# $L^p$ Inequalities For The Derivative Of Polynomials

**Roshan Lal**
*Assistant Professor, Department of Mathematics,*
*Government P.G. College Dakpatthar,*
*Dehradun, Uttarakhand, India*
*rlkeshtwal@gmail.com*

**Abstract**

In this paper, we obtain Zygmund type of integral inequality in the reverse direction for the class of polynomials having all its zeros in the disk $|z| \leq k$, $k \leq 1$, with s-fold zeros at the origin. Our result generalizes the earlier known results.

**Keywords**: Polynomials, Minkowski's inequality, zeros, inversive polynomials.

## 1 INTRODUCTION AND STATEMENT OF RESULTS

Let $p(z) = \sum\limits_{j=0}^{n} a_j z^j$ be a polynomial of degree n and $p^{'}(z)$ be its derivative, then for $r > 0$,

$$\left\{ \int\limits_0^{2\pi} |p^{'}(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq n \left\{ \int\limits_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \tag{1.1}$$

and for every $R \geq 1$ and for $r > 0$,

$$\left\{ \int_0^{2\pi} |p^{'}(Re^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq R^n \left\{ \int\limits_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}. \tag{1.2}$$

Inequalities (1.1) and (1.2) are sharp and equality holds for polynomial $p(z) = \alpha z^n, \alpha \neq 0$.

Inequality (1.1) for $r \geq 1$ is due to Zygmund [17], who proved it for all trigonometric polynomials of degree n and not only for those which are of the form $p(e^{i\theta})$. Arestov [1] proved that (1.1) remains true for $0 < r < 1$ as well. As far as inequality (1.2) is concerned, it is difficult to trace its origin. We can deduce it from a well known result of Hardy [8], according to which for every function f (z), analytic in $|z| < t_0$ and for every $r > 0$,

$$\left\{ \int\limits_0^{2\pi} |f(te^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}$$

is non-decreasing function of $t$ for $0 < t < t_0$. If $p(z)$ is a polynomial of degree n, then $f(z) = z^n \overline{p(\frac{1}{\bar{z}})}$ is again a polynomial, known as inversive polynomial, is an entire function and by Hardy's result for $r > 0$, we have

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(te^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}$$

for $t = \frac{1}{R} < 1$. This is equivalent to (1.2).

If we let $r \to \infty$ in (1.1) and (1.2) and make use of well-known fact from analysis (see [15], [16] ) that

$$\lim_{r\to\infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} = \max_{|z|=1} |p(z)|, \tag{1.3}$$

we get the following inequalities

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)| \tag{1.4}$$

and

$$\max_{|z|=R\geq 1} |p(z)| \leq R^n \max_{|z|=1} |p(z)|. \tag{1.5}$$

Inequality (1.4) is a classical result due to Bernstein [2] where as (1.5) is a well - known inequality deduced from Maximum Modulus Principle (see [10], [14]).

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequality (1.1) and (1.2) can be sharpened. In fact, in this case the following results are also known.

**Theorem A.** If $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for each $r > 0$,

$$\left\{ \int_0^{2\pi} |p'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq nC_r \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \tag{1.6}$$

where

$$C_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\alpha})|^r d\alpha \right\}^{-\frac{1}{r}}.$$

**Theorem B.** If $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$ , then for every $R > 1$ and $r > 0$,

$$\left\{\int_0^{2\pi} |p(Re^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq E_r \left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}, \tag{1.7}$$

where

$$E_r = \frac{\left\{\int_0^{2\pi} |1 + R^n e^{i\alpha}|^r d\alpha\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi} |1 + e^{i\alpha}|^r d\alpha\right\}^{\frac{1}{r}}}$$

In both the inequalities (1.6) and (1.7), equality occurs for $p(z) = \alpha z^n + \beta$, $|\alpha| = |\beta|$.

For $r \geq 1$, Theorem A was proved by de-Bruijn [4] and later independently proved by Rahman [11]. For the special case $r = 2$ , it was proved by Lax [9]. On the other hand, Theorem B was proved by Boas and Rahman [3] for $r \geq 1$. Rahman and Schmeisser [12] showed that both the inequalities (1.6) and (1.7) remains valid for $0 < r < 1$ as well.

For the class of polynomials having no zeros in the disc $|z| < k$, $k \geq 1$, Govil and Rahman [7] proved the next inequality (1.8) for $r \geq 1$, later it was shown by Gardner and Weems [6], and independently by Rather [13] that inequality (1.8) also holds for $0 < r < 1$.

**Theorem C.** If $p(z)$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $r > 0$,

$$\left\{\int_0^{2\pi} |p'(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq nF_r \left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}, \tag{1.8}$$

where

$$F_r = \left\{\frac{1}{2\pi}\int_0^{2\pi} |k + e^{i\alpha})|^r d\alpha\right\}^{-\frac{1}{r}}.$$

Next two results are due to Dewan, Bhat and Pukhta [5].

**Theorem D.** If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for each $r \geq 1$,

$$\left\{\int_0^{2\pi} |p'(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq nS_r \left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}, \tag{1.9}$$

where

$$S_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |S_c + e^{i\alpha})|^r d\alpha \right\}^{-\frac{1}{r}},$$

and

$$S_c = k^2 \left\{ \frac{\frac{1}{n}|\frac{a_1}{a_0}| + 1}{\frac{1}{n}|\frac{a_1}{a_0}|k^2 + 1} \right\}$$

In this paper, for the class of polynomials having all its zeros in $|z| \leq k,\ k \leq 1$, with s-fold zeros at the origin, we obtain Zygmund [17] type of integral inequality but in the reverse direction. More precisely, we prove

**Theorem 1.** If $p(z) = z^s \sum_{j=0}^{n-s} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \leq 1$, with a zero of order $s$ at $z = 0$. Then for $\beta$ with $|\beta| < k^{n-s}$ and $r \geq 1$

(1.10)
$$\left\{ \int_0^{2\pi} |p'(e^{i\theta}) + \frac{sm}{k^n}\overline{\beta} e^{i(s-1)\theta}|^r d\theta \right\}^{\frac{1}{r}} \geq \{n - (n-s)C_r^{(k)}\} \left\{ \int_0^{2\pi} |p(e^{i\theta}) + \frac{m}{k^n}\overline{\beta} e^{is\theta}|^r d\theta \right\}^{\frac{1}{r}},$$

where

$$m = \min_{|z|=k} |p(z)|,$$

$$C_r^{(k)} = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |S + e^{i\alpha})|^r d\alpha \right\}^{-\frac{1}{r}},$$

and

$$S = \frac{(\frac{1}{n-s})|\frac{a_{n-s-1}}{a_{n-s}}| + 1}{k^2 + (\frac{1}{n-s})|\frac{a_{n-s-1}}{a_{n-s}}|}$$

By taking $k = 1$ and $\beta = 0$ inTheorem 1, we obtain

**Corollary 2.** If $p(z)$ is a polynomial of degree n having all its zeros $|z| \leq 1$, with a zero of order $s$ at $z = 0$. Then for $r \geq 1$

$$\left\{ \int_0^{2\pi} |p'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \geq \{n - (n-s)C_r^{(1)}\} \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \tag{1.11}$$

where

$$C_r^{(1)} = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\alpha})|^r d\alpha \right\}^{-\frac{1}{r}}.$$

By letting $r \to \infty$ in Theorem 1, we obtain

**Corollary 3.** Let $p(z) = z^s \sum_{j=0}^{n-s} a_j z^j$ be a polynomial of degree n having all its zeros in $|z| \le k, \ k \le 1$, with a zero of order $s$ at $z = 0$. Then for $\beta$ with $|\beta| < k^{n-s}$,

$$\max_{|z|=1} |p^{'}(z) + \frac{sm}{k^n}\overline{\beta} z^{s-1}| \ge \left( \frac{S + nS}{1 + S} \right) \max_{|z|=1} |p(z) + \frac{m}{k^n}\overline{\beta} z^s|, \tag{1.12}$$

where m and S are as defined in Theorem 1.

By choosing argument of $\beta$ suitably and $|\beta| \to k^{n-s}$ in Corollary 3, we obtain the following result.

**Corollary 4.** Let $p(z) = z^s \sum_{j=0}^{n-s} a_j z^j$ be a polynomial of degree n having all its zeros in $|z| \le k, \ k \le 1$, with a zero of order $s$ at $z = 0$. Then

$$\max_{|z|=1} |p^{'}(z)| \ge \left( \frac{S + nS}{1 + S} \right) \max_{|z|=1} |p(z)| + \frac{(n-s)S}{(1+S)} \frac{m}{k^s}, \tag{1.13}$$

where m and S are as defined in Theorem 1.

## 2 PROOF OF THE THEOREM.

**Proof of the theorem 1.** Let $p(z) = z^s \sum_{j=0}^{n-s} a_j z^j = z^s \phi(z)$, (say)

where $\phi(z)$ is a polynomial of degree $n - s$, with the property that $\phi(0) \neq 0$.

Then

$$\begin{aligned} q(z) &= z^n \overline{p(\frac{1}{\overline{z}})} \\ &= z^{n-s} \overline{\phi(\frac{1}{\overline{z}})} \end{aligned}$$

is also a polynomial of degree $n - s$ and have no zeros in $|z| < \frac{1}{k}, \ \frac{1}{k} \ge 1$.

Now, if

$$\begin{aligned} m_0 &= \min_{|z|=\frac{1}{k}} |q(z)| = \min_{|z|=\frac{1}{k}} |z^n \overline{p(\frac{1}{\overline{z}})}| \\ &= \frac{1}{k^n} \min_{|z|=k} |p(z)| = \frac{m}{k^n}, \end{aligned}$$

then, by Rouche's theorem, the polynomial

$$q(z) + m_0 \beta z^{n-s} \quad , \qquad |\beta| < k^{n-s}$$

of degree $n-s$, will also have no zeros in $|z| < \frac{1}{k}$, $\frac{1}{k} \geq 1$.
Hence, by Theorem D, we have for $r \geq 1$ and $|\beta| < k^{n-s}$,

$$\left\{ \int_0^{2\pi} |q^{'}(e^{i\theta}) + \frac{m}{k^n} \beta e^{i(n-s-1)\theta}(n-s)|^r d\theta \right\}^{\frac{1}{r}} \leq (n-s) C_r^{(k)} \left\{ \int_0^{2\pi} |q(e^{i\theta}) + \frac{m}{k^n} \beta e^{i(n-s)\theta}|^r d\theta \right\}^{\frac{1}{r}},$$

which implies

(2.1)
$$\left\{ \int_0^{2\pi} |np(e^{i\theta}) - e^{i\theta} p^{'}(e^{i\theta}) + \overline{\beta} \frac{m}{k^n}(n-s) e^{is\theta}|^r d\theta \right\}^{\frac{1}{r}} \leq (n-s) C_r^{(k)} \left\{ \int_0^{2\pi} |p(e^{i\theta}) + \frac{m}{k^n} \overline{\beta} e^{is\theta}|^r d\theta \right\}^{\frac{1}{r}}$$

Now by Minkowski's inequality, we have for $r \geq 1$ and $|\beta| < k^{n-s}$,

$$\begin{aligned} n \left\{ \int_0^{2\pi} |p(e^{i\theta}) + \overline{\beta} \frac{m}{k^n} e^{is\theta}|^r d\theta \right\}^{\frac{1}{r}} &\leq \left\{ \int_0^{2\pi} |np(e^{i\theta}) + \frac{m}{k^n} \overline{\beta}(n-s) e^{is\theta} - e^{i\theta} p^{'}(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \\ &\quad + \left\{ \int_0^{2\pi} |e^{i\theta} p^{'}(e^{i\theta}) + \frac{sm}{k^n} \overline{\beta} e^{is\theta}|^r d\theta \right\}^{\frac{1}{r}}, \end{aligned}$$

which implies by using inequality (2.1)

$$\begin{aligned} n \left\{ \int_0^{2\pi} |p(e^{i\theta}) + \overline{\beta} \frac{m}{k^n} e^{is\theta}|^r d\theta \right\}^{\frac{1}{r}} &\leq (n-s) C_r^{(k)} \left\{ \int_0^{2\pi} |p(e^{i\theta}) + \frac{m}{k^n} \overline{\beta} e^{is\theta}|^r d\theta \right\}^{\frac{1}{r}} \\ &\quad + \left\{ \int_0^{2\pi} |p^{'}(e^{i\theta}) + \frac{sm}{k^n} \overline{\beta} e^{i(s-1)\theta}|^r d\theta \right\}^{\frac{1}{r}}, \end{aligned}$$

from which Theorem 1 follows.

## References


[1] V.V. Arestov, On inequalities for trigonometric polynomials and their derivative, IZV. Akad. Nauk. SSSR. Ser. Math., 45(1981), 3-22.

[2] S. Bernstein, Lecons Sur Les Proprietes extrremales et la meillure approximation des functions analytiques d'une functions reele , Paris, 1926.

[3] R.P. Boas Jr. and Q.I. Rahman, $L^p$ inequalities for polynomials and entire functions, Arch. Rational Mech. Anal., 11(1962), 34-39.

[4] N.G. de-Bruijn, Inequalties concerning polynomial the complex domain, Nederl. Akad. Wetench. Proc. Ser.A, 50(1947) 1265-1272; Indag. Math., 9(1947) 591-598.

[5] K.K. Dewan, A. Bhat and M.S. Pukhta, Inequalities concerning the $L^p$ norm of a polynomial, J. Math. Anal. Appl., 224(1998),14-21.

[6] R.B. Gardner and A. Weems, A Bernstein-type inequality for a certain class of polynomials, J. Math. Anal. Appl., 219(1998),472-478.

[7] N.K. Govil and Q.I. Rahman, Functions of exponential type not vanishing in a half-plane and related polynomials, Trans. Amer. Math. Soc., 137(1969), 501-517.

[8] G.H. Hardy, The mean value of the modulus of an analytic function, Proc. London Math. Soc., 14(1915), 269-277.

[9] P.D.Lax, Proof of a conjecture of P.Erdö s on the derivative of a polynomial, Bull. Amer. Math. Soc., 50(1944) 509-513.

[10] G.Pōlya and G.Szegö , Problems and Theorems in Analysis vol-1, Springler-Verlag, Berlin, 1972.

[11] Q.I. Rahman, Functions of exponential type, Trans. Amer. Math. Soc., 115 (1992), 337-343.

[12] Q.I. Rahman and G. Schmeisser, $L^p$ inequalities for polynomials, J. Approx. Theory, 53(1988), 26-32.

[13] N.A. Rather, Extremal properties and location of the zeros of polynomials, Ph.D. Thesis, University of Kashmir, 1998.

[14] M.Riesz, Ü ber enin Satz der Herrn Serge Bernstein, Acta Math., 40(1918) 337-347.

[15] W.Rudin, Real and Complex Analysis, Tata McGraw-Hill publishing Company, (Reprinted in India),1977.

[16] A.E. Taylor, Introduction to functional analysis, John Wiley and Sons, Inc. New York, 1958.

[17] Zygmund, A remark on conjugate series, Proc. London Math. Soc., 34(1932), 392-400.