Non-differentiable symmetric duality under generalized invexity

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Abstract

In the present paper, a pair of Mond-Weir type non-differentiable mutiobjective second-order programming problems, involving two kernel functions, where each of the objective functions contain support function, is formulated. We prove Weak, Strong and Converse duality theorem for the second-order symmetric dual programs under η-pseudoinvexity conditions.

Keywords: Non-differentiable Multiobjective programming; Second-order symmetric duality; Efficiency; Support function; η-pseudoinvexity.

1. Introduction

Support functions play a significant role in optimization theory. We use subdifferentials in presence of support functions which makes any optimization problem to nonsmooth. subidfferentials of support function have a neat representation and are used to study the non smooth optimization problems. The study of nondifferential problems arises from the fact that even though the objective function or constraint function of the primal problem are non smooth, we can associate dual problems which are differentiable.
The duality in linear programming is symmetric. However, the Symmetric duality in nonlinear programming was first introduced by Dorn [5]. He introduced the concept of symmetric duality for mathematical programming problems. Later on Dantzig et al. [4] formulated a pair of symmetric dual nonlinear programs involving convex/concave functions. These results were extended by Bazaraa and Goode [2] over arbitrary cones.
Nanda and Das [14] also studied the symmetric dual nonlinear programming problem for arbitrary cones assuming the functions to be pseudoinvex. Kim et al. [11] studied a pair of multiobjective symmetric dual programs for pseudo-invex functions and arbitrary cones. Khurana [10] formulated a pair of differentiable multiobjective symmetric dual programs of Mond-Weir type over arbitrary cones in which the objective function is optimized with respect to an arbitrary closed convex cone by assuming the involved functions to be cone-pseudoinvex and strongly cone-pseudoinvex.
Mangasarian [12] introduced the concept of second-order duality for nonlinear problems. Since many authors [6, 7, 8, 13, 15, 18] have worked on second-order symmetric duality. Suneja et al. [17] studied a pair of Mond-Weir type second-order multiobjective symmetric dual programs and proved the duality results under η-bonvexity/η-pseudobonvexity

assumptions. Yang et al.[18] extended the results of Suneja et al. [17] to the nondifferentiable case. Second order symmetric duality for Mond-Weir type duals involving nondifferentiable function has been discussed by Hou and Yang [9] and by Ahmad and Husain [1].

The symmetric dual problems in the above papers involve only one kernel function. Recently, Wolfe and Mond-Weir type second-order differentiable symmetric dual models involving two kernel functions have been studied in [5]. In this paper, we present nondifferentiable symmetric dual multiobjective problems involving two kernel functions.

2. PREREQUISITES

We consider the following multiobjective programming problem :

(P) Minimize

$$K(x) = \{K_1(x), K_2(x), ..., K_k(x)\}$$

$$\text{subject to :} \quad x \in X = \{x \in R^n \mid G_j(x) \leqq 0, j = 1, 2, ..., m\},$$

where $G : R^n \to R^m$ and $K : R^n \to R^k$.

All vectors shall be considered as column vectors. For a function $f : R^n \times R^m \to R^k$, let $\nabla_x f(\nabla_y f)$ denote the $n \times k$ $(m \times k)$ matrix of first order derivative and $\nabla_{xy} f_i$ denote the $n \times m$ matrix of second order derivative.

For a, b $\in R^n$,
a $\geqq b \Leftrightarrow a_i \geqq b_i, i = 1, 2, ..., n$,
a $\geqslant b \Leftrightarrow a \geqq b$ and $a \neq b$,
$a > b \Leftrightarrow a_i > b_i, i = 1, 2, ..., n$.

Definition 2.1. A point $\bar{x} \in X$ is said to be an efficient solution of (P) if there exists no $x \in X$ such that $K(x) \leq K(\bar{x})$.

Definition 2.2 [5]. The function K is $\eta-$invex at $u \in R^n$ if there exists a vector valued function $\eta : R^n \times R^n \to R^n$ such that

$$K(x) - K(u) - \eta^T(x, u)\nabla K(u) \geqq 0, \forall x \in R^n.$$

Definition 2.3 [3, 6]. The function K is pseudoinvex at $u \in R^n$ with respect to $\eta : R^n \times R^n \to R^n$ such that

$$\eta^T(x, u)\nabla K(u) \geqq 0 \Rightarrow K(x) \geqq K(u), \forall x \in R^n.$$

Definition 2.4.[**?**]. Let S be a compact convex set in R^n. The support function $s(x|S)$ of S is defined by

$$s(x|S) = \max\{x^T y : y \in S\}.$$

The subdifferential of $s(x|S)$ is given by

$$\partial s(x|S) = \{z \in S : z^T x = s(x|S)\}.$$

For any convex set $S \subset R^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leqq 0 \ \text{ for all } \ z \in S\}.$$

It is readily verified that for a compact convex set S, y is in $N_S(x)$ if and only if $s(y|S) = x^T y$.

3. MOND-WEIR TYPE SYMMETRIC DUALITY

We now consider the following pair of Mond Weir type nondifferentiable multiobjective problems with k-objectives and establish weak, strong and converse duality theorems.

Primal (MWP):

$$\text{Minimize} \quad H(x, y, \lambda, h, p) = \{f_1(x, y) + s(x|D_1) - (y^T w_1),$$
$$f_2(x, y) + s(x|D_2) - (y^T w_2), ..., f_k(x, y) + s(x|D_k) - (y^T w_k)\}$$

$$\text{(0.1)} \qquad \text{Subject to} \quad \sum_{i=1}^{k} \{ \lambda_i(\nabla_y f_i(x, y) - w_i)\} + \nabla_{yy}(h^T g(x, y))p \leqq 0,$$

$$\text{(0.2)} \qquad y^T \sum_{i=1}^{k} \{ \lambda_i(\nabla_y f_i(x, y) - w_i)\} + \nabla_{yy}(h^T g(x, y))p \geqq 0,$$

$$\text{(0.3)} \qquad \lambda > 0, x > 0, w_i \in R^m.$$

Dual (MWD):

$$\text{Maximise} \quad G(u, v, \lambda, h, q) = \{f_1(u, v) - s(v|E_1) + (u^T z_1),$$
$$f_2(u, v) - s(v|E_2) + (u^T z_2), ..., f_k(u, v) - s(v|E_k) + (u^T z_k)\}$$

$$\text{(0.4)} \qquad \text{Subject to} \quad \sum_{i=1}^{k} \{\lambda_i(\nabla_x f_i(u, v) + z_i)\} + \nabla_{xx}(h^T g(u, v))q \geqq 0,$$

(0.5) $$u^T \sum_{i=1}^{k} \{\lambda_i(\nabla_x f_i(u,v) + z_i)\} + \nabla_{xx}(h^T g(u,v))q \leqq 0,$$

(0.6) $$\lambda > 0, v > 0, z_i \in R^n,$$

where,
(i) $f_i, i = 1, 2, ..., k : R^n \times R^m \to R$ are a twice differentiable function of x and y,
(ii) $g : R^n \times R^m \to R_r$ are a thrice differentiable function of x and y,
(iii) $\lambda_i \in R$, $h \in R_r$, $p_i \in R^m$, $q_i \in R^n$, and
(iv)D_i and E_i, i=1, 2, ..., k are compact convex sets in R^n and R^m, respectively.

Also we take $p = (p_1, p_2, ..., p_k), q = (q_1, q_2, ..., q_k), w = (w_1, w_2, ..., w_k)$ and $z = (z_1, z_2, ..., z_k)$.

Any problem, say (MWD), in which λ is fixed to be $\bar{\lambda}$ will be denoted by (MWD)$_{\bar{\lambda}}$.

Theorem 3.1 (Weak duality). Let (x, y, λ, h, z, p) be feasible for (WP) and (u, v, λ, h, w, q) be feasible for (MWD). Let

(i)$\sum_{i=1}^{k}[\lambda_i^T f_i(., v) + (.)^T z_i]$ be pseudoinvex at u with respect to η_1 for fixed v and z,
(ii)$\sum_{i=1}^{k}[\lambda_i^T f_i(x, .) - (.)^T w_i]$ be pseudoinvex at y with respect to η_2 for fixed x and w,
(iii)$\eta_1(x, u) + u \geqq o$ and $\eta_2(v, y) + y \geqq 0$, and
(iv)$\eta_1^T(x, u)(\nabla_{xx}(h^T g)(u, v)q) \leqq 0$ and $\eta_2^T(v, y)(\nabla_{yy}(h^T g)(x, y)p) \geqq 0$.
Then

$$\sup(MD) \leqq \inf(MP)$$

or

$$\sum_{i=1}^{k} \lambda_i[f_i(u,v) - s(v|E_i) + (u^T z_i)] \ngeq \sum_{i=1}^{k} \lambda_i[f_i(x,y) + s(x|D_i) - (y^T w_i)].$$

Proof. Suppose, to the contrary, that

$$\sum_{i=1}^{k} \lambda_i[f_i(u,v) - s(v|E_i) + (u^T z_i)] \geq \sum_{i=1}^{k} \lambda_i[f_i(x,y) + s(x|D_i) - (y^T w_i)]. \tag{9}$$

From the dual constraint (5) and hypothesis (iii),we get

$$\eta^T(x,u)[\sum_{i=1}^{k}\{\lambda_i(\nabla_x f_i(u,v) + z_i)\} + \nabla_{xx}(h^T g(u,v))q]$$
$$\geqq -u^T[\sum_{i=1}^{k}\{\lambda_i(\nabla_x f_i(u,v) + z_i)\} + \nabla_{xx}(h^T g(u,v))q]. \tag{10}$$

Now inequalities (6), (10) along with hypothesis(iv), yields

$$\eta^T(x,u)[\sum_{i=1}^{k}\{\lambda_i(\nabla_x f_i(u,v)+z_i)\}] \geqq, 0$$

which by pseudoinvexity of $\sum_{i=1}^{k}\lambda_i^T[f_i(.,v)+(.)^T z_i]$, we have

$$\sum_{i=1}^{k}\lambda_i[f_i(x,v)+(x^T z_i)-f_i(u,v)-(u^T z_i)] \geqq 0. \tag{11}$$

Similarly by pseudoinvexity of $\sum_{i=1}^{k}\lambda_i^T[f_i(x,.)-(.)^T w_i]$, the primal constraints (1), (2) and hypotheses (iii) and (iv), we obtain

$$\sum_{i=1}^{k}\lambda_i[f_i(x,y)-(y^T w_i)-f_i(x,v)+(v^T w_i)] \geqq 0. \tag{12}$$

Adding inequalities (11) and (12), we get

$$\sum_{i=1}^{k}\lambda_i[f_i(x,y)+x^T z_i-y^T w_i] \geqq \sum_{i=1}^{k}\lambda_i[f_i(u,v)+u^T z_i-v^T w_i].$$

Finally since $x^T z_i \leqq s(x|D_i)$ and $v^T w_i \leqq s(v|E_i)$, we obtain

$$\sum_{i=1}^{k}\lambda_i[f_i(x,y)+s(x|D_i)-y^T w_i] \geqq \sum_{i=1}^{k}\lambda_i[f_i(u,v)+s(v|E_i)-v^T w_i].$$

Which contradict inequality (19).Thus, the result holds. □

Theorem 3.2 (Strong duality). Let $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p})$ be an efficient solution for (WP). Suppose that

(i)

$\nabla_{yy}(\bar{h}^T g)(\bar{x},\bar{y})$ is nonsingular,
(ii) the set $\{\nabla_y f_i(\bar{x},\bar{y})-\bar{w}_i, i=1,\ldots,k\}$ is linearly independent, and
(iii) $\nabla_{yy}(\bar{h}^T g)(\bar{x},\bar{y})\bar{p} \notin \text{span}\{\nabla_y f_1(\bar{x},\bar{y})-E_1,...,\nabla_y f_k(\bar{x},\bar{y})-E_k\}\backslash\{0\}$.

Then $(\bar{x},\bar{y},\bar{z},\bar{q}=0)$ is feasible for $(\text{MWD})_{\bar{\lambda}}$, and the objective function values of (MWP) and $(\text{MWD})_{\bar{\lambda}}$ are equal. Also, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of $(\text{MWP})_{\bar{\lambda}}$ and $(\text{MWD})_{\bar{\lambda}}$, then $(\bar{x},\bar{y},\bar{z},\bar{q}=0)$ is an efficient solution for $(\text{MWD})_{\bar{\lambda}}$.

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ is an efficient solution for (MWP), by the Fritz-John necessary optimality conditions of [12], there exist $\bar{\alpha} \in R^k$, $\bar{\beta} \in R^m$, $\bar{\xi} \in R^k$, $\bar{\delta} \in R$ and $\bar{z}, \bar{\eta} \in R^n$ such that the following conditions are satisfied at $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$:

$$\sum_{i=1}^{k}(\nabla_x f_i(\bar{x}, \bar{y}) + \bar{\delta}_i)\bar{\alpha}_i + \{\sum_{i=1}^{k} \lambda_i(\nabla_{yx} f_i(\bar{x}, \bar{y})) + \nabla_x(\nabla_{yy}(\bar{h}^T g)(\bar{x}, \bar{y})\bar{p})\}^T(\bar{\beta} - \bar{\gamma}\bar{y}) - \bar{\eta} = 0 \tag{13}$$

$$\sum_{i=1}^{k}(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{w}_i)(\bar{\alpha}_i - \bar{\gamma}\bar{\lambda}_i) + \{\sum_{i=1}^{k} \lambda_i(\nabla_{yy} f_i(\bar{x}, \bar{y})) + \nabla_y(\nabla_{yy}(\bar{h}^T g)(\bar{x}, \bar{y})\bar{p})\}^T(\bar{\beta} - \bar{\gamma}\bar{y}) - \bar{\gamma}(\nabla_{yy}(\bar{h}^T g)(\bar{x}, \bar{y})\bar{p}) = 0], \tag{14}$$

$$(\bar{\beta} - \bar{\gamma}\bar{y})^T[\nabla_y f_i(\bar{x}, \bar{y}) - \bar{w}_i] - \bar{\xi} = 0, \tag{15}$$

$$(\bar{\beta} - \bar{\gamma}\bar{y})^T \nabla_h(\nabla_{yy}(\bar{h}^T g)(\bar{x}, \bar{y})\bar{p}) = 0, \tag{16}$$

$$(\bar{\beta} - \bar{\gamma}\bar{y})^T \nabla_{yy}(\bar{h}^T g)(\bar{x}, \bar{y}) = 0, \tag{17}$$

$$\bar{\alpha}_i\bar{y} + (\bar{\beta} - \gamma\bar{y})\bar{\lambda}_i \in N_{E_i}(\bar{w}_i), i = 1, 2, ..., k \tag{18}$$

$$\bar{\delta}_i \in D_i, \bar{\delta}_i^T \bar{x} = s(\bar{x}|D_i), i = 1, 2, ..., k, \tag{19}$$

$$\bar{\lambda}^T \bar{\xi} = 0, \tag{20}$$

$$\bar{\eta}^T \bar{x} = 0, \tag{21}$$

$$\bar{\beta}^T[\sum_{i=1}^{k}\{\ \bar{\lambda}_i(\nabla_y f_i(x, y) - \bar{w}_i)\} + \nabla_{yy}(\bar{h}^T g(x, y))\bar{p}] = 0, \tag{22}$$

$$\bar{\gamma}\bar{y}^T[\sum_{i=1}^{k}\{\ \bar{\lambda}_i(\nabla_y f_i(x, y) - \bar{w}_i)\} + \nabla_{yy}(h^T g(x, y))\bar{p}] = 0, \tag{23}$$

$$(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{\xi}, \bar{\eta}) \geqq 0, \tag{24}$$

$$(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{\xi}, \bar{\eta}) \neq 0. \tag{25}$$

As $\bar{\lambda} > 0$, from (20) we conclude $\bar{\xi} = 0$. By hypothesis (i), equation (17) implies

$$\bar{\beta} = \gamma\bar{y}. \tag{26}$$

Now, we claim that $\bar{\gamma} > 0$. Indeed, if $\bar{\gamma} = 0$, then (26) gives $\beta \bar{=} 0$ and equation (14),hypothesis (ii) gives $\bar{\alpha} = 0$ which along with equations (13),(18) gives $\bar{\eta} = 0$, $\bar{\delta} = 0$, which is a contradiction to $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{\xi}, \bar{\eta}) \neq 0$. Hence

$$\gamma > 0. \tag{27}$$

Also from (27), we have

$$\bar{y} = \frac{\beta}{\gamma} \geqq 0. \tag{28}$$

Now, from (14) and (26), we have

$$\sum_{i=1}^{k} (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{w}_i)(\bar{\alpha}_i - \gamma\bar{\lambda}_i) - \bar{\gamma}(\nabla_{yy}(\bar{h}^T g)(\bar{x}, \bar{y})\bar{p}) = 0. \tag{29}$$

Using the hypothesis (iii), the above relation implies $\bar{\gamma}(\nabla_{yy}(\bar{h}^T g)(\bar{x}, \bar{y})\bar{p}) = 0$, which by hypothesis (i) and (27) yields

$$\bar{p} = 0. \tag{30}$$

Therefore equation (29) gives

$$\sum_{i=1}^{k} (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{w}_i)(\bar{\alpha}_i - \gamma\bar{\lambda}_i) = 0.$$

Since the set $\{\nabla_y f_i - \bar{w}_i, i = 1, \ldots, k\}$ is linearly independent, the above equation implies

$$\bar{\alpha}_i = \gamma\bar{\lambda}_i, i = 1, 2, ..., k. \tag{31}$$

Since $\bar{\lambda}_i > 0$, i=1, 2, ...,k, the above equation implies $\alpha_i > 0, i = 1, 2, ..., k.$
Using (26) in (13), we get
$\sum\limits_{i=1}^{k} (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{\delta}_i)\bar{\alpha}_i = \bar{\eta} \geqq 0.$
Thus $(\bar{x}, \bar{y}, \bar{z} = \bar{\delta}, \bar{q} = 0)$ is feasible for $(\text{MWD})_{\bar{\lambda}}$.
Now from above equation, $\sum\limits_{i=1}^{k} (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{z}_i)\bar{\alpha}_i = \bar{\eta}$, which by (31) gives

$$\sum_{i=1}^{k} \bar{\lambda}_i[\nabla_x f_i(\bar{x}, \bar{y}) + \bar{z}_i] = \frac{\bar{\eta}}{\bar{\gamma}} \geqq 0, \tag{32}$$

and

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i[\nabla_x f_i(\bar{x}, \bar{y}) + \bar{z}_i] = \frac{\bar{x}^T\bar{\eta}}{\bar{\gamma}} = 0. \tag{33}$$

or using (19)

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i[\nabla_x f_i(\bar{x}, \bar{y})] = -\bar{x}^T\bar{z}_i = -s(\bar{x}|D_i). \tag{34}$$

Further, from (18), (26), (27) and (31), we have for $i = 1, 2, ..., k$,
$\bar{\lambda}_i \bar{y} \in N_{E_i}(\bar{w}_i)$ or $\bar{y} \in N_{E_i}(\bar{w}_i)$, using $\bar{\lambda}_i > 0$.
Since E_i is a compact convex set in R^m, $\bar{y}^T \bar{w}_i = s(\bar{y}|E_i),\ i = 1, 2, ..., k$. Hence two objective function values are equal. Now it follows from Theorem 3.1 that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q})$ is an efficient solution for $(\text{MWD})_{\bar{\lambda}}$. □

Theorem 3.3 (Converse duality). Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{h}, \bar{z}, \bar{q})$ be an efficient solution for (MWD). Suppose that

(i) $\nabla_{xx}(\bar{h}^T g)(\bar{u}, \bar{v})$ is nonsingular,
(ii) the set $\{\nabla_x f_i(\bar{u}, \bar{v}) + \bar{z}_i, i = 1, \ldots, k\}$ is linearly independent, and
(iii) $\nabla_{xx}(\bar{h}^T g)(\bar{u}, \bar{v})\bar{q} \notin \text{span}\{\nabla_x f_1(\bar{u}, \bar{v}) + \bar{z}_1, ..., \nabla_x f_k(\bar{u}, \bar{v}) + \bar{z}_k\} \backslash \{0\}$.

Then $(\bar{u}, \bar{v}, \bar{z}, \bar{p} = 0)$ is feasible for $(\text{MWP})_{\bar{\lambda}}$, and the objective function values of $(\text{MWP})_{\bar{\lambda}}$ and (MWD) are equal. Also, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of $(\text{MWP})_{\bar{\lambda}}$ and $(\text{MWD})_{\bar{\lambda}}$, then $(\bar{u}, \bar{v}, \bar{w}, \bar{q})$ is an efficient solution for $(\text{MWP})_{\bar{\lambda}}$.

Proof. Follows on the lines of Theorem 3.2. □

4. Self Duality

A mathematical problem is said to be self dual if it is formally identical with its dual, that is, the dual can be recast in the form of the primal. In general, the problem (MP) is not self dual. It is so if $m = n$, $D_i = E_i,\ i = 1, 2, ..., k$, the vector function f and g are skew-symmetric, i.e., $f(x, y) = -f(y, x)$ and $g(x, y) = -g(y, x)$ as shown below :

By recasting the dual problem (MD) as minimization problem, we have

$$\text{minimize}\{-\{f_1(u, v) + s(v|E_1) - (u^T z_1),$$
$$-f_2(u, v) + s(v|E_2) - (u^T z_2), ..., -f_k(u, v) + s(v|E_k) - (u^T z_k)\}$$

$$\text{Subject to } \sum_{i=1}^{k}\{\lambda_i(\nabla_x f_i(u, v) + z_i)\} + \nabla_{xx}(h^T g(u, v))q \geqq 0,$$
$$u^T \sum_{i=1}^{k}\{\lambda_i(\nabla_x f_i(u, v) + z_i)\} + \nabla_{xx}(h^T g(u, v))q \leqq 0,$$

$$\lambda > 0, v > 0, z_i \in R^n,$$

Since f is skew symmetric, $\nabla_x f_i(u,v) = -\nabla_y f_i(v,u)$ and $\nabla_{xx} g_i(u,v) = -\nabla_{yy} g_i(v,u)$ for $i = 1, 2, ..., k$, and so the above problem becomes

$$\text{minimize}\{f_1(v,u) + s(v|E_1) - u^T w_1, ..., f_k(v,u) + s(v|E_k) - u^T w_k$$

$$\text{subject to} \qquad -\sum_{i=1}^{k} \lambda_i(\nabla_y f_i(v,u) - z_i + \nabla_{yy} h^T g(v,u) q) \geqq 0,$$

$$u^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(v,u) - w_i + \nabla_{yy} f_i(v,u) q_i) \geqq 0,$$

$$\lambda > 0, x > 0, w_i \in R^m.$$

which is the primal problem (MP). Therefore (MP) is self dual. Thus, if $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ is feasible for (MP), then $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{z}, \bar{p})$ is feasible for (MD) and conversely.

Conclusion

A pair of symmetric dual programs has been formulated by considering the optimization under the assumptions of generalized pseudoinvexity. It may be noted that the symmetric duality between (MWP) and (MWD) can be utilized to establish mixed symmetric duality in integer over cone and other related programming problems.

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