary condition, thus they are global boundary conditions and hence these boundary conditions no longer contribute topologically invariant indices. In particular, $\mathrm{ind}\big(D^E_+, P^E_{+,\geq 0}\big)$ defined in (5) now depends on the induced geometric data on ∂M.

The first main result of $[P7]$ provides an explicit formula for $\mathrm{ind}\big(D^E_+, P^E_{+,\geq 0}\big)$ which generalizes Atiyah-Singer index theorem for (D^E_+) in the case M is closed. The significant thing that besides the usual geometric term which can be obtained from the local computation, there occurs in this index formula a new term contributed from the boundary ∂M, namely the $\eta-$ invariant associated to the Dirac operator $D^E_{\partial M,+}$ on ∂M.

Recall that $D^E_{\partial M,+} : \Gamma\big((S_+(TM)\otimes E)\big|_{\partial M}\big) \to \Gamma\big((S_+(TM)\otimes E)\big|_{\partial M}\big)$ is elliptic and formally self-adjoint. By standard elliptic theory, one sees that for any complex number s with $\mathrm{Re}(s) >> 0$, the following $\eta-$ function of $D^E_{\partial M,+}$ is well defined,

(8) $\eta(D^E_{\partial M,+}, s) = \sum\limits_{\lambda\in \mathrm{spec}(D^M_{\partial M,+}),\lambda\neq 0} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}$ where $\mathrm{sgn}(\lambda) = 1$ if $\lambda > 0$, $= -1$ if $\lambda \neq 0$.

η is a spectral function depending only on the restriction of the geometric data $(g^{TM}, g^E, \bigtriangledown^E)$ on ∂M. It was proved in $[P7]$ that $\eta(D^E_{\partial M,+}, s)$ can be extended to a meromorphic function of s over $\mathbb{C}$, which is holomorphic at $s = 0$. Then naturally one calls $\eta(D^E_{\partial M,+}, 0)$ the $\eta-$ invariant of $D^E_{\partial M,+}$ and is denoted simply by $\eta(D^E_{\partial M,+})$. Taking the zero eigenvalue of $D^E_{\partial M,+}$ into account one can define

(9) $\bar\eta(D^E_{\partial M,+}) = \frac{\dim(\ker(D^E_{\partial M,+}))+\eta(D^E_{\partial M,+})}{2}$

which is called the reduced $\eta-$ invariant of $D^E_{\partial M,+}$. One can now state the A-P-S index formula for $(D^E_+, P^E_{+,\geq 0})$ (P7–I, formula 4.3, p62).

Theorem 0.1. *(APS) The following identity holds:*

(10) $\mathrm{ind}(D^E_+, P^E_{+,\geq 0}) = \int_M \hat A(TM, \bigtriangledown TM)\mathrm{ch}(E, \bigtriangledown^E) - \bar\eta(D^E_{\partial M,+})$

where $\hat A(TM, \bigtriangledown TM)$ is the Hirzebruch $\hat A$- characteristic form associated to Levi-Civita connection $\bigtriangledown^M$ of Riemann metric g^{TM} and $ch(E, \bigtriangledown^E)$ is the Chern characteristic form associated to the hermitian connection $\bigtriangledown^E$ on E, defined by

(11) $\hat A(TM, \bigtriangledown TM) = \det^{\frac12}\left(\frac{R^{TM}/4\pi}{\sinh(R^{TM}/4\pi)}\right)$ and

(12) $\mathrm{ch}(E, \bigtriangledown^E) = \mathrm{tr}\big(\exp(\frac{\sqrt{-1}}{2\pi})(\bigtriangledown^E)^2\big)$ with

$R^{TM} = (\bigtriangledown TM)^2$ is the curvature of g^{TM} on manifold M and.
$\Omega = (\bigtriangledown^E)^2$ is the curvature of hermitian connection $\bigtriangledown^E$ on E.

II. Now some remarks are in order on this theorem.

(a) If M is closed then $\partial M = \phi$ and (10) becomes $\mathrm{ind}(D_+^E) = \langle \hat{A}(TM)\mathrm{ch}(E), [M]\rangle$ which is the original global Atyah-Singer index theorem for D_+^E (BAMS69 (1963) 422-433).

(b) The product structure reduction near ∂M for g^{TM}, g^E and $\bigtriangledown^E$ can always be realized with out losing any generality in the index formula (cf.P.B.Glkey on the index of geometrical operators for Riemannian manifolds with boundary, Adv in Maths 102 (1993) p129-183).

(c) This theorem was generalized for any first order elliptic differential operator on manifolds with boundary (P7-I Theorem 3.10) formally. However, only for the geometric operators one gets the explicit local expression for the interior contribution. On the otherhand, all geometric operators of interest can locally be expressed as a kind of twisted Dirac operators and hence the above stated APS theorem for twisted Dirac operators is the main result (Theorem 0.1).

(d) Infact the aim of [P7] is to solve a conjecture of Hirzebruch (Hilbert modular surfaces, L'ensignment Math 19 (1973 183-281) concerning the computation of the signature of Hilbert modular varieties having cusp singularities. Then later on using Theorem 0.1 for the signature operators this Hirzebruch conjecture was solved by Atiyah, Donnelly and Singer (all these collaborated with Patodi) in 1983 (M.F.A, H.D. and IMS: Eta invariants, signature defects of cusps and values of L-functions Ann. of Maths 118 (1983), 131-177).
An independent proof for Hirzebruch conjecture was also given by Muller (Signature defects of cusps of modular varieties and values of L-series at $s = 1$ JDG 20 (1984) 55-119).
These two papers established Patodi's Theorem 0.1 enjoys a solid place in Number Theory also.
Now we make some observations on Theorem 0.1 from purely index point of view.

(e) One observes from (10) that $\bar{\eta}(D^E_{\partial M,+})$ depends on the geometric data $(g^{TM}, g^E, \bigtriangledown^E)|_{\partial M}$ and hence it is not a topological invariant and $\bar{\eta}(D^E_{\partial M,+})$ does not admit a local expression. That is not expressible as an integration over M of some local geometrical terms.

(f) From (10), modulo $\mathbb{Z}$, $\bar{\eta}(D^E_{\partial M,+})$ depends smoothly on $(g^{TM}, g^E, \bigtriangledown^E)|_{\partial M}$ and so using Chern-Weil theory, one may calculate explicitly the variation of the $\eta-$ invariant with respect to smooth deformations of $(g^{TM}, g^E, \bigtriangledown^E)|_{\partial M}$.

(g) If $(g^{TM}, g^E, \bigtriangledown^E)|_{\partial M}$ changes then $\dim\big(\ker D^E_{\partial M,+}\big)$ may jump and hence by (10), $\mathrm{ind}(D_+^E, P^E_{+,\geq 0})$ depends in general on $(g^{TM}, g^E, \bigtriangledown^E)|_{\partial M}$. Alternatively note when $\dim\big(\ker D^E_{\partial M,+}\big)$ jumps, $P^E_{+,\geq 0}$ also jumps and we get a different elliptic boundary value problem which may give a possibly different index.

(h) in view of (g) above the problem of characterizing the variation of the index of Theorem 0.1 with different elliptic boundary conditions was investigated by APS in P7-

III paper by introducing the concept of "Spectral flow". Infact, Theorem 0.1 and the associated $\eta-$ invariants and the spectral flow have played later on very important roles in many aspects of geometry, topology, number theory and Mathematical Physics (which needs further elaboration).