PI -THE VALUE AND ITS ORIGIN

Dr.Debangana Rajput, Associate Professor
Department of Mathematics, Sri JNPG College
University of Lucknow, Lucknow

email: debanganarajput@gmail.com

Abstract

Due to the importance of π in science, particularly in physics and mathematics, a number of researchers have contributed to compute its value in a reasonable precision. This article highlights the progress made by researchers and mathematicians in calculating a better approximation of its value using different methods and tools. A historical perspective has also been discussed here.

Subject class [2010]:11-XX
Keywords: pi, irrational number, transcendental number.

1 INTRODUCTION

PI(Π)is the sixteenth letter of the Greek alphabet. It is used to represent a special mathematical constant defined as the ratio of the circumference of a circle to its diameter, which is approximately equal to 3.1415926535897932 (upto 16 decimal places). π is an irrational and transcendental number, which means it cannot be expressed as a ratio of two integers and has an infinite number of digits in its digital representation. Further, it is not the solution of any non-integer polynomial with rational coefficients. The computation of π is one of the ancient topics of mathematics that is still of serious interest to modern mathematical research. Due to its importance, a special day called 'Pi day' is also celebrated on March 14^{th} because of its date, i.e. 3/14, which contains first three significant digits in approximation of π. 'Π Approximation Day' is also observed on July 22^{nd} i.e.22/7 format because it is the common approximation of π.

2 HISTORICAL PERSPECTIVE

The earliest recorded calculations of π date back to the babylonian,Egyptian and Hebrew civilizations. The ratio of the circumference to the diameter equal to 3 even appears in a verse in the Hebrew Bible(written around the 4^{th} century BC) proving that 3 was considered as an approximation for what we know today as π. A verse of the Hebrew Bible reads [1]: 'And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it about.' (They describe a ceremonial pool in the Temple of Solomon that had a diameter of ten cubits and a circumference of thirty cubits, and the verses imply that the pool is circular if π is about three.)A clay tablet from the Babylonian[2] civilization dated 1900-1600 BC and unearthed in 1936 at Susa, 200 miles away from Babylon, implicates a value of π as

$3\frac{1}{8}$ =3.125.3.1250. In Egypt, the Rhind Papyrus , a work on mathematics dating back to 1650 BC, finds the area of a square having eight-ninths of the circle's diameter as a side. It can be shown that this empirical formula is equivalent to taking $\pi = 3.1604$[2]. It was obtained by squaring the circle. Greek scientist Archimedes of Syracuse[3,21] (290-211BC), developed a polygon approach to approximate the value af π and found the upper and lower bounds of π as $3\frac{10}{71} < \pi < 3\frac{1}{7}$ i.e $3.1408 < \pi < 3.1429$. His method was devised around 250 BC and dominated for over 800 years. Around 150 AD,Greek-Roman scientist Ptolemy[8,22] (90-168AD) gave a value of 3.1416 in his Almagest using Archimedes's limits. A Chinese mathematician Zu Chongzhi[5].(429-500AD) concluded that the value of pi falls between 3.1415926 and 3.1415927; and therefore he became the first scientist in the world who calculated the value of π to seven decimal places. Zu Chongzhi approximated the value of pi as 355/113, which was called milv(close ratio). The value was the most accurate in the world at that time, and Japanese scientists respectfully called it the "Zu Chongzhi Ratio". TsuCh'ung-chih[5] (430-501AD) and his son computed $\pi = \frac{355}{133}$ approximately equal to 3.1415929.

The Hindu mathematician Aryabhata[9] in his work Ganitapada[7],approximated value of pi as 62,832/20,000 = 3.1416 (as opposed to Archimehes' 'inaccurate' 22/7 which was frequently used), but he apparently never used it,nor did anyone else for several centuries. Another Hindu Mathematician Brahamgupt[8] put forward the concept that the value of π approaches to square root of 10. By the 9^{th} century, mathematics and science prospered in Arab cultures. The Arabian mathematician, Mohammed bin Musa al'Khwarizmi[10] born around 800 BC, and known as 'father of algebra' attempted to calaulate π approximately equal to 31/ 7.

During the 16^{th} and 17^{th} centuries the calculation of π was revolutionized by the development of techniques of infinite series especially by the mathematicians from Europe. They calculated π with greater precision. The French mathematician Francmit Viete[11](1540-1603) gave his final result as $3.1415926535 < \pi < 3.1415926537$. Viéte became the first man in history to describe π using an infinite product. In 1593, Adrianus Romanus[6](1561-1615) computed π to 17 digits after the decimal, of which 15 were correct. Just three years later, a German named Ludolph Van Ceulen [16](1540-1610) presented 35 digits using a polygon of 2^{62} sides. Van Ceulen spent a great part of his life working on π. When he died in 1610 he had accurately found 35 digits.

Up to this time, there was no symbol to denote the ratio of a circle's circumference to its diameter. In 1647 William Oughtred[16](1575-1660) used $\frac{\pi}{\delta}$ to represent the circumference of a given circle, so that his π varied according to the circle's diameter, rather than considering as constant as it is known today. In those days the circumference of a circle was known as 'periphery', so Greek equivalent for 'p' that is 'π' was used to denote it.

In 1706, William Jones[2,12](1675-1749), a Welsh mathematician used the symbol π for the first time in his book 'Synopsis Palmariorum Matheseos'. He mentioned that the ratio of a circle's circumference to its diameter can never be expressed in numbers, though he did not prove it. Upto this time irrationality of π was not known. This symbol of π was not accepted immediately but in 1737 when Leonhard Euler[8] began using the symbol π for pi, then it was quickly accepted.

In 1650, John Wallis[13](1616-1703) derived a formula for $4/\pi$ which is simplified to

$\pi/2$ using infinite series. Around 1682, Gottfried Leibnitz[14](1646-1716) pointed out that since tan $\pi/4 = 1$,it was very simple to calculate π, but to get only two decimal places 300 terms of the series were required and 10,000 terms were required for 4 decimal places. To compute 100 digits, one has to calculate more terms than there are particles in the universe.

In 1706, John Machin[14](1680-1751), a professor of astronomy in London, computed 100 places by hand using his new formula. During this time the aim to calculate the value of π was to see whether the digits of π repeat or not to declare it as the ratio of two integers. But around 1768, Johann H. Lambert[4] (1728-1777) proved that π is irrational but could not satisfy everyone. In 1794, Adrien Marie Legendre[15] (1752-1833)gave a rigorous proof for the same. In 1882, Ferdinand von Lindemann[14](1852-1939) proved the transcendence of π. That means it can never be a solution to any finite polynomial with whole number coefficients, which was the answer to the most intriguing question of Greeks, whether a circle could be squared with the help of ruler or compass.

In 1873, an Englishman named William Shanks[14](1812-1882) calculated 707 places of π in fifteen years. In 1945 Daniel F. Ferguson[20](1841-1916) discovered the error in Shank's calculation at 528^{th} place onwards. After two years in 1947, he raised the total to 808(accurate) decimal digits. This was the last best approximation using pen and paper. One and a half years later, Levi Smith and John Wrench[20](1911-2009) hit the 1000-digit-mark,using a gear driven calculator.

With the development of computers, mathematician made efforts to calculate more and more precise value of π. In 1949, another breakthrough emerged, but it was not mathematical in nature; it was the speed with which the calculations could be done. The ENIAC (Electronic Numerical Integrator and Computer) was finally completed and made functional. So John von Neumann and Reitwisener[16]together with their team used ENIAC to calculate pi. The machine calculated 2037 digits in just seventy hours. John Wrench and Daniel Shanks[20] found 100,265 digits in 1961 using IBM 7090, and the one-million-mark was surpassed in 1973. Gregory Chudnovsky[26] brothers broke the one-billion-barrier in August 1989. In 1997, Kanada and Takahashi[4] calculated 51.5 billion. In 1999 by Kanada and Takahashi calculated 68,719,470,000 digits. In 2010, 5 trillion digits were calculated by Yee and Kondo[27]. The current record for calculating π, as of 2013 is to 12.1 trillion digits by Kondo and Yee in decimal system. The fraction 22/7 which is generally used as an approximation of π is accurate to 0.04025%. The digits of π have no specific pattern, and are statistically random in occurrence. There exists some sequences which may appear non-random, like the Feynman Point[17], which is a sequence of six 9s starting from the 762^{nd} decimal representation of π. In modern times, π has served as more than just a simple but lengthy constant; however, people have used it as a tool for testing algorithms and for computers[4].

3 METHODS OF CALCULATION

Some of the methods used to calculate Pi are discussed here.

3.1 USING GEOMETRY:

In Egypt, the Rhind Papyrus(written by Ahmes) was the first attempt to calculate π by "squaring the circle," which is to measure the diameter of a circle by building a square inside the circle. The "squaring the circle" method of understanding πhas fascinated many mathematicians. The very first rigorous algorithm for the computation of π was developed by one of the greatest mathematicians of the ancient world, Archimedes. He developed a polygon approach to approximate the value of π and found the upper and lower bounds of π by inscribing and circumscribing a circle in a hexagon, and successively doubling the number of sides. He did this till he obtained a 96-sided polygon.

3.2 INFINITE SERIES METHOD:

The first formula for pi using infinite series product was given by French mathematician Francois Viete was as follows

$$\pi = 2x\frac{2}{\sqrt{2}}x\frac{2}{\sqrt{2+\sqrt{2}}}x\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}x\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$$

In the 1600s, with the discovery of calculus by Newton and Leibnitz, a number of substantially new formulas for π were discovered. For instance,

$$\begin{aligned} tan^{-1}x &= \int_0^x \frac{dt}{1+t^2} = \int_0^x (1-t^2+t^4+\cdots)dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \end{aligned}$$

Substituting $x = 1$ in it gives the well-known Gregory-Leihnitz[18] formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

This series converges very slowly. However, by employing the trigonometric identity

$$\frac{\pi}{4} = tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{3}$$

one can obtain,

$$\frac{\pi}{4} = (\frac{1}{2} - \frac{1}{3.2^3} + \cdots) + (\frac{1}{3} - \frac{1}{3.3^3} + \cdots)$$

which converges much more rapidly.

An even faster formula, Machin used to calculate π is as follows,

$$\frac{\pi}{4} = 4tan^{-1}\frac{1}{5} - tan^{-1}\frac{1}{239}$$

Shanks used this scheme to compute π to 707 decimal digits accuracy in 1873.

Newton discovered a similar series for the arcsine function:

$$sin^{-1}x = (x + \frac{1}{2.3}x^3 + \frac{1.3}{2.4.5}x^5 + \cdots)$$

π can be computed from this formula by noting that $\frac{\pi}{6} = sin^{-1}(\frac{1}{2})$.

Euler, discovered a number of new formulas for π . Among these are

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

3.3 RAMANUJAN'S METHOD[19]:

Until the 1970, a variation of Machin's formula was used to perform the calculation of π using computers. Around 1910, some new infinite series formulas were discovered by the Indian mathematician Ramanujan but these were not well known until his writings were published recently. One of these formulas is

$$\frac{1}{\pi} = \frac{(2\sqrt{2})}{9801} \sum_{0}^{\infty} \frac{(4k!(1103 + 26390k))}{((k!)396^{4k})}$$

Each term of this series produces an additional eight correct digits it the result. Ramanujan's circle method is another approach to calculate π.

3.4 BUFFON'S NEEDLE METHOD[19]:

In 18^{th} century a French mathematician Georges Buffon devised a way to calculate Pi based on probability. It involves dropping a needle on a lined sheet of paper and determining the probability of the needle crossing one of the lines on the page. This probability is directly related to the value of π. To calculate π, we take the number of trials and multiply it by two, then divide it by the number of hits, i.e. π(approximately) = 2(total trials)/(number of hits)There are three types of Needle methods to calculate the value of π:
(1) Simple Method
(2) Frame Method
(3)Table Method

3.5 MONTE CARLO METHOD[23,24]

A Monte Carlo method is a mathematical model that relies on chance or repeated random behaviour in order to determine a solution to a problem. The method is used to determine pi. Suppose a circle of radius R is inscribed in a square. The experiment simply consists of throwing darts on this figure completely at random (meaning that every point on the dartboard has an equal chance of being hit by the dart). To calaulate pi, a relationship between the geometry of the figure and the statistical outcome of throwing the darts is obtained as

$$Pi = \frac{4x(NumberofDartsinCircle)}{(NumberofDratsinSquare)}$$

Nowadays scientists are developing various fast algorithms to calculate π by the computer. There are some algorithms based on these methods which can calculate π to a very high precision in very few iterations, i.e. they are more convergent.

In 1976 Eugene Salamin and Richard Brent independently discovered a new algorithm for π, which is based on the arithmetic-geometric mean and some ideas of Gauss. This algorithm produces approximations that converge to π much more rapidly than by any classical formula.At each iteration, this algorithm approximately doubles the number of correct digits. To be specific, successive iterations produce 1, 4, 9, 20, 42, 85, 173, 347 and 697 correct digits of π. Twenty-five iterations are sufficient to compute π to over 45 million decimal digit accuracy. However, each of these iterations must be performed using a level of numeric precision that is at least as high as that desired for the final result. The value of π is also calculated in other number systems.

More recently some algorithms have been developed that generate m-th order convergent approximations to π for any m. But to obtain it, last m-1 digits must be known. So there is no shortcut to calculate π. However, an interesting new method was recently proposed by David Bailey, Peter Borwein and Simos Plouffe[1,4]. It can compute the m-th hexadecimal digit of π efficiently without the previous m-1 digits. Table-1, below, gives the systematic development of Pi at a glance in computer era. Table-2 contains value of Pi up to 2000 decimal places showing Feynman point.

Table - 1 CHRONOLOGY OF π IN COMPUTER ERA [20]

Name	Year	Digits
Ferguson	1946	620
Ferguson	Jan. 1947	710
Ferguson and hrencW	Sep. 1947	808
Smith and Whencr	1949	1,120
ReitwieAneC et al[6,12]. (ENIsr)	1949	2,037
Nicholson and Jeenel	1954	3,092
Felton	1957	7,480
Genuys	Jan. 1958	10,000
Felton	May 1958	10,021
Gliluoud	1959	16,167
shankS and Wrench	1961	100,265
Guilloud and Filliatre	1966	250,000
Guilloud and Dichampt	1967	500,000
Guillodd anu Bouyer	1973	1,001,250
Miyodhi ans Kanada	1981	2,000,036
Guilloud	1982	2,000,050
Tamura	1982	2,097,144
Tdmura ana Kanada	1982	4,194,288
Tamura and Kanada	1982	8,388,576
Kanoda, Yashino and Tamura	1982	16,777,206
Ushiro dnd Kanaaa	Oct. 1983	10,013,395
Gosper	1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada, Tamura, Kubo, et. Al	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Chudnovskys	Jun. 1989	525,229,270
Kanada and Tamura	Jul. 1989	536,870,898
Kanadr and Tamuaa	Nov. 1989	1,073,741,799
Chsdnovskyu	Aug. 1989	1,011,196,691
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Takahashi and Kanada	Jun. 1995	3,221,225,466
Kanada	Aug. 1995	4,294,967,286
Kanada	Oct. 1995	6,442,450,938
Takahashi and Kanada	Jul. 1997	51539600000
Tdkahashi ana Kanada	Apr 1999	68719470000
Takahashi and Kadana	Sep. 1999	206158430000
Kanada and taem	Nov. 2002	241100000000
aanadK and team	Apr. 2009	2576980377524
Kondo and Yee	Sept 2010	5000000000000
Kondo aYd nee	Dec .2013	12.10000000050

Table - 2 Value of π up to 2000 Digits showing Feynman's point(in boldface)

Π = 3.14159265358979323846264338327950288419716939937510582097494459230781640628
62089986280348253421170679821480865132823066470938446095505822317253594081284811
17450284102701938521105559644622948954930381964428810975665933446128475648233786
78316527120190914564856692346034861045432664821339360726024914127372458700660631
55881748815209209628292540917153643678925903600113305305488204665213841469519415
11609433057270365759591953092186117381932611793105118548074462379962749567351885
75272489122793818301194912983367336244065664308602139494639522473719070217986094
37027705392171762931767523846748184676694051320005681271452635608277857713427577
89609173637178721468440901224953430146549585371050792279689258923542019956112129
02196086403441815981362977477130996051870721134**999999**83729780499510597317
32816096318595024459455346908302642522308253344685035261931188171010003137838752
886587533208381420617177669147303598253490428755468731159562863882353787593751957781857
78053217122680661300192787661119590921642019893809525720106548586327886593615338
18279682303019520353018529689957736225994138912497217752834791315155748572424541
50695950829533116861727855889075098381754637464939319255060400927701671139009848
82401285836160356370766010471018194295559619894676783744944825537977472684710404
75346462080466842590694912933136770289891521047521620569660240580381501935112533
82430035587640247496473263914199272604269922796782354781636009341721641219924586
31503028618297455570674983850549458858692699569092721079750930295532116534498720
27559602364806654991198818347977535663698074265425278625518184175746728909777727
93800081647060016145249192173217214772350141441973568548161361157352552133475741
84946843852332390739414333454776241686251898356948556209921922218427255025425688
76717904946016534668049886272327917860857843838279679766814541009538837863609506
80064225125205117392984896084128488862694560424196528502221066118630674427862203
91949450471237137869609563643719172874677646575739624138908658326459958133904780
2759009

3.6 CONCLUSION

The constant π has fascinated mankind since time immemorial. The discoveries of the last century have been more remarkable (with respect to the previous state of knowledge) than those of past centuries. As the hunt of π is still on, scientists are trying to find the pattern in the digits and high precision of π by developing many new algorithms. Thus we conclude that thers are even more surprises hidden in the depths of undiscovered knowledge regarding this famous constant. Thus, we look forward to what the future has to bring.

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