Certain Criteria for Univalency of an Integral Operator involving Convolutions

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Abstract

In this paper, an integral operator involving convolutions is investigated for its univalency in the open unit disk. For this, certain conditions on the convolutions are considered. Our results, in particular, provide the univalency of a new integral operator involving several linear operators. Several previously obtained results also follow from our results.

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1 Introduction

Let $\mathcal{A}$ denotes a class of functions analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$, normalized by the conditions $f(0) = 0 = f'(0) - 1$. A subclass of univalent functions in $\mathcal{A}$ is denoted by $\mathcal{S}$.

A convolution (Hadamard product) $*$ of $f \in \mathcal{A}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

and $g \in \mathcal{A}$ of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.2}$$

is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = g(z) * f(z). \tag{1.3}$$

Note that the convolution $'*'$ preserves the class $\mathcal{A}$.

We have following results:

Lemma 1.1. *[16]Let $f \in \mathcal{A}$. If for all $z \in \mathbb{U}$,*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1, \tag{1.4}$$

then f is univalent in $\mathbb{U}$.

Lemma 1.2. *[19, Corollary 2.3., p.3] Let $f \in \mathcal{A}$. If for $m > 0$*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 - \frac{m-1}{2} |z|^{m+1} \right| < \frac{m+1}{2} |z|^{m+1}, \tag{1.5}$$

for all $z \in \mathbb{U}$. Then the function f is analytic and univalent in $\mathbb{U}$.

Let $\mathcal{B}(m)$ be the class of functions $f \in \mathcal{A}$ satisfying condition (1.5). Clearly, $\mathcal{B}(1) =: \mathcal{B}$ becomes the class of functions satisfying condition (1.4).

Silverman [22] introduced a subclass G_b of the class $\mathcal{A}$ for $0 < b \leq 1$, by

$$G_b = \left\{ f \in \mathcal{A} : \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < b \left| \frac{zf'(z)}{f(z)} \right| \ (z \in \mathbb{U}) \right\}. \tag{1.6}$$

In this paper, we consider for $\mu \in \mathbb{C} \setminus \{0\}$ and for $i = 1, ..., n, \alpha_i, \gamma_i \in \mathbb{C}$, $f_i, g_i, h_i \in \mathcal{A}$, an integral $I_n^{\alpha_i,\gamma_i}(f_i, g_i, h_i; \mu)(z)$ $(z \in \mathbb{U})$ which is as follows:

$$I_n^{\alpha_i,\gamma_i}(f_i, g_i, h_i; \mu)(z) = \left[\mu \int_0^z t^{\mu-1} \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(t)}{t} \right)^{\alpha_i} \left((f_i * h_i)'(t) \right)^{\gamma_i} dt \right]^{\frac{1}{\mu}}, \tag{1.7}$$

where powers taken have their principal values.

Several linear operators are defined and studied so far in the *Geometric Function Theory* (see [23] for detail). It can easily be seen that these linear operators can be written as the convolutions (see for example [24, 25]). Here, for instance,
let $f_i * g_i = \mathcal{I}_{g_i}(f_i)$ and $f_i * h_i = \mathcal{J}_{h_i}(f_i)$ for each $i = 1, ..., n$, the integral $I_n^{\alpha_i,\gamma_i}(f_i, g_i, h_i; \mu)(z)$ defined above by (1.7) actually involve $2n$ linear operators, and it reduces to several integral operators studied earlier (see [8, 14]). Clearly, for $\gamma_i = 0$ $(i = 1, ..., n)$ and for various values of μ, the integral $I_n^{\alpha_i,\gamma_i}(f_i, g_i, h_i; \mu)(z)$ reduces to the integrals studied in [1, 5, 7, 9, 11, 12, 20, 21, 26, 27] which include other integrals cited in these work.

We obtain univalency conditions of the integral operator $I_n^{\alpha_i,\gamma_i}(f_i, g_i, h_i; \mu)(z)$ by using following results:

Lemma 1.3. *[13] Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$, M fixed. If f has a zero with multiplicity greater then m at $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \ \ (z \in \mathbb{U}_R).$$

the equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

Lemma 1.4. *[17]Let $f \in \mathcal{A}$ and $\mu \in \mathbb{C}$. If $\Re\mu > 0$ and*

$$\frac{1-|z|^{2\Re\mu}}{\Re\mu}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator

$$F_\mu(z) = \left[\mu \int_0^z t^{\mu-1} f'(t)dt\right]^{\frac{1}{\mu}},$$

is regular and univalent in $\mathbb{U}$.

Lemma 1.5. *[27]Let $\mu \in \mathbb{C}$ with $\Re\mu > 0$, $c \in \mathbb{C}$ with $|c| \leq 1$, $c \neq -1$. If $f \in \mathcal{A}$ satisfies*

$$\left|c\,|z|^{2\mu} + \left(1-|z|^{2\mu}\right)\frac{zf''(z)}{\mu f'(z)}\right| \leq 1 \ \ (z \in \mathbb{U}),$$

then the integral operator

$$F_\mu(z) = \left[\mu \int_0^z t^{\mu-1} f'(t)dt\right]^{\frac{1}{\mu}}$$

is regular and univalent in $\mathbb{U}$.

2 Main results

Theorem 2.1. *Let for each $i = 1, ..., n$, $f_i, g_i, h_i \in \mathcal{A}$ be such that for $m_i > 0$, $f_i * g_i \in \mathcal{B}(m_i)$ and $f_i * h_i \in \mathcal{B}(m_i)$ alongwith*

$$f_i * h_i \in G_{b_i} \ \ (0 < b_i \leq 1) \tag{2.1}$$

and

$$|(f_i * g_i)(z)| \leq M_i, \quad |(f_i * h_i)(z)| \leq N_i \ \ (M_i \geq 1, N_i \geq 1; z \in \mathbb{U}). \tag{2.2}$$

If for some $\alpha_i, \gamma_i \in \mathbb{C}$ $(i = 1, ..., n)$ and for $\mu \in \mathbb{C}$ $(\Re\mu > 0)$,

$$\Re\mu \geq \sum_{i=1}^{n}\left[|\alpha_i|\left\{(m_i+1)M_i+1\right\} + |\gamma_i|(b_i+1)\left\{(m_i+1)N_i+1\right\} + b_i|\gamma_i|\right], \tag{2.3}$$

then the integral operator $I_n^{\alpha_i,\gamma_i}(f_i, g_i, h_i; \mu)(z)$ defined by (1.7) is univalent.

Proof. Let

$$G(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{(f_i * g_i)(t)}{t}\right)^{\alpha_i}\left((f_i * h_i)'(t)\right)^{\gamma_i} dt. \tag{2.4}$$

Then $G \in \mathcal{A}$ and on differentating (2.4), we get

$$G'(z) = \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(z)}{z} \right)^{\alpha_i} \left((f_i * h_i)'(z) \right)^{\gamma_i}. \tag{2.5}$$

On differentiating logarithmically (2.5), we obtain

$$\frac{zG''(z)}{G'(z)} = \sum_{i=1}^{n} \left[\alpha_i \left(\frac{z (f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right) + \gamma_i \left(\frac{z (f_i * h_i)''(z)}{(f_i * h_i)'(z)} \right) \right].$$

Hence,

$$\begin{aligned} &\left| \frac{zG''(z)}{G'(z)} \right| \\ \leq \; &\sum_{i=1}^{n} \left[|\alpha_i| \left| \frac{z (f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right| + |\gamma_i| \left| \frac{z (f_i * h_i)''(z)}{(f_i * h_i)'(z)} - \frac{z (f_i * h_i)'(z)}{(f_i * h_i)(z)} + 1 \right| \right. \\ &\left. + |\gamma_i| \left| \frac{z (f_i * h_i)'(z)}{(f_i * h_i)(z)} - 1 \right| \right], \end{aligned}$$

which on using the hypothesis (2.1) from (1.6), yields

$$\begin{aligned} &\left| \frac{zG''(z)}{G'(z)} \right| \\ \leq \; &\sum_{i=1}^{n} \left[|\alpha_i| \left| \left(\frac{z (f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right) \right| + |\gamma_i| (b_i + 1) \left| \left(\frac{z (f_i * h_i)'(z)}{(f_i * h_i)(z)} - 1 \right) \right| + b_i |\gamma_i| \right] \\ (2.6) \leq \; &\sum_{i=1}^{n} \left[|\alpha_i| \left(\left| \frac{z^2 (f_i * g_i)'(z)}{[(f_i * g_i)(z)]^2} \right| \left| \frac{(f_i * g_i)(z)}{z} \right| + 1 \right) \right. \\ &\left. + |\gamma_i| (b_i + 1) \left(\left| \frac{z^2 (f_i * h_i)'(z)}{[(f_i * h_i)(z)]^2} \right| \left| \frac{(f_i * h_i)(z)}{z} \right| + 1 \right) + b_i |\gamma_i| \right]. \end{aligned}$$

Now, from (2.2), we have by the General Schwarz Lemma 1.3 for the multiplicity 1 of the zero at $z = 0$ of the functions $f_i * g_i$ and $f_i * h_i$:

$$|(f_i * g_i)(z)| \leq M_i |z| \quad \text{and} \quad |(f_i * h_i)(z)| \leq N_i |z| \quad (i = 1, ..., n; z \in \mathbb{U})$$

Thus from (1.5) of Lemma 1.2, we find

$$\left|\frac{zG"(z)}{G'(z)}\right|$$

$$\leq \sum_{i=1}^{n}\left[|\alpha_i|\left|\frac{z^2\left(f_i*g_i\right)'(z)}{\left[\left(f_i*g_i\right)(z)\right]^2}-1-\frac{m_i-1}{2}|z|^{m_i+1}\right|M_i+\left(1+\frac{m_i-1}{2}|z|^{m_i+1}\right)M_i\right.$$

$$+1+|\gamma_i|\left(b_i+1\right)\left\{\left|\frac{z^2\left(f_i*h_i\right)'(z)}{\left[\left(f_i*h_i\right)(z)\right]^2}-1-\frac{m_i-1}{2}|z|^{m_i+1}\right|N_i\right.$$

$$\left.\left.+\left(1+\frac{m_i-1}{2}|z|^{m_i+1}\right)N_i+1\right\}+b_i|\gamma_i|\right]$$

$$\text{(2.7)}\leq \sum_{i=1}^{n}\left[|\alpha_i|\left\{\frac{m_i+1}{2}|z|^{m_i+1}M_i+\left(1+\frac{m_i-1}{2}|z|^{m_i+1}\right)M_i+1\right\}\right.$$

$$\left.+|\gamma_i|\left(b_i+1\right)\left\{\frac{m_i+1}{2}|z|^{m_i+1}N_i+\left(1+\frac{m_i-1}{2}|z|^{m_i+1}\right)N_i+1\right\}+b_i|\gamma_i|\right].$$

Thus, for $\mu\in\mathbb{C}$ $(\Re\mu>0)$,

$$\frac{1-|z|^{2\Re\mu}}{\Re\mu}\left|\frac{zG''(z)}{G'(z)}\right|$$

$$\text{(2.8)}\qquad \leq \frac{1}{\Re\mu}\sum_{i=1}^{n}\left[|\alpha_i|\left\{\left(m_i+1\right)M_i+1\right\}+|\gamma_i|\left(b_i+1\right)\left\{\left(m_i+1\right)N_i+1\right\}\right.$$

$$\left.+b_i|\gamma_i|\right],$$

which by (2.3) proves

$$\frac{1-|z|^{2\Re\mu}}{\Re\mu}\left|\frac{zG''(z)}{G'(z)}\right|\leq 1\ \ (z\in\mathbb{U}).$$

On applying Lemma 1.4 for the function $G(z)$, we obtain that the integral operator $I_n^{\alpha_i,\gamma_i}\left(f_i,g_i,h_i;\mu\right)(z)$ defined by (1.7) is univalent. This proves Theorem 2.1. □

Taking $\alpha_i=1-\frac{1}{\beta_i},\gamma_i=\frac{1}{\beta_i}$, and $f_1=...=f_n=\frac{z}{1-z}$ $(z\in\mathbb{U})$ in Theorem 2.1, we get following result:

Corollary 2.1. *Let for each* $i=1,...,n,g_i,h_i\in\mathcal{A}$ *be such that for* $m_i>0$, $g_i,h_i\in\mathcal{B}\left(m_i\right)$ *alongwith*

$$h_i\in G_{b_i}\ \ (0<b_i\leq 1)$$

and

$$|g_i(z)|\leq M_i,\quad |h_i(z)|\leq N_i\ \ (M_i\geq 1,N_i\geq 1;z\in\mathbb{U}).$$

If for some $\beta_i\in\mathbb{C}$ $(\beta_i\neq 0,i=1,...,n)$ *and for* $\mu\in\mathbb{C}$ $(\Re\mu>0)$,

$$\Re\mu\geq\sum_{i=1}^{n}\frac{1}{|\beta_i|}\left[|\beta_i-1|\left\{\left(m_i+1\right)M_i+1\right\}+\left(b_i+1\right)\left\{\left(m_i+1\right)N_i+1\right\}+b_i\right],$$

then the integral operator

$$I_n^{\beta_i}(g_i, h_i; \mu)(z) = \left[\mu \int_0^z t^{\mu-1} \prod_{i=1}^{n} \left(\frac{g_i(t)}{t}\right)^{1-\frac{1}{\beta_i}} (h_i'(t))^{\frac{1}{\beta_i}}\, dt\right]^{\frac{1}{\mu}}, \tag{2.9}$$

is univalent.

Remark 2.1. If we take $m_i = 1$ in Corollary 2.1, we get a similar result obtained by Oprea and Breaz [14, Theorem 5, p.217].

Taking $\alpha_i = 1 - \beta_i, \gamma_i = \beta_i$, and $g_1 = ... = g_n = \frac{z}{1-z}, h_1 = ... = h_n = \frac{z}{1-z}$ $(z \in \mathbb{U})$ in Theorem 2.1, we get following result:

Corollary 2.2. *Let for each $i = 1, ..., n, f_i \in \mathcal{A}$ be such that for $m_i > 0$, $f_i \in \mathcal{B}(m_i)$ alongwith*

$$f_i \in G_{b_i} \ \ (0 < b_i \leq 1)$$

and

$$|f_i(z)| \leq M_i \ \ (M_i \geq 1; z \in \mathbb{U}).$$

If for some $\beta_i \in \mathbb{C}$ $(i = 1, ..., n)$ and for $\mu \in \mathbb{C}$ $(\Re\mu > 0)$,

$$\Re\mu \geq \sum_{i=1}^{n} \left[|\beta_i - 1| \{(m_i + 1) M_i + 1\} + |\beta_i| (b_i + 1) \{(m_i + 1) N_i + 1\} + b_i |\beta_i|\right],$$

then the integral operator

$$I_n^{\beta_i}(f_i; \mu)(z) = \left[\mu \int_0^z t^{\mu-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{1-\beta_i} (f_i'(t))^{\beta_i}\, dt\right]^{\frac{1}{\mu}}, \tag{2.10}$$

is univalent.

Remark 2.2. The Corollary 2.2 coincides with the result [8, Theorem 2, p.119] if for each $i = 1, ..., n, f_i \in \mathcal{B}$.

Remark 2.3. Taking $\gamma_i = 0$ in Theorem 2.1, several results obtained earlier follow for special values of parameters m_i, M_i, α_i, μ and for special forms of convolution $f_i * g_i$, for example see [2, Theorem 1, p.138], [5, Theorem 2.3, p.43], [6, Theorem 1, p.89], [9, Theorem 3.6, p.133], also [15, 21].

We use Lemma 1.5 in our next result which is as follows:

Theorem 2.2. *Let for each $i = 1, ..., n, f_i, g_i, h_i \in \mathcal{A}$ be such that for $m_i > 0$, $f_i * g_i \in \mathcal{B}(m_i)$ and $f_i * h_i \in \mathcal{B}(m_i)$ alongwith satisfy equation (2.1) and (2.2). If for some $\alpha_i, \gamma_i \in \mathbb{C}$ $(i = 1, ..., n)$ and for $c, \mu \in \mathbb{C}$ $(\Re\mu > 0, |c| \leq 1, c \neq -1)$,*

$$|c| \leq 1 - \frac{1}{\Re\mu} \sum_{i=1}^{n} \left[|\alpha_i| \{(m_i + 1) M_i + 1\} + |\gamma_i| (b_i + 1) \{(m_i + 1) N_i + 1\} + b_i |\gamma_i|\right], \tag{2.11}$$

then the integral operator $I_n^{\alpha_i, \gamma_i}(f_i, g_i, h_i; \mu)(z)$ defined by (1.7) is univalent.

Proof. From (2.7), we get for $c, \mu \in \mathbb{C}$ $(\Re\mu > 0, |c| \leq 1, c \neq -1)$,

$$\begin{aligned} & \left| c\,|z|^{2\mu} + \left(1 - |z|^{2\mu}\right) \frac{zG''(z)}{\mu G'(z)} \right| \\ \leq \quad & |c| + \frac{1}{\Re\mu} \left[|\alpha_i| \left\{ (m_i + 1) M_i + 1 \right\} + |\gamma_i| (b_i + 1) \left\{ (m_i + 1) N_i + 1 \right\} \right. \\ & \left. + b_i |\gamma_i| \right] \end{aligned} \tag{2.12}$$

which by (2.11) proves

$$\left| c\,|z|^{2\mu} + \left(1 - |z|^{2\mu}\right) \frac{zG''(z)}{\mu G'(z)} \right| \leq 1 \ \ (z \in \mathbb{U})$$

and hence on applying Lemma 1.5 for the function $G(z)$, it proves that the integral operator $I_n^{\alpha_i, \gamma_i}(f_i, g_i, h_i; \mu)(z)$ defined by (1.7) is univalent. □

Taking $\alpha_i = 1 - \frac{1}{\beta_i}, \gamma_i = \frac{1}{\beta_i}$, and $f_1 = ... = f_n = \frac{z}{1-z}$, in Theorem 2.2, we get following result:

Corollary 2.3. *Let for each $i = 1, ..., n, g_i, h_i \in \mathcal{A}$ be such that for $m_i > 0$, $g_i, h_i \in \mathcal{B}(m_i)$ alongwith*

$$h_i \in G_{b_i} \ \ (0 < b_i \leq 1)$$

and

$$|g_i(z)| \leq M_i, \quad |h_i(z)| \leq N_i \ \ (M_i \geq 1, N_i \geq 1; z \in \mathbb{U}).$$

If for some $\beta_i \in \mathbb{C}$ $(\beta_i \neq 0, i = 1, ..., n)$ and for $c, \mu \in \mathbb{C}$ $(\Re\mu > 0, |c| \leq 1, c \neq -1)$,

$$|c| \leq 1 - \frac{1}{\Re\mu} \sum_{i=1}^{n} \frac{1}{|\beta_i|} \left[|\beta_i - 1| \left\{ (m_i + 1) N_i + 1 \right\} + (b_i + 1) \left\{ (m_i + 1) N_i + 1 \right\} + b_i \right],$$

then the integral operator $I_n^{\beta_i}(g_i, h_i; \mu)(z)$ defined by (2.9) is univalent in $\mathbb{U}$.

Remark 2.4. The Corollary 2.3 for $m_i = 1$, coincides with the result of Oprea and Breaz [14, Theorem 7, p.221].

Taking $\alpha_i = 1 - \beta_i, \gamma_i = \beta_i$ and $g_1 = ... = g_n = \frac{z}{1-z}, h_1 = ... = h_n = \frac{z}{1-z}$ $(z \in \mathbb{U})$ in Theorem 2.2, we get following result:

Corollary 2.4. *Let for each $i = 1, ..., n, f_i \in \mathcal{A}$ be such that for $m_i > 0$, $f_i \in \mathcal{B}(m_i)$ alongwith*

$$h_i \in G_{b_i} \ \ (0 < b_i \leq 1),$$

and

$$|f_i(z)| \leq M_i \ \ (M_i \geq 1; z \in \mathbb{U}).$$

If for some $\beta_i \in \mathbb{C}$ $(i = 1, ..., n)$ *and for* $c, \mu \in \mathbb{C}$ $(\Re\mu > 0, |c| \leq 1,$ *and* $c \neq -1)$,

$$|c| \leq 1 - \frac{1}{\Re\mu} \sum_{i=1}^{n} \left[(|\beta_i| \, b_i + 1) \left\{ (m_i + 1) \, M_i + 1 \right\} + |\beta_i| \, b_i \right],$$

then the integral operator

$$I_n^{\beta_i} \left(f_i; \mu \right) (z) = \left[\mu \int_0^z t^{\mu - 1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{1-\beta_i} \left(f_i'(t) \right)^{\beta_i} dt \right]^{\frac{1}{\mu}},$$

is univalent.

Remark 2.5. For $m_i = 1$, the Corollary 2.4 coincides with the result of Frasin [8, Theorem 4, p.122].

Remark 2.6. Taking $\gamma_i = 0$ in Theorem 2.2, we may get several results proved earlier for special values of the parameters m_i, M_i, α_i, μ and for special forms of convolution $f_i * g_i$ ([4, Theorem 4, p.43], [5, Theorem 2.9, p.47], [10, Theorem 1, p.1143], [18, Theorem 3, p.2137], also [3, 28]).

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