Some Curvature Tensor on (k, μ) -Contact Space Form

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Abstract

The object of present paper is to study (k, μ) -Contact Space Forms admitting a conharmonic curvature tensor $(C\ )$ and a concircular curvature tensor $(\widetilde{C}\ )$. Next we study (k, μ) -Contact Space Forms satisfying $C.S = 0$ and $\widetilde{C}.S = 0$.

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1 INTRODUCTION

The notion of (k, μ)-contact metric manifold was introduced by D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [1]. A class of contact metric manifolds with contact metric structure(ϕ, ξ, η, g) in which the curvature tensor R satisfies the condition

$$R(X,Y)\xi = (KI + \mu h)(\eta(Y)X - \eta(X)Y), \tag{1.1}$$

For all $X, Y \in TM$, is called (k, μ)-contact metric manifolds. The sectional curvature $K(X, \phi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a ϕ-sectional curvature. If the (k, μ)-contact metric manifold M has constant ϕ-sectional curvature c, then it is called a (k, μ)-contact space form and is denoted by $M(c)$. In the recent paper [2], De and Samui studied the structure of some classes of (k, μ)-contact space form. Also, (k, μ)-contact space forms have been studied by K. Arslan, R. Ezentas, I. Mihai, C. Murathan and C. Ozgur, [3] and A. Akbar, A. Sarkar [4] and many others.

Motivated by the above studies, in this paper we consider (k, μ)-contact space forms and obtain some interesting results. The paper is organized as follows:

In section 2, we give necessary details about (k, μ)-contact space forms and some definitions. Section 3 deals with the study of conharmonically flat (k, μ)-contact space forms and (k, μ)-contact space forms satisfying $C.S = 0$. In section 4, deals with the study of concircularlly flat (k, μ)-contact space forms and (k, μ)-contact space forms satisfying $\widetilde{C}.S = 0$.

2 PRELIMINARIES

A $(2n+1)$ dimensional differentiable manifold M is called almost contact manifold [5] , if there is an almost contact structure (ϕ,ξ,η) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η satisfying

$$\phi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\circ\xi = 0,\ \eta\circ\phi = 0$$

Let g be a compatible Riemannian metric with (ϕ,ξ,η), that is,

$$g(X,Y) = g(\phi X,\phi Y) + \eta(X)\eta(Y)$$

or equivalently

$$g(X,Y) = \eta(X), \qquad g(\phi X,Y) = -g(X,\phi Y)$$

for all $X,Y \in TM$, An Almost contact metric structure becomes a contact metric structure if

$$g(X,\phi Y) = d\eta(X,Y), \qquad \text{for all } X,Y \in TM$$

Given a contact metric manifold $M(\phi,\xi,\eta,g)$, we define a $(1,1)$ tensor field h by $h = \frac{1}{2}L_\xi\phi$ where L denotes the Lie defferentiation. Then h is symmetric and satisfies

$$\begin{aligned} h\xi &= 0, & h\phi + \phi h = 0 \\ \nabla\xi &= -\phi - \phi h, & \operatorname{trace}(h) = \operatorname{trace}(\phi h) = 0 \end{aligned} \tag{2.1}$$

where ∇ is the Levi-civita connection. A contact metric manifold is said to be an η-Einstein manifold if

$$S(X,Y) = a_1 g(X,Y) + b_1\eta(X)\eta(Y)$$

where a_1, b_1 are smooth functions on M and $X,Y \in TM$. An Almost contact metric manifold is Sasakian if and only if

$$(\nabla_X\phi)Y = g(X,Y)\xi - \eta(Y)X$$

On a Sasakian manifold the following relation holds

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \qquad \text{for all } X,Y \in TM.$$

Blair, Koufogiorgos and Papantoniou [1] considered the (k,μ)-nullity condition and gave several reasons for studying it. The (k,μ)-nullity distribution $N(k,\mu)$ ([1], [6]) of a contact metric manifold M is defned by

$$N(k,\mu) : p \longmapsto N_p(k,\mu) = [U \in T_pM \mid R(X,Y)U = (kI + \mu h)(g(Y,U)X - g(X,U)Y)]$$

for all $X, Y \in TM$, where $(k,\mu) \in R^2$.A contact metric manifold M with $\xi \in N(k,\mu)$ is called a (k,μ)-contact metric manifold. Then we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

for all $X, Y \in TM$. For (k,μ)-contact metric manifold, it follows that $h^2 = (k-1)\phi^2$.This class contains Sasakian manifolds for $k = 1$ and $h = 0$. In fact, for a (k,μ)-contact metric manifold, the condition of being Sasakian manifold, K-contact manifold, $k = 1$ and $h = 0$ are equivalent. If $\mu = 0$, then the (k,μ)-nullity distribution $N(k,\mu)$ is reduced to k-nullity distribution $N(k)$ [7]. If $\xi \in N(k)$, and then we call a contact metric manifold M as an $N(k)$-contact metric manifold.

The sectional curvature $K(X,\phi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a ϕ-sectional curvature. If the (k,μ)-contact metric manifold M has constant ϕ-sectional curvature c,then it is called a (k,μ) -contact space form and is denoted by $M(c)$. The curvature tensor of $M(c)$ is given by [8] as:

(2.2)

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y] \\ &+\frac{c-1}{4}[2g(X,\phi Y)\phi Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X] \\ &+\frac{c+3-4k}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi] \\ &\frac{1}{2}[g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX \\ &+g(\phi Y,\phi Z)hX - g(\phi X,\phi Z)hY + g(hX,Z)\phi^2 Y - g(hY,Z)\phi^2 X] \\ &+\mu[\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY + g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi], \end{aligned}$$

For all $X, Y, Z \in T(M)$, where $c + 2k = -1 = k - \mu$ if $k \leq 1$ From (2.2) we obtain the following properties of (k,μ)-space forms:

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.3}$$

$$R(X,\xi)\xi = k[X - \eta(X)\xi] + \mu hX \tag{2.4}$$

$$R(\xi,Y)Z = k[g(Y,Z)\xi - \eta(Z)Y] + \mu[g(hY,Z)\xi - \eta(Z)hY] \tag{2.5}$$

$$\begin{aligned} S(Y,Z) &= \frac{1}{2}[c(n+1) + 3(n-1) + 2k]g(Y,Z) \\ &+\frac{1}{2}[-c(n+1) - 3(n-1) + 2k(2n-1)]\eta(Y)\eta(Z) \\ &+(2n-2+\mu)g(hY,Z) \end{aligned} \tag{2.6}$$

$$S(Y,hZ) = \frac{1}{2}[c(n+1)+3(n-1)+2k]g(Y,hZ) +(k-1)[2n-2+\mu]g(Y,Z) -(k-1)[2n-2+\mu]\eta(Y)\eta(Z) \tag{2.7}$$

$$S(Y,\xi) = 2nk\eta(Y) \tag{2.8}$$

$$S(\xi,\xi) = 2nk \tag{2.9}$$

$$QY = \frac{1}{2}[c(n+1)+3(n-1)+2k]Y +\frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)]\eta(Y)\xi +[2n-2+\mu]hY \tag{2.10}$$

$$Q\xi = 2nk\xi \tag{2.11}$$

Definition 2.1. The conharmonic curvature tensor C of type $(1,3)$ on (k,μ)-contact metric form M of dimension $(2n+1)$ is defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \tag{2.12}$$

For any vector field X,Y,Z on M. The manifold is said to be conharmonically flat if C vanishes identically on M.

Definition 2.2. The concircular curvature tensor $\widetilde{C}$ of type $(1,3)$ on (k,μ)-contact metric form M of dimension $(2n+1)$ is defined by

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{2.13}$$

$(n>1)$For any vector field X,Y,Z on M, where R is the curvature tensor and r is the scalar curvature.

From (2.12) using (2.3), (2.6), (2.8), (2.9), (2.10),and (2.11) we have

$$C(X,Y)\xi = a[\eta(Y)X - \eta(X)Y + b[\eta(Y)hX - \eta(X)hY] \tag{2.14}$$

$$C(\xi,Y)\xi = a[\eta(Y)\xi - Y] - bhY \tag{2.15}$$

$$C(\xi,Y)Z = a[g(Y,Z)\xi - \eta(Z)Y] + b[g(hY,Z)\xi - \eta(Z)hY] \tag{2.16}$$

$$C(\xi,Y)hZ = ag(Y,hZ)\xi + bg(hY,hZ)\xi \tag{2.17}$$

where

$$a = k - \frac{2nk}{2n-1} - \frac{1}{2(2n-1)}[c(n+1) + 3(n-1) + 2k]$$

and

$$b = \mu - \frac{1}{2n-1}[2n-2+\mu]$$

Again from (2.13) using (2.3), (2.6), (2.8), (2.9), (2.10),and (2.11) we have

$$\widetilde{C}(X,Y)\xi = a_0[\eta(Y)X - \eta(X)Y + b_0[\eta(Y)hX - \eta(X)hY] \tag{2.18}$$

$$\widetilde{C}(\xi,Y)\xi = a_0[\eta(Y)\xi - Y] - b_0hY \tag{2.19}$$

$$\widetilde{C}(\xi,Y)Z = a_0[g(Y,Z)\xi - \eta(Z)Y] + b_0[g(hY,Z)\xi - \eta(Z)hY] \tag{2.20}$$

$$\widetilde{C}(\xi,Y)hZ = a_0g(Y,hZ)\xi + b_0g(hY,hZ)\xi \tag{2.21}$$

where $a_0 = k - \frac{r}{2n(2n+1)}$ and $b_0 = \mu$

3 Conhormonically flat (k,μ)-contact space forms

From the definition of Conhormonically flat (k,μ)-contact space forms we have $C(X,Y)=0$. Applying this in (2.12), we get

$$R(X,Y)Z=\frac{1}{2n-1}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY] \tag{3.1}$$

Taking the inner product with W of (3.1), we obtain

$$g(R(X,Y)Z,W)=\frac{1}{2n-1}[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)+g(Y,Z)g(QX,W)-g(X,Z)g(QY,W)] \tag{3.2}$$

Putting $X=W=\xi$ in (3.2) and using $(2.5),(2.6),(2.8),(2.10)$ and (2.11), we have

$$g(hY,Z)=\frac{1}{\mu(2n-1)}S(Y,Z)+\frac{k}{\mu(2n-1)}g(Y,Z)+\frac{\eta(Y)\eta(Z)}{\mu}\{\frac{-k-2nk}{(2n-1)}\} \tag{3.3}$$

By using (3.3) in (2.6), we obtain

$$S(Y,Z)=a_1g(Y,Z)+b_1\eta(Y)\eta(Z) \tag{3.4}$$

where

$$a_1=\frac{\frac{1}{2}[c(n+1)+3(n-1)+2k]\mu(2n-1)+(2n-2+\mu)k}{2(n-1)(\mu-1)}$$

and

$$b_1=\frac{\frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)]\mu(2n-1)+(2n-2+\mu)(-k-2nk)}{2(n-1)(\mu-1)}$$

From (3.4), we can state the following:

Theorem 3.1. *A $(2n+1)$-dimensional conhormonically flat (k,μ)-contact space form is an η-Einstein manifold.*

4 (k,μ)-contact space forms satisfying $C.S=0$

Let $M(c)$ be a$(2n+1)$ dimensional (k,μ)-contact space forms satisfying $C.S=0$,which implies that $S(C(X,Y)U,V)+S(U,C(X,Y)V)=0$. By putting $U=X=\xi$, we get $S(C(\xi,Y)\xi,V)+S(\xi,C(\xi,Y)V)=0$. By using $(2.6),(2.7),(2.8),(2.12)$ and (2.13), we obtain

$$g(hY,V)=A_1g(Y,V)+B_1\eta(Y)\eta(Z) \tag{4.1}$$

where,

$$A_1 = \frac{\frac{1}{2}a[c(n+1)+3(n-1)+2k]+b(k-1)(2n-2+\mu)-2nka}{2nkb-\frac{1}{2}b[c(n+1)+3(n-1)+2k]-a(2n-2+\mu)}$$

and

$$B_1 = \frac{\frac{1}{2}a[-c(n+1)-3(n-1)+2k(2n-1)]-b(k-1)(2n-2+\mu)}{2nkb-\frac{1}{2}b[c(n+1)+3(n-1)+2k]-a(2n-2+\mu)}$$

By using (4.1) in (2.6), we get

$$S(Y,V) = A_2 g(Y,V) + B_2 \eta(Y)\eta(V) \tag{4.2}$$

where

$$A_2 = \frac{1}{2}[c(n+1)+3(n-1)+2k] + (2n-2+\mu)A_1$$

and

$$B_2 = \frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)] + (2n-2+\mu)B_1$$

From (4.2), we can state the following:

Theorem 4.1. *A $(2n+1)$ dimensional (k,μ)-contact space forms satisfying $C.S = 0$ is an η-Einstein manifold.*

5 Concircularlly flat (k,μ)-contact space form:

From the definition of Concircularlly flat (k,μ)-contact space forms we have $\widetilde{C}(X,Y) = 0$. Applying this in (2.13), we get

$$R(X,Y)Z = \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{5.1}$$

Taking the inner product with W of (5.1), we obtain

$$g(R(X,Y)Z,W) = \frac{r}{2n(2n+1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \tag{5.2}$$

Putting $X = W = \xi$ in (5.2) and using (2.5) we have

$$g(hY,Z) = \frac{1}{\mu}[-k + \frac{r}{2n(2n+1)}][g(Y,Z) - \eta(Y)\eta(Z)] \tag{5.3}$$

By using (5.3) in (2.6), we obtain

$$S(Y,Z) = a_2 g(Y,Z) + b_2 \eta(Y)\eta(Z) \tag{5.4}$$

where

$$a_2 = \frac{1}{2}[c(n+1) + 3(n-1) + 2k] + \frac{1}{\mu}[-k + \frac{r}{2n(2n+1)}](2n-2+\mu)$$

and

$$b_2 = \frac{1}{2}[-c(n+1) - 3(n-1) + 2k(2n-1)] - \frac{1}{\mu}[-k + \frac{r}{2n(2n+1)}](2n-2+\mu)$$

From (5.4), we can state the following:

Theorem 5.1. *A $(2n+1)$ dimensional (k,μ)-contact space forms satisfying $\widetilde{C}.S = 0$ is an η-Einstein manifold.*

6 (k,μ)-contact space forms satisfying $\widetilde{C}.S = 0$

Let $M(c)$ be a$(2n+1)$ dimensional (k,μ)-contact space forms satisfying $\widetilde{C}.S = 0$,which implies that $S(\widetilde{C}(X,Y)U,V) + S(U,\widetilde{C}(X,Y)V) = 0$. By putting $U = X = \xi$, we get $S(\widetilde{C}(\xi,Y)\xi,V) + S(\xi,\widetilde{C}(\xi,Y)V) = 0$. By using $(2.6), (2.7), (2.8), (2.12)$ and (2.13), we obtain

$$g(hY,V) = \widetilde{A_1} g(Y,V) + \widetilde{B_1} \eta(Y)\eta(Z) \tag{6.1}$$

where,

$$\widetilde{A_1} = \frac{\frac{1}{2}a_0[c(n+1) + 3(n-1) + 2k] + b_0(k-1)(2n-2+\mu) - 2nka_0}{2nkb_0 - \frac{1}{2}b_0[c(n+1) + 3(n-1) + 2k] - a_0(2n-2+\mu)}$$

and

$$\widetilde{B_1} = \frac{\frac{1}{2}a_0[-c(n+1) - 3(n-1) + 2k(2n-1)] - b_0(k-1)(2n-2+\mu)}{2nkb_0 - \frac{1}{2}b_0[c(n+1) + 3(n-1) + 2k] - a_0(2n-2+\mu)}$$

Now, using (6.1) in (2.6),we get

$$S(Y,V) = \widetilde{A_2} g(Y,V) + \widetilde{B_2} \eta(Y)\eta(V) \tag{6.2}$$

where,

$$\widetilde{A_2} = \frac{1}{2}[c(n+1) + 3(n-1) + 2k] + (2n-2+\mu)\widetilde{A_1}$$

and

$$\widetilde{B_2} = \frac{1}{2}[-c(n+1) - 3(n-1) + 2k(2n-1)] + (2n-2+\mu)\widetilde{B_1}$$

From (6.2), we can state the following:

Theorem 6.1. *A $(2n+1)$ dimensional (k,μ)-contact space forms satisfying $\widetilde{C}.S = 0$ is an η-Einstein manifold.*

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