An Almost $(0,2)$ - Interpolation on the nodes of $x\pi_n(x)$

Neha Mathur
Department of Mathematics,
Career Convent Degree College, Lucknow
email: neha_mathur13@yahoo.com

Abstract

The problem dealt in this paper was motivated by a beautiful result due to Sharma and Szabados [5], where they have given the existence of a few interpolation processes, on a mixed set of nodes. Here we have considered the problem of existence, uniqueness, explicit representation and convergence of an almost (0, 2)- interpolation on the zeros of $x\pi_n(x) = x(1-x^2)P'_{n-1}(x)$, $P_{n-1}(x)$ denotes the $(n-1)^{th}$ Legendre polynomial.

Subject class [2010]:Primary 05C38, 15A15; Secondary 05A15, 15A18
Keywords:interpolation, $\pi_n(x)$

1 Introduction

Let $(n+1)$ points in the interval $[-1,1]$ be such that

$$-1 = x_{n,n} < x_{n-1,n} < \cdots < x_{1,n} = 1, \tag{1.1}$$

where $\{x_{i,n}\}_{i=1}^{n}$ and $x_{0,n}$ are the zeros of $x\pi_n(x) = x(1-x^2)P'_{n-1}(x)$, $P_{n-1}(x)$ denotes the $(n-1)^{th}$ Legendre polynomial with $P_{n-1}(1) = 1$. Further for arbitrarily given real numbers:

$$\{\alpha_{i,n}\}_{i=0}^{n} \text{and} \{\beta_{i,n}\}_{i=1}^{n} \tag{1.2}$$

we seek to find a polynomial $S_n(x)$ of minimal possible degree satisfying the conditions:

$$S_n(x_{i,n}) = \alpha_{i,n}; \quad i = 0,1,2,\cdots,n \tag{1.3}$$

$$S''_n(x_{i,n}) = \beta_{i,n}; i = 1,2,\cdots,n \tag{1.4}$$

then we call $S_n(x)$ as an Almost (0,2) interpolation polynomial.

The problem of $(0,2)$ interpolation on the zeros of $\pi_n(x) = (1-x^2)P'_{n-1}(x)$ was initiated by J. Balázs and P. Turán [1]. Since then led to number of extensions and generalizations most of which are listed in the book on Birkhoff Interpolation [2] by Lorentz et. al. The problem dealt in this paper was motivated by a beautiful result due to Sharma and Szabados [5], where they have given the existence of a few interpolation processes, on a mixed set of nodes. They have shown that the problem of $(0,2)$ interpolation on the zeros of $\pi_n(x)$ is regular when n is even. If n is odd the problem is not regular. In this paper, we consider n to be even and consider an additional node $x = 0$ making the problem analogous to that discussed in [5] for n odd. We shall discuss the existence, uniqueness and explicit representation of the interpolatory polynomial $S_n(x)$ when n is even (but the number of nodes is odd). Quantitative estimate of the interpolatory polynomial has also been obtained.

In section 2, we give preliminaries. Existence, uniqueness and the explicit representation of the interpolatory polynomials have been dealt with in Section 3. Sections 4 and 5 are devoted to the estimation of the fundamental polynomials and proof of the convergence problem respectively.

2 Preliminaries

The differential equation satisfied by $P_{n-1}(x)$ [6] is

$$(1-x^2)P''_{n-1}(x) - 2xP'_{n-1}(x) + n(n-1)P_{n-1}(x) = 0 \tag{2.1}$$

and that by $\pi_{n-1}(x)$ is

$$(1-x^2\pi''_n(x) + n(n-1)\pi_{n-1}(x) = 0 \tag{2.2}$$

which obviously gives

$$\pi''_n(x_j) = 0 \ , \ \ j = 2, 3, \cdots, n-1. \tag{2.3}$$

For $k = 1, 2, \cdots, n$, we have

$$\ell_k(x) = \frac{\pi_n(x)}{(x-x_k)\pi'_n(x_k)} \tag{2.4}$$

and for $k = 1, 2, \cdots, n$

$$L_k(x) \quad = \quad \frac{x\pi_n(x)}{x_k(x-x_k)\pi'_n(x_k)}. \tag{2.5}$$

Also, for $-1 \leq x \leq 1$ [6], we have

$$\left|(1-x^2)^{1/4}P_{n-1}(x)\right| \leq \sqrt{\frac{2}{\pi(n-1)}}, \tag{2.6}$$

$$\left|(1-x^2)^{3/4}P'_{n-1}(x)\right| \leq \sqrt{2n}, \tag{2.7}$$

$$\left|P'_m(x)\right| \leq \frac{m(m+1)}{2} \tag{2.8}$$

and

$$|\pi_n(x)| \leq \sqrt{\frac{2n}{\pi}}. \tag{2.9}$$

Let $0 < \theta_2 < \theta_4 \cdots < \theta_2 n < \pi$ be the zeros $P_{n-1}(cos\theta)$, then [6] we have

$$2(k-\frac{3}{4})\frac{\pi}{n-1} < \theta_{2k} \ , \ 0 < \theta_{2k} < \frac{\pi}{2} \tag{2.10}$$

and

$$\left|1-x_k^2\right| > \frac{k^2}{4(n-1)^2}. \tag{2.11}$$

Also by [1]

$$|P_{n-1}(x_k)| > \sqrt{\frac{1}{8k\pi}}. \tag{2.12}$$

and

$$\ell_j^2(x) \leq \sum_{k=1}^{n} \ell_k^2(x) \leq 1. \tag{2.13}$$

3 Existence, Uniqueness and Explicit Representation of the Interpolatory Polynomials

We shall prove the following:

Theorem 3.1. *Let n be even and the (n+1) points in [-1, 1] be given by* (1.1)*, then to the prescribed numbers* (1.2)*, there exists a unique polynomial $S_n(x)$ of degree $\leq 2n$ satisfying the conditions* (1.3)- (1.4)*. In particular, if $S_n(x_i) = 0$ for $i = 0, 1, \cdots, n$ and $S_n''(x_i) = 0$ for $i = 1, \cdots, n$ then $S_n(x) \equiv 0$. But if n is odd, there is in general no polynomial of degree $\leq 2n$, which satisfies the conditions* (1.3)- (1.4)*. If there exists such a polynomial, then they are infinitely many.*

Proof. Let $Q(x)$ be another polynomial of degree $\leq 2n$ satisfying the conditions (1.3) - (1.4) viz.

$$\begin{aligned} Q(x_{i,n}) &= \alpha_{i,n}; \quad i = 0, 2, \cdots, n \\ Q''(x_{i,n}) &= \beta_{i,n}; i = 1, 2, \cdots, n \end{aligned}$$

then we have

$$S_n(x_{i,n}) - Q(x_i) = 0; \quad i = 0, 2, \cdots, n \tag{3.1}$$

$$S_n''(x_{i,n}) - Q''(x_{i,n}) = 0; i = 1, 2, \cdots, n \tag{3.2}$$

which implies

$$S_n(x) - Q(x) = x\pi_n(x)q_{n-1}(x) \tag{3.3}$$

where $q_{n-1}(x)$ is a polynomial of degree $\leq n-1$. On applying (3.2) we get

$$\{xq_{n-1}(x)\}'_{x=x_i} = 0; i = 2, 3, \cdots, n-1,$$

which implies

$$\{xq_{n-1}(x)\}' = (c_1x + c_2)P'_{n-1}(x),$$

where c_1 and c_2 are constants. Thus,

$$xq_{n-1}(x) = \int_{-1}^{x} (c_1x + c_2)P'_{n-1}(x)dx + c_3. \tag{3.4}$$

Putting this value in (3.3) and applying the conditions (3.1)-(3.1) we get $c_1 = c_2 = c_3 = 0$ leading to $q_{n-1}(x) = 0$. Hence $S_n(x) \equiv Q(x)$, which proves that for n even $S_n(x)$ exists uniquely.

For n odd, c_1 and c_2 remain undetermined and $c_3 = 0$, leading to infinitely many polynomials. □

The uniquely determined polynomial $S_n(x)$ satisfying the conditions (1.3) and (1.4), is given by

$$S_n(x) = \sum_{i=0}^{n} \alpha_i A_i(x) + \sum_{i=1}^{n} \beta_i B_i(x) \tag{3.5}$$

where $\{A_i(x)\}_{i=0}^{n}$ and $\{B_i(x)\}_{i=1}^{n}$ are the fundamental polynomials each of degree $\leq 2n$, which are uniquely determined by the following conditions:
For $i = 0, 2, \cdots, n$

$$\begin{cases} A_i(x_j) = \delta_{ij}, & j = 0, 1, \cdots, n \\ A''_i(x_j) = 0, & j = 1, 2, \cdots, n \end{cases} \tag{3.6}$$

and for $i = 1, 2, \cdots, n$

$$\begin{cases} B_i(x_j) = 0, & j = 0, 1, 2, \cdots, n \\ B''_i(x_j) = \delta_{ij}, & j = 1, 2, \cdots, n \end{cases} \tag{3.7}$$

The explicit representation of the fundamental polynomials are given in the following:

Theorem 3.2. *For n even, the fundamental polynomials $B_1(x), B_n(x)$satisfying the conditions* (3.7)*, can be represented as*

$$B_1(x) = \pi_n(x)\left[\int_{-1}^{x}(c_4x + c_5)\,P'_{n-1}(x)dx + c_6\right] \tag{3.8}$$

where

$$c_4 = -\frac{1}{n^2(n-1)^2}\left[3 - \int_{-1}^{0}P'_{n-1}(x)dx\right]^{-1}, \tag{3.9}$$

$$c_6 = 2c_4\int_{-1}^{0}P'_{n-1}(x)dx, \tag{3.10}$$

$$c_5 = c_4 - \frac{1}{2}c_6 \tag{3.11}$$

and

$$B_n(x) = \pi_n(x)\left[\int_{1}^{x}(c_7x + c_8)\,P'_{n-1}(x)dx + c_8\right], \tag{3.12}$$

where

$$c_7 = -\frac{1}{n^2(n-1)^2}\left[3 + \int_{0}^{1}P'_{n-1}(x)dx\right]^{-1}$$

$$c_9 = -2c_7\int_{0}^{1}P'_{n-1}(x)dx$$

$$c_8 = -c_7 - \frac{1}{2}c_9.$$

Theorem 3.3. *The fundamental polynomials $A_0(x)$ satisfying the conditions* (3.6) *can be represented as*

$$A_0(x) = \frac{\pi_n(x)}{\pi_n(0)} + c_{11}B_1(x) + c_{12}B_n(x) \tag{3.13}$$

where

$$c_{11} = -\frac{n}{2}\left[\int_{-1}^{0}P_{n-1}(x)dx\right]^{-1}.$$

and

$$c_{12} = \frac{n}{2}\left[\int_{0}^{1}P_{n-1}(x)dx\right]^{-1}.$$

Theorem 3.4. *For n even, the fundamental polynomials* $\{B_i(x)\}_{i=2}^{n-1}$ *satisfying the conditions* (3.7)*, can be represented as*

$$B_i(x) = \frac{\pi_n(x)}{2\pi_n'(x_i)} \int_{-1}^{x} \ell_i(x)dx + c_{1i}B_1(x) + c_{2i}A_0(x) \tag{3.14}$$

where

$$c_{1i} = -\frac{\pi_n''(1)}{2\pi_n'(x_i)} \int_{-1}^{1} \ell_i(x)dx \tag{3.15}$$

and

$$c_{2i} = -\frac{\pi_n(0)}{2\pi_n'(x_i)} \int_{-1}^{0} \ell_i(x)dx. \tag{3.16}$$

Theorem 3.5. *The fundamental polynomials* $\{A_i(x)\}_{i=2}^{n-1}$ *satisfying the conditions* (3.6)*, can be represented as*

$$\begin{aligned} A_i(x) &= L_i^2(x) - \frac{2}{x_i^2}B_i(x) + c_{3i}B_1(x) + c_{4i}A_0(x) \\ &\quad -\frac{\pi_n(x)}{x_i\pi_n'(x_i)} \int_{-1}^{x} \frac{xL_i'(x) - L_i(x)}{x - x_i}dx, \end{aligned} \tag{3.17}$$

where

$$c_{3i} = -\frac{n^2(n-1)^2}{2x_i\pi_n'(x_i)} \int_{-1}^{1} \frac{xL_i'(x) - L_i(x)}{x - x_i}dx \tag{3.18}$$

and

$$c_{4i} = -\frac{\pi_n(0)}{x_i\pi_n'(x_i)} \int_{-1}^{0} \frac{xL_i'(x) - L_i(x)}{x - x_i}dx. \tag{3.19}$$

Theorem 3.6. *The fundamental polynomials* $A_p(x), p = 1, n$ *with* $x_1 = 1, x_n = -1$*, satisfying the conditions* (3.6)*, can be represented as,*

$$\begin{aligned} A_p(x) &= x\left[\frac{x + 3x_p}{4}\ell_p^2(x) - \frac{1 - x^2}{4}\ell_p(x)\ell_p'(x)\right] \\ &+c_{13p}B_p(x) + c_{14p}A_0(x) - \frac{\pi_n(x)}{4\pi_n'(x_p)^2}\Big[(x + x_p)^2P_{n-1}'(x) \\ &-2(x + x_p)P_{n-1}(x) + 2\int_{-x_p}^{x} P_{n-1}(x)dx\Big], \end{aligned} \tag{3.20}$$

where

$$
\begin{aligned}
c_{13,1} &= -\frac{3}{16}n(n-1)(n^2-n+10)-\frac{1}{8}(n+1)(n-2)(n^2-n+1),\\
c_{13,n} &= \frac{11}{8}n(n-1)+\frac{3}{16}n^2(n-1)^2+\frac{1}{8}(n+1)(n-2)(n^2-n-1)
\end{aligned}
$$

and

$$c_{14p} = -\frac{\pi_n(0)}{4n^2(n-1)^2}\Big[P'_{n-1}(0)+2\int_{x_p}^{0}P_{n-1}(x)dx\Big].$$

The polynomial $S_n(x)$, for n even satisfies the following quantitative estimate:

Theorem 3.7. *Let $f \in C^2[-1,1]$, then*

$$S_n(f,x) = \sum_{i=0}^{n} f(x_i)A_i(x) + \sum_{i=1}^{n} f''(x_i)B_i(x) \tag{3.21}$$

satisfies the relation:

$$
\begin{aligned}
|f(x)-S_n(f,x)| = O(1)\Bigg[&\delta_n^2(x)\omega\left(f'',\delta_n(x)\right)\\
&+\frac{1}{n^2}\sum_{k=2}^{n/2}\sqrt{k}\,\omega\left(f'',\delta_n(x_k)\right)+\frac{1}{n^{3/2}}\sum_{k=2}^{n/2}\frac{1}{x_k^2}\omega\left(f'',\delta_n(x_k)\right)\Bigg],
\end{aligned}
\tag{3.22}
$$

where O(1) is independent of n *and* x*.*

We will prove only our main Theorem 3.7 as the proof of other Theorems is quite similar to that of theorems in [4]. In order to prove the theorem, we shall need the estimates of the fundamental polynomials.

4 estimation of the fundamental polynomials

We may need the following result proved in [1]:

Lemma 4.1. *For $k = 2,3,\cdots,n-2 (n>2)$*

$$\left|\int_{-1}^{x}\ell_k(x)dx\right| \le \begin{cases} \frac{8}{|\pi'_n(x_k)|}+\frac{2(1-x_k^2)}{(x_k-x)|\pi'_n(x_k)|} & ,x\neq 1\\ \frac{16k\pi}{n(n-1)} \quad , & x=1.\end{cases} \tag{4.1}$$

Lemma 4.2. *For $B_p(x), p=1,n$ given in Theorem 3.2, we have*

$$|B_p(x)| \le \frac{1}{n^{3/2}(n-1)^2}. \tag{4.2}$$

Proof. We prove the lemma for $p = 1$ as for $p = n$ the lemma follows on same lines. From (3.8), we have

$$|B_1(x)| \leq |\pi_n(x)| \left[\left| \int_{-1}^{x} (c_4 x + c_5)\, P'_{n-1}(x) dx \right| + |c_6| \right] \tag{4.3}$$

Since

$$3 - \int_{-1}^{0} P_{n-1}(x) dx = 3 - \frac{P'_{n-1}(0)}{n(n-1)} > 3 - \frac{n^{1/2}}{n(n-1)} > 2 \tag{4.4}$$

therefore, by (3.9), we have

$$|c_4| \leq \frac{4}{n^2(n-1)^2}. \tag{4.5}$$

The estimates of c_5 and c_6 given by (3.11) and (3.10) can be obtained similarly. Thus by (4.3) the Lemma follows. □

Lemma 4.3. *For $-1 \leq x \leq 1$ and $k = 2, 3, \cdots, n-1$, we have*

$$|B_k(x)| \leq \frac{(1 - x_k^2)\, |\ell_k(x)| \sqrt{k}}{n(n-1)} + \frac{k^{3/2}}{n^{3/2}(n-1)^2}\,, \quad 2 \leq k \leq \frac{n}{2}$$

and

$$|B_k(x)| \leq \frac{(1 - x_k^2)\, |\ell_k(x)| \sqrt{n-k}}{n(n-1)} + \frac{(n-k)^{3/2}}{n^{3/2}(n-1)^2}\,, \quad \frac{n}{2} + 1 \leq k \leq n-1.$$

Proof. Obviously it suffices to prove the first assertion. Let $x < x_k < 1$. By (3.14), we have

$$|B_k(x)| \leq \left| \frac{\pi_n(x)}{2\pi'_n(x_k)} \right| \left| \int_{-1}^{x} \ell_k(x) dx \right| + |c_{1k} B_1(x)| + |c_{2k} A_0(x)|\,. \tag{4.6}$$

From (3.15) using (2.12) and Lemma 4.1, we have

$$|c_{1k}| \leq \sqrt{8k\pi}\, n(n-1) \left| \int_{-1}^{1} \ell_k(x) dx \right| \leq (8k\pi)^{3/2}. \tag{4.7}$$

Also by (3.16), it follows that

$$|c_{2k}| \leq \frac{\sqrt{8k\pi}}{2n^{3/2}(n-1)^2}. \tag{4.8}$$

Hence the lemma follows at once by (4.6)-(4.8), $|r_0(x)| = O(1)$ and Lemmas 4.1 and 4.2. □

Lemma 4.4. *For* $-1 \le x \le 1$ *and* $k = 2, 3, \cdots, n-1$*, we have*

$$|A_k(x)| \le \frac{c}{x_n^2}\left(\sqrt{k} + \frac{\sqrt{n}}{\sqrt{k}}\right), \;\; 1 \le k \le \frac{n}{2}$$

and

$$|A_k(x)| \le \frac{c}{x_{n-k}^2}\left(\sqrt{n-k} + \frac{\sqrt{n}}{\sqrt{n-k}}\right), \;\; \frac{n}{2}+1 \le k \le n-1$$

Proof. We confine ourselves to the case $2 \le k \le \frac{n}{2}$ and $-1 < x < x_k$. By (3.17), we have

$$\begin{aligned} |A_k(x)| \;\; \le \;\; & L_k^2(x) + \frac{2}{x_k^2}|B_k(x)| + |c_{3k}B_1(x)| + |c_{4k}A_0(x)| \\ & + \left|\frac{\pi_n(x)}{x_k\pi_n'(x_k)}\int_{-1}^{x}\frac{xL_k'(x) - L_k(x)}{x - x_k}dx\right|, \\ & \le L_k^2(x) + \frac{2}{x_k^2}|B_k(x)| + I_1 + I_2 + I_3 \end{aligned} \tag{4.9}$$

Since

$$\int_{-1}^{x}\frac{xL_k'(x) - L_k(x)}{x - x_k}dx = \frac{1}{x_k}\int_{-1}^{x}\frac{x^2\ell_k'(x)}{x - x_k}dx = \frac{1}{x_k}\Big[(x + x_k)\ell_k(x)$$
$$+\int_{-1}^{x}\ell_k(x)dx + x_k^2\left\{\frac{\ell_k(x)}{x - x_k} + \int_{-1}^{x}\frac{\ell_k(x)}{(x - x_k)^2}dx\right\}\Big].$$

Thus

$$\begin{aligned} I_3 \;\; \le \;\; & \left|\frac{x^2\pi_n(x)\ell_k(x)}{x_k^2(x - x_k)\pi_n'(x_k)}\right| + \left|\frac{\pi_n(x)}{x_k^2\pi_n'(x_k)}\right|\left|\int_{-1}^{x}\ell_k(x)dx\right| \\ & +2\left|\frac{\pi_n(x)}{\pi_n'(x_k)}\right|\left|\int_{-1}^{x}\frac{\ell_k(x)}{(x - x_k)^2}dx\right| \end{aligned}$$

which reduces to

$$\begin{aligned} I_3 \;\; \le \;\; & L_k^2(x) + \ell_k^2(x) + \left|\frac{\pi_n(x)}{x_k^2\pi_n'(x_k)}\right|\left|\int_{-1}^{x}\ell_k(x)dx\right| \\ \le \;\; & \frac{\sqrt{8\pi}}{(n-1)x_k^2}\left[\frac{\sqrt{8\pi}}{\sqrt{n}} + \frac{(1 - x_k^2)}{\sqrt{k}}|\ell_k(x)| + \frac{\sqrt{n}}{k^{3/2}}|\ell_k(x)|\right] \end{aligned} \tag{4.10}$$

Also, since by [1]

$$\int_{-1}^{x}\frac{\ell_k'(x)}{x - x_k}dx = -\frac{1}{(1 - x_k^2)P_{n-1}^2(x_k)}$$

thus, we have

$$\begin{aligned}\int_{-1}^{1}\frac{xL_k'(x)-L_k(x)}{x-x_k}dx &= \int_{-1}^{x}\ell_k(x)dx+x_k^2\int_{-1}^{x}\frac{\ell_k'(x)}{x-x_k}dx\\ &\leq \frac{16k\pi}{n(n-1)}+\frac{8\pi n^2}{k}.\end{aligned}$$

Thus

$$\begin{aligned}I_1 &= |c_{3k}B_1(x)|\\ &\leq \frac{\sqrt{8k\pi}}{|x_k|\sqrt{n}(n-1)}\left[\frac{16k\pi}{n(n-1)}+\frac{8\pi n^2}{k}\right]\leq c\sqrt{\frac{n}{k}}\frac{1}{|x_k|}\end{aligned}\tag{4.11}$$

where c is a constant independent of k and n. For the estimate of I_2 we have

$$I_2=|c_{4k}A_0(x)|\leq c\frac{\sqrt{k}}{|x_k|}\tag{4.12}$$

because $|A_0(x)|\leq c$ and $|c_{4k}|\leq c\frac{\sqrt{k}}{|x_k|}$ hence, the Lemma follows by using results (4.10)-(4.12) in (4.9). □

5 Proof of the Main theorem 3.7

In order to prove our main Theorem 3.7, we need the following important result of I.E. Gopengaus [3]: Let $f\in C^r[-1,1]$, then for $n\ geq 4r+5$, there exists a polynomial $Q_n(x)$ of degree at most n such that for all $x\in[-1,1]$ and for $k=0,1,\cdots,r$

$$\left|f^{(k)}(x)-Q_n^{(k)}(x)\right|\leq c_k\left(\delta_n(x)\right)^{(r-k)}\omega\left(f^{(r)},\delta_n(x)\right)\tag{5.1}$$

where $\delta_n(x)=\frac{\sqrt{1-x^2}}{n}$ and c_ks are constants independent of f,n and x.

From the uniqueness of $S_n(x)$ in (3.5) it follows that every polynomial $Q_n(x)$ of degree $\leq 2n$ satisfies the relation

$$Q_n(x) = \sum_{k=0}^{n}Q_n(x_k)A_k(x)+\sum_{k=1}^{n}Q_n''(x_k)B_k(x)\tag{5.2}$$

Thus

$$\begin{aligned}&|S_n(x)-f(x)|\leq|S_n(x)-Q_n(x)|+|Q_n(x)-f(x)|\\ &+\sum_{k=0}^{n}|f(x_k)-Q_n(x_k)|\,|A_k(x)|+\sum_{k=1}^{n}\left|f''(x_k)-Q_n''(x_k)\right||B_k(x)|+|Q_n(x)-f(x)|\end{aligned}$$

Since $|f(x_k) - Q_n(x_k)| = 0$ for $k = 1, n$, thus we have

$$
\begin{aligned}
|S_n(x) - f(x)| &\le |f(x_0) - Q_n(x_0)|\,|A_0(x)| \\
&+ \sum_{k=2}^{n-1} |f(x_k) - Q_n(x_k)|\,|A_k(x)| + \sum_{k=1}^{n} \left|f''(x_k) - Q_n''(x_k)\right| |B_k(x)| \\
&\equiv J_1 + J_2 + J_3. \qquad (5.3)
\end{aligned}
$$

From (5.1) for $r = 2$ and $k = 0$, we have

$$J_3 \le c\delta_n^2(x)\omega\left(f'', \delta_n(x)\right). \qquad (5.4)$$

Again by (5.1) for $r = 2$, $k = 2$ and Lemma 4.3 we have

$$
\begin{aligned}
J_2 &\le \frac{c}{n^2} \sum_{k=2}^{n-1} \frac{\sqrt{k}}{n(n-1)} \left\{(1 - x_k^2)\,|\ell_k(x)| + \frac{k}{\sqrt{n}(n-1)}\right\} \omega\left(f'', \delta_n(x_k)\right) \\
&\le \frac{c}{n^2} \sum_{k=2}^{n-2} \sqrt{k}\omega\left(f'', \delta_n(x_k)\right). \qquad (5.5)
\end{aligned}
$$

Also by (5.1) for $r = 2$, $k = 0$ and Lemma 4.4 we have

$$
\begin{aligned}
J_1 &\le \frac{c}{n^2}\omega\left(f'', 1/n\right) + \frac{c}{n^2} \sum_{k=2}^{n/2} \frac{(1 - x_k^2)}{x_k^2} \omega\left(f'', \delta_n(x_k)\right) \left\{\sqrt{k} + \sqrt{\frac{n}{k}}\right\} \\
&\le \frac{c}{n^2}\omega\left(f'', 1/n\right) + \frac{c}{n^{3/2}} \sum_{k=2}^{n/2} \frac{1}{x_k^2} \omega\left(f'', \delta_n(x_k)\right) \qquad (5.6)
\end{aligned}
$$

Using equations (5.4)-(5.6) in (5.3), the Theorem follows.

References

[1] J. Balázs and P. Turán, Notes on interpolation II, Acta. Math. Acad. Sci. Hungar. 8(1957), 201-215.

[2] G.G. Lorentz, K. Jetter and S. D. Riemenschineider, Birkhoff Interpolation, addison Wesley Pub. Co., Mass., 1983.

[3] I. E. Gopengauz; On the Theorem of A. F. Timan on Approximation of Continuous Functions on a line segment, Math. Zametski, 2(1967), 163-172.

[4] I. Joó; On Pál Interpolation, Annals Univ. Sci. Budapest, Sect. Math., 37 (1994), 247-262.

[5] J. Szabados and A. Sharma; Remarks on $(0, 2)$ interpolation, Ganita, 46(2)(1995), 87-110.

[6] G. Szegö; Orthogonal Polynomials, Amer. Math. Soc., Coll. Publ., 1959.