Region of Variability for a Class of Strongly Starlike Analytic Functions

Poonam Sharma[1] **& Rajesh Kumar Maurya**[2]

[1] *Department of Mathematics & Astronomy,*
Lucknow University, Lucknow, Uttar Pradesh
[2] *Department of Mathematics,*
Govt. Post Graduate College Gopeshwar, (Chamoli) Uttarakhand

poonambaba@gmail.com & rajeshkrmaurya@gmail.com

Abstract

In this paper, we determine region of variability for $\log \frac{f(z_0)}{z_0}$, where z_0 is a non zero fixed complex number in the unit disk $\mathbb{U}$ and f varies over a class of strongly starlike functions determined by the subordination codition $\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}$ $(z \in \mathbb{U})$.

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1 Introduction

Let $\mathcal{H}[a, n]$ denotes the class of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...,$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denotes a subclass of $\mathcal{H}[0, 1]$ whose members are of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \quad (z \in \mathbb{U}). \tag{1.1}$$

Here we think of $\mathcal{H}$ as topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{U}$.

We say that an analytic function $f(z)$ is subordinate to another analytic function $g(z)$ and write $f \prec g$, if and only if there exists a Schwarz class function w analytic in $\mathbb{U}$ such that $w(0) = 0$ and $|w(z)| < 1$, $\forall$ $z \in \mathbb{U}$ with $f(z) = g(w(z))$. In particular if $g(z)$ is univalent in $\mathbb{U}$. We have the following equivalence

$$f \prec g \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

Let us denote

$$q_f(z) = z\frac{f'(z)}{f(z)} \tag{1.2}$$

and let $SS^*(\beta)$ denotes a class of strongly starlike functions of order β, defined by

$$SS^*(\beta) = \{\ f \in \mathcal{A}:\ |\arg\ q_f(z)| \leq \beta\frac{\pi}{2},\ 0 < \beta \leq 1\}.$$

The class of strongly starlike functions $SS^*(1)$ becomes the well known class $S^*$of starlike functions.

In this paper, we consider the class $\mathcal{LS}^*$ defined by

$$\mathcal{LS}^* = \{f \in \mathcal{A} : |q_f^2 - 1| < 1\}$$

which is associated with the right half of the lemniscate of Bernoulli [2]. Observe that $\mathcal{L} = \{w \in \mathbb{C} : Rew > 0, |w^2 - 1| < 1\}$ is the interior of the right half of the lemniscate of Bernoulli: $\gamma : (x^2 + y^2)^2 - 2(x^2 - y^2) = 0$. It is easy to see that $f \in \mathcal{LS}^*$ if and only if

$$q_f(z) \prec q_0(z) = \sqrt{1+z},\ q_0(0) = 1. \tag{1.3}$$

Moreover

$$\mathcal{L} \subset \{w : |\arg\ w| < \frac{\pi}{4}\}.$$

Thus, we have following inclusion relation

$$\mathcal{LS}^* \subset SS^*(\frac{1}{2}) \subset S^*.$$

We determine the region of variability $V(z_0, \lambda)$ for the function $\log\frac{f(z_0)}{z_0}$, where z_0 is a non zero fixed complex number in the unit disk $\mathbb{U}$ and $f \in \mathcal{LS}^*$. In recent years, the region of variability $V(z_0, \lambda)$ for functions belonging to various classes of $\mathcal{A}$ is studied by several authors (see in [1, 3, 4, 5, 6]).

2 Main Results

Let

$$B_0 = \{w \in \mathcal{H}, w(0) = 0, w^{'}(0) \neq 0 \text{ and } |w(z)| < 1 \text{ for } z \in \mathbb{U}\}.$$

From (1.1) and from the definition of $q_f(z)$, we have $q_f(0) = 1$ and from (1.3)

$$q_f^2(z) = 1 + w_f(z) \tag{2.1}$$

for some $w_f(z) \in B_0$ and conversely.

2.1 The region $V(z_0, \lambda)$

For some $\lambda \in \overline{\mathbb{U}}$ with $w'_f(0) = \lambda$ and $z_0 \in \mathbb{U},\ z_0 \neq 0$, we have

(2.2) $$V(z_0, \lambda) = \{\log \frac{f(z_0)}{z_0} :\ f \in \mathcal{LS}^*\},$$

where $V(z_0, \lambda)$ is region of variability as f varies over $\mathcal{LS}^*$, logarithm of $\frac{f(z_0)}{z_0}$ is single valued.

Lemma 2.1. *Let f be an analytic function in $\mathbb{U}$ with*

$$f(z) = z^k + \sum_{n=1}^{\infty} a_{n+k} z^{n+k}.$$

If

$$Re\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0\ \ (z \in \mathbb{U}),$$

then $f \in (S^*)^k$.

2.2 Basic Properties of $V(z_0, \lambda)$

Proposition 2.1. *We have*
1) $V(z_0, \lambda)$ is compact.
2) $V(z_0, \lambda)$ is convex.
3) for $|\lambda| = 1$

$$V(z_0, \lambda) = 2(\sqrt{1 + \lambda z_0} - 1) + 2\log\frac{2}{(\sqrt{1 + \lambda z_0} + 1)}$$

4) for $|\lambda| < 1$ and $z_0 \in \mathbb{U}\backslash\{0\}$, $V(z_0, \lambda)$ has $2(\sqrt{1 + \lambda z_0} - 1) + 2\log\frac{2}{(\sqrt{1+\lambda z_0}+1)}$ as interior point.

Proof. 1) Since $\mathcal{LS}^*$ is compact subset of $\mathcal{A}$, any bounded sequence of functions in $\mathcal{LS}^*$ converges in it, hence corresponding sequence in $V(z_0, \lambda)$ also converges in $V(z_0, \lambda)$, it follows that $V(z_0, \lambda)$ is also compact.

2) If $f_1,\ f_2 \in \mathcal{LS}^*$ and $0 \leq t \leq 1$, then the function

$$\begin{aligned} \frac{f_t\,(z_0)}{z_0} &= \exp\{(1-t)\log\frac{f_1(z_0)}{z_0} + t\log\frac{f_2(z_0)}{z_0}\} \\ &= f_1^{1-t}(z_0) f_2^t(z_0). \end{aligned}$$

and

$$z\frac{f_t'(z)}{f_t(z)} = (1-t)z\frac{f_1'(z)}{f_1(z)} + tz\frac{f_2'(z)}{f_2(z)}.$$

Since we have

$$z\frac{f_1'(z)}{f_1(z)} \prec \sqrt{1+z} \quad \text{and} \quad z\frac{f_2'(z)}{f_2(z)} \prec \sqrt{1+z}$$

which implies that

$$\begin{aligned} z\frac{f_t'(z)}{f_t(z)} &\prec (1-t)\sqrt{1+z}+t\sqrt{1+z} \\ &= \sqrt{1+z}. \end{aligned}$$

Hence,

$$f_t \in \mathcal{LS}^*.$$

Also because of the representation of f_t, we easily see that the set $V(z_0,\lambda)$ is convex.

3) If $|\lambda| = 1 = |w_f'(0)|$, then it follows from classical Schwarz lemma that

$$w_f(z) = \lambda z$$

and we have from (2.1) that

$$q_f^2(z) = 1+\lambda z \Rightarrow z\frac{f'(z)}{f(z)} = \sqrt{1+\lambda z}$$

consequently

$$\log\frac{f(z_0)}{z_0} = 2(\sqrt{1+\lambda z_0}-1) + 2\log\frac{2}{(\sqrt{1+\lambda z_0}+1)} \ .$$

4) For $|\lambda| < 1$ and $a \in \overline{\mathbb{U}}$, we define

$$\delta(z,\lambda) = \frac{z+\lambda}{1+\overline{\lambda}z} \tag{2.3}$$

and the function $H_{a,\lambda}(z)$ by

$$\log\frac{H_{a,\lambda}(z)}{z} = \left\{\int_0^z \frac{\sqrt{1+\delta(a\varepsilon,\lambda)\varepsilon}-1}{\varepsilon}\mathrm{d}\varepsilon\right\}. \tag{2.4}$$

First we claim that $H_{a,\lambda}(z) \in \mathcal{LS}^*$, for this we may easily get by simple computation that

$$z\frac{H_{a,\lambda}'(z)}{H_{a,\lambda}(z)} = \sqrt{1+\delta(az,\lambda)z}$$

as $\delta(az,\lambda)$ lies in the unit disk $\mathbb{U}$ and hence, $H_{a,\lambda}(z) \in \mathcal{LS}^*$and the claim follows. Also we observe that

$$w_{H_{a,\lambda}}(z) = \delta(az,\lambda)z.$$

Next we claim that the mapping $\mathbb{U} \ni a \to \log \frac{H_{a,\lambda}(z_0)}{z_0}$ is a non-constant analytic function of a for each fixed $z_0 \in \mathbb{U}/\{0\}$ and $\lambda \in \mathbb{U}$, to do this we put

$$\begin{aligned} h(z_0) &= \frac{4}{1-\lambda\overline{\lambda}} \frac{\partial}{\partial a}\{\log H_{a,\lambda}(z_0)\}|_{a=0} \\ &= \int_0^{z_0} \frac{2\varepsilon}{\sqrt{1+\lambda\varepsilon}} \mathrm{d}\varepsilon \\ &= z_0^2 - \frac{1}{3}\lambda z_0^3 + ..., \end{aligned}$$

so that

$$Re\left(z_0 \frac{h''(z_0)}{h'(z_0)}\right) = \frac{1}{2} Re\left(1 + \frac{1}{1+\lambda z_0}\right) > 0 \ .$$

By Lemma 2.1 there exists a function $h_0 \in S^*$ with $h(z) = h_0^2(z)$ the univalence of h_0 together with the condition $h_0(0) = 0$, implies that $h(z_0) \neq 0$ for $z_0 \in \mathbb{U}\backslash\{0\}$. Consequently, the mapping

$$\mathbb{U} \ni a \to \log \frac{H_{a,\lambda}(z_0)}{z_0}$$

is a non-constant analytic function of a therefore it is an open mapping, thus $V(z_0, \lambda)$ is an open set and

$$\log \frac{H_{a,\lambda}(z_0)}{z_0} = 2(\sqrt{1+\lambda z_0} - 1) + 2\log \frac{2}{(\sqrt{1+\lambda z_0}+1)} \qquad (\lambda \in \mathbb{U}),$$

is an interior of

$$\left\{\log \frac{H_{a,\lambda}(z_0)}{z_0} : a \in \mathbb{U}\right\} \subset \ V(z_0, \lambda).$$

We remark that, since $V(z_0, \lambda)$ is a compact convex subset of $\mathbb{C}$ and has non empty interior, the boundary $\partial V(z_0, \lambda)$ is a Jordan curve and $V(z_0, \lambda)$ is union of $\partial V(z_0, \lambda)$ and its inner domain. □

Theorem 2.1. *For $\lambda \in \mathbb{U}$ and $z_0 \in \mathbb{U}\backslash\{0\}$, the boundary $\partial V(z_0, \lambda)$ is the Jordan curve given by*

$$(-\pi, \pi] \ni \theta \to \log \frac{H_{e^{i\theta},\lambda}(z_0)}{z_0} = \int_0^{z_0} \frac{\sqrt{1+\delta(e^{i\theta}\varepsilon, \lambda)\varepsilon} - 1}{\varepsilon} d\varepsilon.$$

If

$$\log \frac{f(z_0)}{z_0} = \log \frac{H_{e^{i\theta},\lambda}(z_0)}{z_0}$$

for some $f \in \mathcal{LS}^*$ and $\theta \in (-\pi, \pi]$, then $f(z) = H_{e^{i\theta},\lambda}(z)$, where $H_{e^{i\theta},\lambda}(z)$ is given by (2.4).

3 Region of Variability

Proposition 3.1. *For $f \in \mathcal{LS}^*$ we have*

$$\left| q_f^2(z) - \frac{1+\lambda z - (z+\overline{\lambda}) z\overline{z}\lambda}{1 - z\overline{z}\lambda\overline{\lambda}} \right| \leq \frac{|z\overline{z}||1-\lambda\overline{\lambda}|}{1 - z\overline{z}\lambda\overline{\lambda}} \qquad (\lambda \in \mathbb{U}) \tag{3.1}$$

for each $z \in \mathbb{U}\backslash\{0\}$. Equality in (3.1) holds if and only if $f(z) = H_{e^{i\theta},\lambda}(z)$ for some $\theta \in (-\pi, \pi]$, where $H_{e^{i\theta},\lambda}(z)$ is given by (2.4).

Proof. Let $f \in \mathcal{LS}^*$. Then from (2.1) there exists a function $w_f \in B_0$ such that

$$q_f^2(z) = 1 + w_f(z),$$

where $q_f(z)$ is as defined by (1.2). It follows from the Schwarz lemma that

$$\left| \frac{\frac{w_f(z)}{z} - \lambda}{1 - \overline{\lambda}\frac{w_f(z)}{z}} \right| \leq |z| \qquad (z \in \mathbb{U})$$

or,

$$\left| \frac{w_f(z) - \lambda z}{z - \overline{\lambda} w_f(z)} \right| \leq |z| \quad (z \in \mathbb{U})$$

which from (2.1) is equivalent to

$$\left| \frac{q_f^2(z) - 1 - \lambda z}{\frac{z+\overline{\lambda}}{\overline{\lambda}} - q_f^2(z)} \right| \leq |\overline{\lambda} z|$$

which implies that

$$\left| q_f^2(z) - \frac{A - B|E|^2}{1 - |E|^2} \right| \leq \frac{|E||B-A|}{1-|E|^2} \tag{3.2}$$

where

$$A = 1 + \lambda z, \; B = \frac{z + \overline{\lambda}}{\overline{\lambda}}, \; E = \overline{\lambda} z. \tag{3.3}$$

So we have

$$\frac{A - B|E|^2}{1-|E|^2} = \frac{1+\lambda z - (z+\overline{\lambda}) z\overline{z}\lambda}{1 - z\overline{z}\lambda\overline{\lambda}}, \quad \frac{|E||B-A|}{1-|E|^2} = \frac{z\overline{z}|1-\lambda\overline{\lambda}|}{1 - z\overline{z}\lambda\overline{\lambda}}, \tag{3.4}$$

which proves the inequality (3.1). Now from the inequality (3.1) and the last two equations in (3.4), we check for the equality

$$\left| q_f^2(z) - \frac{1+\lambda z - (z+\overline{\lambda}) z\overline{z}\lambda}{1 - z\overline{z}\lambda\overline{\lambda}} \right| = \frac{|z\overline{z}||1-\lambda\overline{\lambda}|}{1 - z\overline{z}\lambda\overline{\lambda}}, \tag{3.5}$$

where

$$\begin{aligned} q_f^2(z) &= 1+\delta(az,\lambda)z \\ &= 1+\frac{az+\lambda}{1+\overline{\lambda}az}z \end{aligned}$$

from (2.3). Thus, we have

$$\left|\left(1+\left(\frac{az+\lambda}{1+\overline{\lambda}az}\right)z\right)-\frac{1+\lambda z-(z+\overline{\lambda})z\overline{z}\lambda}{1-z\overline{z}\lambda\overline{\lambda}}\right|=\frac{|z\overline{z}||1-\lambda\overline{\lambda}|}{1-z\overline{z}\lambda\overline{\lambda}}. \tag{3.6}$$

On solving (3.6) for a, we get

$$|a|=\left|\frac{1-\lambda z}{1-\overline{z}\overline{\lambda}}\right|=1\Rightarrow a=e^{i\theta},$$

hence equality occurs for any $z\in\mathbb{U}$ in (3.1) when $f=H_{e^{i\theta},\lambda}$ for some $\theta\in(-\pi,\pi]$. Conversely if equality occurs for some $z\in\mathbb{U}\backslash\{0\}$ in (3.1), then equality must hold in (3.2), thus from the well known Schwarz lemma $\exists\ \theta\in(-\pi,\pi]$ such that

$$w_f(z)=z\delta(e^{i\theta}z,\lambda)$$

for $\forall\ z\in\mathbb{U}$ this implies $f=H_{e^{i\theta},\lambda}$. □

Corollary 3.1. *Let $V(z_0,\lambda)$ be given by (2.2). Let $\gamma: z(t)$, $0\le t\le 1$ be a C^1- curve in $\mathbb{U}$ with $z(0)=0$, $z(1)=z_0$. Then*

$$V(z_0,\lambda)\subset\{w\in\mathbb{C}:|w|\ \le R(\lambda,\gamma)\},$$

where

$$\begin{aligned} R(\lambda,\gamma) &= \left|\int_0^1\left(\sqrt{\frac{1+\lambda z-(z+\overline{\lambda})z\overline{z}\lambda+z\overline{z}(1-\lambda\overline{\lambda})e^{i\theta}}{1-z\overline{z}\lambda\overline{\lambda}}}-1\right)\frac{z'(t)}{z(t)}dt\right| \\ \forall\ \theta &\in (-\pi,\pi]. \end{aligned}$$

Proof. For $f\in\mathcal{LS}^*$ we have equality (3.5) in Proposition 3.1 for the extremal function, hence, for any $\theta\in(-\pi,\pi]$,

$$q_f^2(z)=\frac{1+\lambda z-(z+\overline{\lambda})z\overline{z}\lambda+z\overline{z}(1-\lambda\overline{\lambda})e^{i\theta}}{1-z\overline{z}\lambda\overline{\lambda}}$$

which implies that

$$q_f(z)=\sqrt{\frac{1+\lambda z-(z+\overline{\lambda})z\overline{z}\lambda+z\overline{z}(1-\lambda\overline{\lambda})e^{i\theta}}{1-z\overline{z}\lambda\overline{\lambda}}}$$

or,

$$\frac{zf'(z)}{f(z)} = \sqrt{\frac{1+\lambda z-(z+\overline{\lambda})z\overline{z}\lambda+z\overline{z}(1-\lambda\overline{\lambda})e^{i\theta}}{1-z\overline{z}\lambda\overline{\lambda}}}.$$

Hence,

$$\begin{aligned}&\int_0^1\left(\frac{f'(z)}{f(z)}-\frac{1}{z(t)}\right)z'(t)\mathrm{d}t\\ = \;&\int_0^1\left[\sqrt{\frac{1+\lambda z-(z+\overline{\lambda})z\overline{z}\lambda+z\overline{z}(1-\lambda\overline{\lambda})e^{i\theta}}{1-z\overline{z}\lambda\overline{\lambda}}}-1\right]\frac{z'(t)}{z(t)}\mathrm{d}t\end{aligned}$$

which implies that

$$\begin{aligned}\log\frac{f(z_0)}{z_0} &= :w\\ &= \int_0^1\left[\sqrt{\frac{1+\lambda z-(z+\overline{\lambda})z\overline{z}\lambda+z\overline{z}(1-\lambda\overline{\lambda})e^{i\theta}}{1-z\overline{z}\lambda\overline{\lambda}}}-1\right]\frac{z'(t)}{z(t)}dt\end{aligned}$$

which implies

$$V(z_0,\lambda)\subset\{w\in\mathbb{C}:|w|\ \le R(\lambda,\gamma)\}.$$

□

Proposition 3.2. *Let* $z_0\in\mathbb{U}\backslash\{0\}$. *Then for* $\theta\in(-\pi,\pi]$ $\log\frac{H_{e^{i\theta},\lambda}(z_0)}{z_0}\in\partial V(z_{0,}\lambda)$. *Furthermore if* $\log\ \frac{f(z_0)}{z_0}=\log\ \frac{H_{e^{i\theta},\lambda}(z_0)}{z_0}$ *for some* $f\in\mathcal{LS}^*$ *and* $\theta\in(-\pi,\pi]$, *then* $f=H_{e^{i\theta},\lambda}$.

Proof. From (2.4)

$$\log\frac{H_{a,\lambda}(z)}{z}=\int_0^z\frac{\sqrt{1+\delta(a\varepsilon,\lambda)\varepsilon}-1}{\varepsilon}d\varepsilon\qquad(\lambda\in\mathbb{U})$$

we easily obtain that

$$\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)}-\frac{1}{z}=\frac{\sqrt{1+\delta(az,\lambda)z}-1}{z}\tag{3.7}$$

which implies that

$$z^2\left(\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)}\right)^2=1+\delta(az,\lambda)z.$$

Hence, on using (3.3), we get

$$q^2_{H_{a,\lambda}}(z)-A=z\delta(az,\lambda)-\lambda z\tag{3.8}$$

and

$$B - q_f^2(z) = \frac{z}{\overline{\lambda}} - z\delta(az, \lambda) \tag{3.9}$$

which on substituting $a = e^{i\theta}$, from (3.3), (3.7), (3.8) and (3.9), we have

$$\begin{aligned} \left| q^2_{H_{a,\lambda}}(z) - \frac{A - B|E|^2}{1 - |E|^2} \right| &= \left| \frac{|z|^2(1 - \lambda\overline{\lambda})(e^{i\theta} + \lambda\overline{z})}{(1 - z\overline{z}\lambda\overline{\lambda})(1 + \overline{\lambda}ze^{i\theta})} \right| \\ &= \frac{|E||B - A|}{1 - |E|^2} \left| \frac{e^{i\theta} + \lambda\overline{z}}{1 + \overline{\lambda}ze^{i\theta}} \right| \end{aligned} \tag{3.10}$$

since $\left| \frac{e^{i\theta}+\lambda\overline{z}}{1+\overline{\lambda}ze^{i\theta}} \right| \leq 1 \quad (\lambda \in \mathbb{U})$, from (3.10) and (3.1) we have

$$q^2_{H_{a,\lambda}}(z) = \frac{|E||B - A|}{1 - |E|^2} \left| \frac{e^{i\theta} + \lambda\overline{z}}{1 + \overline{\lambda}ze^{i\theta}} \right| e^{i\phi} + \frac{1 + \lambda z - (z + \overline{\lambda})z\overline{z}\lambda}{1 - |E|^2}, \quad \phi \in \mathbb{R}$$

which implies that

$$q_{H_{a,\lambda}}(z) = \left(\frac{|E||B - A|}{1 - |E|^2} \left| \frac{e^{i\theta} + \lambda\overline{z}}{1 + \overline{\lambda}ze^{i\theta}} \right| e^{i\phi} + \frac{1 + \lambda z - (z + \overline{\lambda})z\overline{z}\lambda}{1 - |E|^2} \right)^{\frac{1}{2}}.$$

Hence,

$$z\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} = \left(\frac{|E||B - A|}{1 - |E|^2} \left| \frac{e^{i\theta} + \lambda\overline{z}}{1 + \overline{\lambda}ze^{i\theta}} \right| e^{i\phi} + \frac{1 + \lambda z - (z + \overline{\lambda})z\overline{z}\lambda}{1 - |E|^2} \right)^{\frac{1}{2}}. \tag{3.11}$$

On integrating (3.11) along $\gamma : z(t),\ 0 \leq t \leq 1$ which is a C^1- curve in $\mathbb{U}$ with $z(0) = 0,\ z(1) = z_0$, we obtain

$$\log \ \frac{H_{e^{i\theta},\lambda}(z_0)}{z_0} = R(\lambda, \gamma) \quad (\lambda \in \mathbb{U}) \tag{3.12}$$

where $R(\lambda, \gamma)$ is defined in Corollary 3.1, thus we have $\log \ H_{e^{i\theta},\lambda}(z_0) \subset V(z_{0,}\lambda) \subset D$, where D is defined by

$$D = \{w \in \mathbb{C} : |w| \leq R(\lambda, \gamma)\}$$

which concludes that $\log \ \frac{H_{e^{i\theta},\lambda}(z_0)}{z_0} \subset \partial V(z_{0,}\lambda)$. Finally, we prove the uniqueness of the curve, suppose that

$$\log \ \frac{f(z_0)}{z_0} = \log \frac{H_{e^{i\theta},\lambda}(z_0)}{z_0}$$

for $f \in \mathcal{LS}^*$ and $\theta \in (-\pi, \pi]$, we introduce

$$\begin{aligned} & h(t) \\ = & \left[\frac{f'(z)}{f(z)} - \frac{1}{z} \right. \\ & \left. - \frac{1}{z(t)} \left\{ \left(\frac{|E||B-A|}{1-|E|^2} \left| \frac{e^{i\theta} + \lambda \overline{z}}{1 + \overline{\lambda} z e^{i\theta}} \right| e^{i\phi} + \frac{1 + \lambda z - (z + \overline{\lambda}) z \overline{z} \lambda}{1 - |E|^2} \right)^{\frac{1}{2}} - 1 \right\} \right] z'(t) \end{aligned}$$

where $\gamma : z(t) = z$, $0 \leq t \leq 1$ is given in Corollary 3.1, then $h(t)$ is continuous function in the interval $[0, 1]$ and

$$\begin{aligned} & \int_0^1 Re h(t) \mathrm{d}t \\ = & \int_0^1 Re \left\{ \left(\frac{f'(z)}{f(z)} - \frac{1}{z} \right) \right. \\ & \left. - \left(\frac{1}{z(t)} \left(\frac{|E||B-A|}{1-|E|^2} \left| \frac{e^{i\theta} + \lambda \overline{z}}{1 + \overline{\lambda} z e^{i\theta}} \right| e^{i\phi} + \frac{1 + \lambda z - (z + \overline{\lambda}) z \overline{z} \lambda}{1 - |E|^2} \right)^{\frac{1}{2}} - \frac{1}{z(t)} \right) \right\} z'(t) \mathrm{d}t \end{aligned}$$

$$Re \left\{ \log \frac{f'(z_0)}{z_0} - R(\lambda, \gamma) \right\}. \tag{3.13}$$

Since logarithm function is single valued therefore from (3.11), (3.12) and (3.13), we have

$$\frac{f'}{f} = \frac{H'_{e^{i\theta}, \lambda}}{H_{e^{i\theta}, \lambda}}$$

on the curve γ, using the identity theorem for analytic function we conclude that last equality holds in $\mathbb{U}$ and hence we have $f = H_{e^{i\theta}, \lambda}$. □

Proof of Theorem 3.1. We need to prove that the closed curve $(-\pi, \pi] \ni \theta \to \log \frac{H_{e^{i\theta}, \lambda}(z_0)}{z_0}$ is simple. Suppose that $\log \frac{H_{e^{i\theta_1}, \lambda}(z_0)}{z_0} = \log \frac{H_{e^{i\theta_2}, \lambda}(z_0)}{z_0}$ for some $\theta_1, \theta_2 \in (-\pi, \pi]$ with $\theta_1 \neq \theta_2$, then from Proposition 3.2, we have ,

$$H_{e^{i\theta_1}, \lambda} = H_{e^{i\theta_2}, \lambda}.$$

Let us define $\tau(z, \lambda) = \frac{z - \lambda}{1 - \overline{\lambda} z}$, from the equality $w_{H_{e^{i\theta}, \lambda}} = z\delta(e^{i\theta} z, \lambda)$, we have

$$e^{i\theta_1} z = \tau \left(\frac{w_{H_{e^{i\theta_1}, \lambda}}}{z}, \lambda \right) = \tau \left(\frac{w_{H_{e^{i\theta_2}, \lambda}}}{z}, \lambda \right) = e^{i\theta_2} z \ ,$$

which is contrary to the fact that $\theta_1 \neq \theta_2$, thus the curve is simple. Since $V(z_0, \lambda)$ is a compact convex subset of $\mathbb{C}$ and has non empty interior, the boundary $\partial V(z_0, \lambda)$ is a simple closed curve. From Proposition 2.1 the curve $\partial V(z_{0,}\lambda)$ contains the curve $(-\pi,\pi] \ni \theta \to \log \frac{H_{e^{i\theta},\lambda}(z_0)}{z_0}$, since a simple closed curve can not contain any simple closed curve other than itself. Thus $\partial V(z_{0,}\lambda)$ is given by $(-\pi,\pi] \ni \theta \to \log \frac{H_{e^{i\theta},\lambda}(z_0)}{z_0} \quad (\lambda \in \mathbb{U})$. $\square$

Remark 3.1. *For $f \in S^*$ class of starlike functions we have $\log \frac{f(z_0)}{z_0} = \log \frac{1}{(1-z_0)^2}$ which is not bounded when $z_0 \in \partial\mathbb{U}$, but in case $f \in \mathcal{LS}^*$, $\log \frac{f(z_0)}{z_0}$ is bounded even when $z_0 \in \partial\mathbb{U}$.*

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