The Bailey Pairs and The Partial Theta Function

Jitendra Singh
Department of Mathematics
Bansal Institute of Engineering and Technology
Lucknow, India

Abstract

In the "Lost" note-book a number of identities and expansion formulae for partial sums of theta functions are mentioned without proof. It is shown that many of the partial theta function identities in Ramanujan's "Lost" note-book can be generalized to infinite family of such identities. In particular in this paper new Bailey pairs have been derived and by using these Bailey pairs, the Bailey chain and a generalization of the Jacobi triple product identity, the theorems on partial theta function have been proved.

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1 Introduction

In 1976 Andrews rediscovered Ramanujan's Lost Note-book which was hidden in a box containing manuscript of well over a hundred pages of hardly decipherable but very beautiful identities from the late G. N. Watson's estate. The first formula given by Andrews (1979) in his first paper on Lost Note-book is the following q - series transformation

$$\sum_{n=0}^{\infty} \frac{q^n}{(1-a)\prod_{j=1}^{n}(1-aq^j)(1-q^j/a)} = \sum_{n=0}^{\infty}(-1)^n a^{3n} q^{n(3n+1)/2}\left(1-a^2q^{2n+1}\right) + \frac{\sum_{n=0}^{\infty}(-1)^n a^{2n+1} q^{n(n+1)/2}}{\prod_{j=0}^{\infty}(1-aq^{j-1})(1-q^j/a)} \tag{1.1}$$

Characteristic of the above identity is that it contains a partial theta product $(1-a)\prod_{j=1}^{n}\left(1-aq^j\right)\left(1-q^j/a\right)$ and a partial theta sum $\sum_{n=0}^{\infty}\left(-a^2\right)^n q^{n(n+1)/2}$.

Partial theta functions play a prominent role in Ramanujan's Lost Note-book. True to himself Ramanujan gave no proof of any partial theta function identity , making it virtually impossible to determine how he discovered them. A study of these partial sums, identities and expansions has been made by Andrews. Andrews observed that some of these partial theta function identities have interesting Number-Theoritic interpretations and Warnaar (2003) generalized to infinite family of partial theta function identities.

Recently Shukla and Singh (2010) found many partial theta function identities and discussed their limiting cases. Shukla and Singh (2013) also derived some new partial theta function identies by using Bailey pairs.

In this paper we have found some new Bailey pairs, and by using these pairs and a generalization of the Jacobi's triple product identity, partial theta function identies have been derived.

In section 3, the theorems on partial theta function have been given. In section 4, we have derived new Bailey pairs. In section 5, proofs of theorems of (3.1), (3.2), (3.3) have been given whereas in section 6 proof of theorem (3.4) and its particular cases have been given.

2 Notations

The following q - notations and some standard results have been used.

For $\left|q^k\right| < 1$,

$$\begin{aligned}
\left(a;q^k\right)_n &= \prod_{j=0}^{n-1}\left(1-aq^{kj}\right), \qquad n \geq 1 \\
\left(a;q^k\right)_0 &= 1, \\
\left(a;q^k\right)_\infty &= \prod_{j=0}^{\infty}\left(1-aq^{kj}\right), \\
(a)_n\,(a;q)_n &= (1-a)(1-aq)\ldots\ldots\ldots\ldots\left(1-aq^{n-1}\right) \\
\left(a_1,a_2,\ldots\ldots\ldots a_m;q^k\right)_n &= \left(a_1;q^k\right)_n\left(a_2;q^k\right)_n\ldots\ldots\ldots\ldots\left(a_m;q^k\right)_n
\end{aligned}$$

A generalised basic hypergeometric series with base q is defined as

$${}_r\Phi_{r-1}\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,b_2,\ldots,b_{r-1}\end{matrix};q,z\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_{r-1};q)_n}z^n, |z|<1$$

Complete theta products and sums are connected by the famous Jacobi's triple product identity

$$\sum_{n=-\infty}^{\infty}(-1)^n a^n q^{n(n-1)/2} = \prod_{j=1}^{\infty}\left(1-aq^{n-1}\right)(1-q^n/a)(1-q^n) = (a,q/a,q;q)_\infty$$

3 Partial Theta Function Identity (Bailey Chain Identities)

The following theorems have been proved in section 5 and section 6 by using the generalization of the Jacobi triple product identity and new Bailey pairs:

3.1 Theorem (3.1):

For k a positive integer, $N_j = n_j + n_{j+1} + ... + n_{k-1}$

$$\sum_{n=0}^{\infty} \frac{\left(q^3;q\right)_{2n} q^n}{(a;q)_{n+1}\,(q/a;q)_{n+2}} \sum_{n_1,n_2,...,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+...+N_{k-1}^2+3(N_1+N_2+...+N_{k-1})}}{(q;q)_{n-N_1}\,(q;q)_{n_1}\,(q;q)_{n_2}\,...\,(q;q)_{n_{k-1}}\,(q^3;q)_{n_{k-1}}}$$
$$= \frac{(-1)^2a^2}{q^{1+2}} \sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}}(\frac{a}{q^2})^n q^{kn^2+(3k-1)n}$$
$$+\frac{1}{(q,a,q/a;q)_{\infty}} \sum_{r=1}^{\infty}(-1)^{r+1}(\frac{a}{q^2})^r q^{\binom{r}{2}} \sum_{n=0}^{\infty} q^{kn^2+(3k-r)n}(1-q^{r(2n+3)})$$

3.2 Theorem (3.2):

For k a positive integer, $N_j = n_j + n_{j+1} + ... + n_{k-1}$

$$\sum_{n=0}^{\infty} \frac{\left(q^4;q\right)_{2n} q^n}{(a;q)_{n+1}\,(q/a;q)_{n+3}} \sum_{n_1,n_2,...,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+...+N_{k-1}^2+4(N_1+N_2+...+N_{k-1})}}{(q;q)_{n-N_1}\,(q;q)_{n_1}\,(q;q)_{n_2}\,...\,(q;q)_{n_{k-1}}\,(q^4;q)_{n_{k-1}}}$$
$$= \frac{(-1)^3a^3}{q^{1+2+3}} \sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}}(\frac{a}{q^3})^n q^{kn^2+(4k-1)n}$$
$$+\frac{1}{(q,a,q/a;q)_{\infty}} \sum_{r=1}^{\infty}(-1)^{r+1}(\frac{a}{q^3})^r q^{\binom{r}{2}} \sum_{n=0}^{\infty} q^{kn^2+(4k-r)n}(1-q^{r(2n+4)})$$

3.3 Theorem (3.3):

For k a positive integer, $N_j = n_j + n_{j+1} + ... + n_{k-1}$

$$\sum_{n=0}^{\infty} \frac{\left(q^5;q\right)_{2n} q^n}{(a;q)_{n+1}\,(q/a;q)_{n+4}} \sum_{n_1,n_2,...,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+...+N_{k-1}^2+5(N_1+N_2+...+N_{k-1})}}{(q;q)_{n-N_1}\,(q;q)_{n_1}\,(q;q)_{n_2}\,...\,(q;q)_{n_{k-1}}\,(q^5;q)_{n_{k-1}}}$$
$$= \frac{(-1)^4a^4}{q^{1+2+3+4}} \sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}}(\frac{a}{q^4})^n q^{kn^2+(5k-1)n}$$
$$+\frac{1}{(q,a,q/a;q)_{\infty}} \sum_{r=1}^{\infty}(-1)^{r+1}(\frac{a}{q^4})^r q^{\binom{r}{2}} \sum_{n=0}^{\infty} q^{kn^2+(5k-r)n}(1-q^{r(2n+5)})$$

3.4 Theorem (3.4):

For k, m a positive integer, $N_j = n_j + n_{j+1} + ... + n_{k-1}$

$$\sum_{n=0}^{\infty} \frac{(q^m;q)_{2n}\, q^n}{(a;q)_{n+1}\,(q/a;q)_{n+m-1}} \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\ldots+N_{k-1}^2+m(N_1+N_2+\ldots+N_{k-1})}}{(q;q)_{n-N_1}\,(q;q)_{n_1}\,(q;q)_{n_2}\ldots(q;q)_{n_{k-1}}\,(q^m;q)_{n_{k-1}}}$$

$$= \frac{(-1)^{m-1}a^{m-1}}{q^{1+2+\ldots+(m-1)}} \sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}}\left(\frac{a}{q^{m-1}}\right)^n q^{kn^2+(mk-1)n}$$

$$+\frac{1}{(q,a,q/a;q)_\infty} \sum_{r=1}^{\infty}(-1)^{r+1}\left(\frac{a}{q^{m-1}}\right)^r q^{\binom{r}{2}} \sum_{n=0}^{\infty} q^{kn^2+(mk-r)n}(1-q^{r(2n+m)})$$

4 The Bailey Pairs

W. N. Bailey (1949) defined the Bailey pair. Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ and $\beta = \{\beta_n\}_{n=0}^{\infty}$. Then the pair of sequences (α, β) is called a Bailey pair relative to a if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}\,(aq;q)_{n+r}} \tag{4.1}$$

A special case of Bailey's lemma [Andrews (1984) and Paule (1985)] which states that if (α, β) is a pair relative to a, then so the pair (α', β') given by

$$\alpha'_n = a^n q^{n^2} \alpha_n \text{ and } \beta'_n = \sum_{r=0}^{n} \frac{a^r q^{r^2} \beta_r}{(q;q)_{n-r}} \tag{4.2}$$

Iterating this leads to what is called the Bailey Chain. All the theorems of section 3, will arise from Bailey chain.

Now we will derive the following Bailey pairs:

$$\alpha_n = \frac{q^{n^2+2n}(1-q^{2n+3})}{1-q^3}, \beta_n = \frac{1}{(q)_n\,(q^3)_n} \text{ relative to } q^3 \tag{4.3}$$

$$\alpha_n = \frac{q^{n^2+3n}(1-q^{2n+4})}{1-q^4}, \beta_n = \frac{1}{(q)_n\,(q^4)_n} \text{ relative to } q^4 \tag{4.4}$$

$$\alpha_n = \frac{q^{n^2+4n}(1-q^{2n+5})}{1-q^5}, \beta_n = \frac{1}{(q)_n\,(q^5)_n} \text{ relative to } q^5 \tag{4.5}$$

$$\alpha_n = \frac{q^{n^2+(m-1)n}(1-q^{2n+m})}{1-q^m}, \beta_n = \frac{1}{(q)_n\,(q^m)_n} \text{ relative to } q^m \tag{4.6}$$

Proof of the Bailey pairs:
To prove the Bailey pairs (4) to (7), we have used the following identity of Slater (1951)

$$\sum_{r=0}^{n} \frac{\left(1-aq^{2r}\right)(-1)^r\, q^{\binom{r+1}{2}}\,(a)_r\,(c)_r\,(d)_r\,a^r}{(a)_{n+r+1}\,(q)_{n-r}\,(q)_r\,(aq/c)_r\,(aq/d)_r\,c^r d^r} = \frac{(aq/cd)_n}{(q)_n\,(aq/c)_n\,(aq/d)_n} \tag{4.7}$$

(i) Taking $a=q^3$, $c=q$, $d\to\infty$ in above identity, we have

$$\sum_{r=0}^{n} \frac{\left(1-q^{2r+3}\right)q^{r^2+2r}}{(q^3)_{n+r+1}\,(q)_{n-r}} = \frac{1}{(q)_n\,(q^3)_n} \tag{4.8}$$

Comparing this with definition (2), the Bailey pair (4) relative to q^3 is obtained.
(ii) Next taking $a=q^4$, $c=q$, $d\to\infty$ in identity (8), we obtain

$$\sum_{r=0}^{n} \frac{\left(1-q^{2r+4}\right)q^{r^2+3r}}{(q^4)_{n+r+1}\,(q)_{n-r}} = \frac{1}{(q)_n\,(q^4)_n} \tag{4.9}$$

On comparing this with definition (2), the Bailey pair (5) relative to q^4 is obtained.
(iii) Again taking $a=q^5$,$c=q$, $d\to\infty$ in identity (8), we get

$$\sum_{r=0}^{n} \frac{\left(1-q^{2r+5}\right)q^{r^2+4r}}{(q^5)_{n+r+1}\,(q)_{n-r}} = \frac{1}{(q)_n\,(q^5)_n} \tag{4.10}$$

Comparing this with definition (2), the Bailey pair (6) relative to q^5 is obtained.
(iv) Further taking $a=q^m$, $c=q$, $d\to\infty$ in identity (8), we obtain

$$\sum_{r=0}^{n} \frac{\left(1-q^{2r+m}\right)q^{r^2+(m-1)r}}{(q^m)_{n+r+1}\,(q)_{n-r}} = \frac{1}{(q)_n\,(q^m)_n} \tag{4.11}$$

Now we compare the above identity with definition (2), and get the Bailey pair (7) relative to q^m.

5 Proof of theorems (3.1), (3.2), and (3.3)

Warnaar (2003) defined the following generalization of Jacobi triple product identity and proved this identity in (2007):

$$1+\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\left(a^n+b^n\right)=(q,a,b;q)_{\infty}\sum_{n=0}^{\infty}\frac{(ab/q;q)_{2n}}{(q,a,b,ab;q)_n} \tag{5.1}$$

Taking $b=q/a$ one recovers the Jacobi triple product identity. By applying some calculations on this identity, Warnaar derived the following very general identity for partial theta functions from which we will derive the theorems of section 3

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(ab;q)_{2n}\,q^n}{(a;q)_{n+1}\,(b;q)_n}\sum_{r=0}^{n}\frac{(-1)^r q^{\binom{r}{2}} f_r\left(1-abq^{2r}\right)}{(q;q)_{n-r}\,(ab;q)_{n+r+1}}-\frac{(q/b;q)_{\infty}}{(a;q)_{\infty}}\sum_{r=0}^{\infty}(q/b)^r f_r\\ =\;&\frac{1}{(q,a,b;q)_{\infty}}\sum_{n=1}^{\infty}\left\{a^n\sum_{r=0}^{\infty}f_r+(q/b)^n\sum_{r=-\infty}^{-1}f_{-r-1}\right\}(-1)^{n+r}q^{\binom{n+r+1}{2}}\end{aligned} \tag{5.2}$$

In order to turn this into a Ramanujan-type partial theta function identity, we will carry out explicitly the sum

$$\beta_n=\sum_{r=0}^{n}\frac{(-1)^r q^{\binom{r}{2}} f_r\left(1-abq^{2r}\right)}{(q;q)_{n-r}\,(ab;q)_{n+r+1}} \tag{5.3}$$

Proof of theorem (3.1): Taking $b=q^3/a$ in identity (14) and dividing both sides by $(1-q/a)\left(1-q^2/a\right)$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\left(q^3;q\right)_{2n}q^n}{(a;q)_{n+1}\,(q/a;q)_{n+2}}\sum_{r=0}^{n}\frac{(-1)^r q^{\binom{r}{2}} f_r\left(1-q^{2r+3}\right)}{(q;q)_{n-r}\,(q^3;q)_{n+r+1}}-\frac{(-1)^2a^2}{q^{1+2}}\sum_{r=0}^{\infty}(a/q^2)^r f_r\\ =\;&\frac{1}{(q,a,q/a;q)_{\infty}}\sum_{n=1}^{\infty}\left\{\sum_{r=0}^{\infty}f_r+q^{-2n}\sum_{r=-\infty}^{-1}f_{-r-1}\right\}(-1)^{n+r}a^n q^{\binom{n+r+1}{2}}\end{aligned} \tag{5.4}$$

Taking

$$\sum_{r=0}^{n}\frac{(-1)^r q^{\binom{r}{2}} f_r\left(1-q^{2r+3}\right)}{(q;q)_{n-r}\,(q^3;q)_{n+r+1}}=\beta_n \tag{5.5}$$

Comparing the above sum with definition (2) and identifying

$$\alpha_n = (-1)^n q^{\binom{n}{2}} f_n (1-q^{2n+3})/(1-q^3) \text{ relative to } q^3 \tag{5.6}$$

Substituting the value of the sum (17) and f_r from (18) in the identity (16), we get the following result:

For (α, β) a Bailey pair relative to q^3 there holds

$$\sum_{n=0}^{\infty} \frac{\beta_n \left(q^3;q\right)_{2n} q^n}{(a;q)_{n+1} (q/a;q)_{n+2}} - \frac{(-1)^2 a^2 (1-q^3)}{q^{1+2}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} \left(a/q^2\right)^n \alpha_n}{(1-q^{2n+3})}$$
$$= \frac{(1-q^3)}{(q,a,q/a;q)_\infty} \sum_{r=1}^{\infty} (-1)^{r+1} \left(a/q^2\right)^r q^{\binom{r}{2}} \sum_{n=0}^{\infty} \alpha_n q^{(1-r)n} \frac{1-q^{r(2n+3)}}{1-q^{2n+3}} \tag{5.7}$$

provided all sums converge.

In above identity, the Bailey pair is relative to q^3, so taking $a = q^3$ in the Bailey chain (3), we obtain

$$\alpha'_n = q^{n^2+3n} \alpha_n \text{ and } \beta'_n = \sum_{r=0}^{n} \frac{q^{r^2+3r} \beta_r}{(q;q)_{n-r}} \tag{5.8}$$

The Bailey pair relative to q^3 is the Bailey pair (4), so iterating the Bailey pair (4) along the Bailey chain (20), we get

$$\alpha_n^{(k)} = \frac{q^{kn^2+(3k-1)n}(1-q^{2n+3})}{1-q^3}, \tag{5.9}$$

$$\beta_n^{(k)} = \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\ldots+N_{k-1}^2+3(N_1+N_2+\ldots+N_{k-1})}}{(q;q)_{n-N_1} (q;q)_{n_1} (q;q)_{n_2} \ldots (q;q)_{n_{k-1}} (q^3;q)_{n_{k-1}}} \tag{5.10}$$

For k a positive integer and $\left(\alpha^{(1)}, \beta^{(1)}\right) = (\alpha, \beta)$, so $\left(\alpha_n^{(k)}, \beta_n^{(k)}\right) = (\alpha_n, \beta_n)$. Substituting $\alpha_n = \alpha_n^{(k)}$and $\beta_n = \beta_n^{(k)}$ in the identity (19), the theorem (3.1) is obtained.

Proof of theorem (3.2): Again taking $b = q^4/a$ in identity (14) and dividing both sides by $(1-q/a)\left(1-q^2/a\right)\left(1-q^3/a\right)$, this yields

$$\sum_{n=0}^{\infty} \frac{\left(q^4;q\right)_{2n} q^n}{(a;q)_{n+1} (q/a;q)_{n+3}} \sum_{r=0}^{n} \frac{(-1)^r q^{\binom{r}{2}} f_r \left(1-q^{2r+4}\right)}{(q;q)_{n-r} (q^4;q)_{n+r+1}} - \frac{(-1)^3 a^3}{q^{1+2+3}} \sum_{r=0}^{\infty} (a/q^3)^r f_r$$
$$= \frac{1}{(q,a,q/a;q)_\infty} \sum_{n=1}^{\infty} \left\{ \sum_{r=0}^{\infty} f_r + q^{-3n} \sum_{r=-\infty}^{-1} f_{-r-1} \right\} (-1)^{n+r} a^n q^{\binom{n+r+1}{2}} \tag{5.11}$$

Let

$$\sum_{r=0}^{n} \frac{(-1)^r q^{\binom{r}{2}} f_r \left(1-q^{2r+4}\right)}{(q;q)_{n-r} \left(q^4;q\right)_{n+r+1}} = \beta_n \tag{5.12}$$

Comparing the above sum with definition (2) and identifying

$$\alpha_n = (-1)^n q^{\binom{n}{2}} f_n (1-q^{2n+4})/(1-q^4) \text{ relative to } q^4 \tag{5.13}$$

Substituting the value of the sum (24) and f_r from (25) in the identity (23), the following result is obtained:

For (α, β) a Bailey pair relative to q^4 there holds

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{\beta_n \left(q^4;q\right)_{2n} q^n}{(a;q)_{n+1} \left(q/a;q\right)_{n+3}} - \frac{(-1)^3 a^3 (1-q^4)}{q^{1+2+3}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} \left(a/q^3\right)^n \alpha_n}{(1-q^{2n+4})} \\ = \;& \frac{(1-q^4)}{(q,a,q/a;q)_\infty} \sum_{r=1}^{\infty} (-1)^{r+1} \left(a/q^3\right)^r q^{\binom{r}{2}} \sum_{n=0}^{\infty} \alpha_n q^{(1-r)n} \frac{1-q^{r(2n+4)}}{1-q^{2n+4}} \end{aligned} \tag{5.14}$$

provided all sums converge.

In above identity the Bailey pair is relative to q^4, so taking $a = q^4$ in the Bailey chain (3), we get

$$\alpha'_n = q^{n^2+4n} \alpha_n \text{ and } \beta'_n = \sum_{r=0}^{n} \frac{q^{r^2+4r} \beta_r}{(q;q)_{n-r}} \tag{5.15}$$

The Bailey pair relative to q^4 is the Bailey pair (5), so iterating the Bailey pair (5) along the Bailey chain (27), we get

$$\alpha_n^{(k)} = \frac{q^{kn^2+(4k-1)n}(1-q^{2n+4})}{1-q^4}, \tag{5.16}$$

$$\beta_n^{(k)} = \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\ldots+N_{k-1}^2+4(N_1+N_2+\ldots+N_{k-1})}}{(q;q)_{n-N_1} (q;q)_{n_1} (q;q)_{n_2} \ldots (q;q)_{n_{k-1}} \left(q^4;q\right)_{n_{k-1}}} \tag{5.17}$$

For k a positive integer and $\left(\alpha^{(1)}, \beta^{(1)}\right) = (\alpha, \beta)$, so $\left(\alpha_n^{(k)}, \beta_n^{(k)}\right) = (\alpha_n, \beta_n)$. Combining this with the identity (26), this yields the theorem (3.2).

Proof of theorem (3.3): Again taking $b = q^5/a$ in the identity (14) and dividing both sides by $(1 - q/a)\left(1 - q^2/a\right)\left(1 - q^3/a\right)\left(1 - q^4/a\right)$, this yields

$$\sum_{n=0}^{\infty} \frac{\left(q^5;q\right)_{2n} q^n}{(a;q)_{n+1}\,(q/a;q)_{n+4}} \sum_{r=0}^{n} \frac{(-1)^r\, q^{\binom{r}{2}} f_r \left(1 - q^{2r+5}\right)}{(q;q)_{n-r}\,(q^5;q)_{n+r+1}} - \frac{(-1)^4 a^4}{q^{1+2+3+4}} \sum_{r=0}^{\infty} (a/q^4)^r f_r$$

$$(5.18) = \frac{1}{(q,a,q/a;q)_\infty} \sum_{n=1}^{\infty} \left\{ \sum_{r=0}^{\infty} f_r + q^{-4n} \sum_{r=-\infty}^{-1} f_{-r-1} \right\} (-1)^{n+r}\, a^n q^{\binom{n+r+1}{2}}$$

Let

$$(5.19) \qquad \sum_{r=0}^{n} \frac{(-1)^r\, q^{\binom{r}{2}} f_r \left(1 - q^{2r+5}\right)}{(q;q)_{n-r}\,(q^5;q)_{n+r+1}} = \beta_n$$

Comparing the above sum with definition (2) and identifying

$$(5.20) \qquad \alpha_n = (-1)^n q^{\binom{n}{2}} f_n (1 - q^{2n+5})/(1 - q^5) \text{ relative to } q^5.$$

Substituting the value of the sum (31) and f_r from (32) in the identity (30), we get the following result:

For (α, β) a Bailey pair relative to q^5 there holds

$$\sum_{n=0}^{\infty} \frac{\beta_n \left(q^5;q\right)_{2n} q^n}{(a;q)_{n+1}\,(q/a;q)_{n+4}} - \frac{(-1)^4 a^4 (1 - q^5)}{q^{1+2+3+4}} \sum_{n=0}^{\infty} \frac{(-1)^n\, q^{-\binom{n}{2}} \left(a/q^4\right)^n \alpha_n}{(1 - q^{2n+5})}$$

$$(5.21) \quad = \quad \frac{(1 - q^5)}{(q,a,q/a;q)_\infty} \sum_{r=1}^{\infty} (-1)^{r+1} \left(a/q^4\right)^r q^{\binom{r}{2}} \sum_{n=0}^{\infty} \alpha_n q^{(1-r)n} \frac{1 - q^{r(2n+5)}}{1 - q^{2n+5}}$$

provided all sums converge.

In above identity the Bailey pair is relative to q^5, so taking $a = q^5$ in the Bailey chain (3), we get

$$(5.22) \qquad \alpha'_n = q^{n^2+5n} \alpha_n \text{ and } \beta'_n = \sum_{r=0}^{n} \frac{q^{r^2+5r} \beta_r}{(q;q)_{n-r}}$$

The Bailey pair relative to q^5 is the Bailey pair (6), so iterating the Bailey pair (6) along the Bailey chain (34), we get

$$(5.23) \qquad \alpha_n^{(k)} = \frac{q^{kn^2+(5k-1)n}(1 - q^{2n+5})}{1 - q^5},$$

$$\beta_n^{(k)} = \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\ldots+N_{k-1}^2+5(N_1+N_2+\ldots+N_{k-1})}}{(q;q)_{n-N_1}(q;q)_{n_1}(q;q)_{n_2}\ldots(q;q)_{n_{k-1}}(q^5;q)_{n_{k-1}}} \tag{5.24}$$

For k a positive integer and $\left(\alpha^{(1)},\beta^{(1)}\right)=(\alpha,\beta)$, so $\left(\alpha_n^{(k)},\beta_n^{(k)}\right)=(\alpha_n,\beta_n)$. Combining this with the identity (33), this yields the theorem (3.3).

6 Proof of theorem (3.4) and verification of theorems (3.1), (3.2) and (3.3) and corresponding identities

In this section, we will prove the theorem (3.4) and discuss its particular cases.

Further taking $b=q^m/a$ in identity (14) and dividing both sides by $(1-q/a)\left(1-q^2/a\right)\ldots\left(1-q^{m-1}/a\right)$, this yields

$$\sum_{n=0}^{\infty}\frac{(q^m;q)_{2n}\,q^n}{(a;q)_{n+1}(q/a;q)_{n+m-1}}\sum_{r=0}^{n}\frac{(-1)^r q^{\binom{r}{2}} f_r\left(1-q^{2r+m}\right)}{(q;q)_{n-r}(q^m;q)_{n+r+1}}-\frac{(-1)^{m-1}a^{m-1}}{q^{1+2+\ldots+(m-1)}}\sum_{r=0}^{\infty}\left(\frac{a}{q^{m-1}}\right)^r f_r$$
$$=\frac{1}{(q,a,q/a;q)_\infty}\sum_{n=1}^{\infty}\left\{\sum_{r=0}^{\infty}f_r+q^{-(m-1)n}\sum_{r=-\infty}^{-1}f_{-r-1}\right\}(-1)^{n+r}a^n q^{\binom{n+r+1}{2}} \tag{6.1}$$

Taking $m=3,4,5$ in identity (37), we obtain the identities (16), (23), (30) respectively. In particular for $m=1,2$, we obtain the identities (3.2) and (7.1) of Warnaar (2003), respectively. Taking

$$\sum_{r=0}^{n}\frac{(-1)^r q^{\binom{r}{2}} f_r\left(1-q^{2r+m}\right)}{(q;q)_{n-r}(q^m;q)_{n+r+1}}=\beta_n \tag{6.2}$$

Now compare the above sum with definition (2) and identify

$$\alpha_n=(-1)^n q^{\binom{n}{2}} f_n(1-q^{2n+m})/(1-q^m) \text{ relative to } q^m., \tag{6.3}$$

Substituting the value of the sum (38) and f_r from (39) in the identity (37), we get the following result:

For (α,β) a Bailey pair relative to q^m there holds

$$\sum_{n=0}^{\infty}\frac{\beta_n(q^m;q)_{2n}\,q^n}{(a;q)_{n+1}(q/a;q)_{n+m-1}}-\frac{(-1)^{m-1}a^{m-1}(1-q^m)}{q^{1+2+\ldots+(m-1)}}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}\left(a/q^{m-1}\right)^n\alpha_n}{(1-q^{2n+m})}$$
$$=\frac{(1-q^m)}{(q,a,q/a;q)_\infty}\sum_{r=1}^{\infty}(-1)^{r+1}\left(a/q^{m-1}\right)^r q^{\binom{r}{2}}\sum_{n=0}^{\infty}\alpha_n q^{(1-r)n}\frac{1-q^{r(2n+m)}}{1-q^{2n+m}} \tag{6.4}$$

provided all sums converge. Taking $m = 3, 4, 5$ in above identity, we obtain the identities (19), (26) and (33) respectively. In particular for $m = 1, 2$ in identity (40), we obtain the corollary (4.1) and corollary (7.1) of Warnaar (2003) respectively. In the identity (40), the Bailey pair is relative to q^m, so taking $a = q^m$ in the Bailey chain (3), we get

$$\alpha'_n = q^{n^2+mn}\alpha_n \text{ and } \beta'_n = \sum_{r=0}^{n} \frac{q^{r^2+mr}\beta_r}{(q;q)_{n-r}} \tag{6.5}$$

The Bailey pair relative to q^m is the Bailey pair (7), so iterating the Bailey pair (7) along the Bailey chain (41), we get

$$\alpha_n^{(k)} = \frac{q^{kn^2+(mk-1)n}(1-q^{2n+m})}{1-q^m}, \tag{6.6}$$

$$\beta_n^{(k)} = \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\ldots+N_{k-1}^2+m(N_1+N_2+\ldots+N_{k-1})}}{(q;q)_{n-N_1}(q;q)_{n_1}(q;q)_{n_2}\ldots(q;q)_{n_{k-1}}(q^m;q)_{n_{k-1}}} \tag{6.7}$$

For k a positive integer and $\left(\alpha^{(1)}, \beta^{(1)}\right) = (\alpha, \beta)$, so $\left(\alpha_n^{(k)}, \beta_n^{(k)}\right) = (\alpha_n, \beta_n)$. Substituting $\alpha_n = \alpha_n^{(k)}$ and $\beta_n = \beta_n^{(k)}$ in the identity (40), the theorem (3.4) is obtained. Taking $m = 3, 4, 5$ in theorem (3.4), we obtain the theorems (3.1), (3.2), (3.3), respectively.

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