Some Integer Sequences and Standard Monomials

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Abstract

For some classes of monomial ideals J in the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k, the number $\dim_k\left(\frac{R}{J}\right)$ of standard monomials of the Artinian quotient $\frac{R}{J}$ has a nice combinatorial interpretation. In this paper, we have studied many monomial ideals in R and obtain explicit formula for enumerating their standard monomials.

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1 Introduction

Let $\mathfrak{S}_n$ be the group of all permutations of $[n] = \{1, 2, \ldots, n\}$. For each permutation $\sigma \in \mathfrak{S}_n$, let $\mathbf{x}^\sigma = \prod_{i=1}^n x_i^{\sigma(i)}$ be the associated monomial in the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k. The monomial ideal $I_{\mathfrak{S}_n} = I(1, 2, \ldots, n) = \langle \mathbf{x}^\sigma : \sigma \in \mathfrak{S}_n \rangle$ in R generated by monomial associated to all permutations $\sigma \in \mathfrak{S}_n$ is called a *permutohedron ideal*. The Alexander dual $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ of $I_{\mathfrak{S}_n}$ with respect to $\mathbf{n} = (n, \ldots, n) \in \mathbb{N}^n$ is given by

$$I_{\mathfrak{S}_n}^{[\mathbf{n}]} = \left\langle \left(\prod_{j \in A} x_j \right)^{n-|A|+1} : \emptyset \neq A \subseteq [n] \right\rangle.$$

Further, standard monomials of $\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}}$ correspond bijectively to the ordinary parking functions of length n, and hence $\dim_k\left(\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}}\right) = (n+1)^{n-1}$. For more on Alexander duals, we refer to [5, 6].

For a sequence $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ with $\lambda_1 \geq \ldots \geq \lambda_n \geq 1$, the monomial ideal $I_\lambda = \left\langle \left(\prod_{j \in A} x_j \right)^{\lambda_{n-|A|+1}} : \emptyset \neq A \subseteq [n] \right\rangle$ has standard monomial basis corresponding to λ-parking functions of length n. We recall that a sequence $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is called a *λ-parking function of length n* if the non-decreasing rearrangement $q_1 \leq q_2 \leq \ldots \leq q_n$

of $\mathbf{p}$ satisfies $q_i < \lambda_{n-i+1}$, $\forall i$. The *ordinary parking functions* are λ-parking functions for $\lambda = (n, n-1, \ldots, 1)$. In other words, a monomial $\mathbf{x}^{\mathbf{p}} \notin I_\lambda$ if and only in $\mathbf{p}$ is a λ-parking function of length n. Now using Steck determinant formula for counting λ-parking functions, we see that the number of standard monomials of $\frac{R}{I_\lambda}$ is given by

$$\dim_k\left(\frac{R}{I_\lambda}\right) = (n!)\det(\Lambda(\lambda_1, \ldots, \lambda_n)),$$

where $\Lambda(\lambda_1, \ldots, \lambda_n) = [m_{ij}]_{1\le i,j\le n}$ is a matrix with $m_{ij} = \begin{cases} \frac{\lambda_{n-i+1}^{j-i+1}}{(j-i+1)!} & i \le j+1, \\ 0 & i > j+1. \end{cases}$ For λ-parking functions and Steck determinant formula, we refer to [3, 7, 8].

More generally, for any finite (oriented) graph G, Postnikov and Shapiro [8] associated a monomial ideal $\mathcal{M}_G$ in R so that the Artinian k-algebra $\frac{R}{\mathcal{M}_G}$ has the standard monomial basis corresponding to the *G-parking functions*. The number $\dim_k\left(\frac{R}{\mathcal{M}_G}\right)$ of G-parking functions equals the number of (oriented) spanning trees of G, and hence $\dim_k\left(\frac{R}{\mathcal{M}_G}\right) = \det(L_G)$, where L_G is an $n \times n$ matrix called *truncated Laplace matrix* of G.

The monomial ideal $J_n = \left\langle x_l^n, (x_i x_j)^{n-(j-i)} : 1 \le l \le n; 1 \le i < j \le n\right\rangle$ in the polynomial ring R has been studied in [4] and it is shown that the integer sequence $\left\{\dim_k\left(\frac{R}{J_n}\right)\right\}_{n\ge 1}$ is same as the sequence ($A000262$) in OEIS [9], whose n^{th} term is the number of partitions of $[n] = \{1, 2, \ldots, n\}$ into sets of lists. Thus

$$\dim_k\left(\frac{R}{J_n}\right) = \det([a_{ij}]_{1\le i,j\le n}); \quad \text{where} \quad a_{ij} = \begin{cases} 1 & \text{if } i > j, \\ i & \text{if } i = j, \\ i-j & \text{if } i < j. \end{cases}$$

In this paper, we study monomial ideals whose standard monomials have nice combinatorial interpretations. Let $I(\mathbf{u})$ be the multipermutohedron ideal corresponding to $\mathbf{u} = (1, \ldots, 1, n) \in \mathbb{N}^n$. The minimal generators of $I(\mathbf{u})$ are precisely the monomials $x_1^n x_2 \cdots x_n$, $x_1 x_2^n \cdots x_n$, $\ldots$, $x_1 x_2 \cdots x_n^n$. Also, the Alexander dual $I(\mathbf{u})^{[\mathbf{n}]}$ of $I(\mathbf{u})$ with respect to $\mathbf{n} = (n, \ldots, n)$ is given by

$$I(\mathbf{u})^{[\mathbf{n}]} = \langle x_j^n,\ x_1 x_2 \cdots x_n : 1 \le j \le n\rangle.$$

Using the cellular resolution of $I(\mathbf{u})^{[\mathbf{n}]}$, we show that

$$\dim_k\left(\frac{R}{I(\mathbf{u})^{[\mathbf{n}]}}\right) = n^n - (n-1)^n.$$

It is now clear that the integer sequence $\left\{\dim_k\left(\frac{R}{I(\mathbf{u})^{[\mathbf{n}]}}\right)\right\}_{n\ge 1}$ coincides with the integer sequence ($A045531$) in OEIS [9], whose nth term is the number of functions $\phi : [n] \to [n]$

with $\phi(1) = 1$ (such functions are called *sticky functions* of $[n]$). It is an interesting problem to construct an explicit bijection between the set of standard monomials of $\frac{R}{I(\mathbf{u})^{[\mathbf{n}]}}$ and the set of sticky functions of $[n]$.

More generally, for positive integers a, b with $b < a$, we consider the monomial ideal $\mathcal{I}_n(a,b) = \langle x_j^a,\ (x_1 x_2 \cdots x_n)^b : 1 \leq j \leq n \rangle$ in R. Clearly, $\mathcal{I}_n(n,1) = I(\mathbf{u})^{[\mathbf{n}]}$.

Theorem 1.1. *The number of standard monomials of* $\frac{R}{\mathcal{I}_n(a,b)}$ *is given by*

$$\dim_k \left(\frac{R}{\mathcal{I}_n(a,b)} \right) = a^n - (a-b)^n.$$

Since $\mathcal{I}_n(a,b) = I_\lambda$ for $\lambda = (a, \ldots, a, b) \in \mathbb{N}^n$, the standard monomials of $\frac{R}{\mathcal{I}_n(a,b)}$ correspond to the λ-parking functions of length n. Hence by Steck determinant formula, we obtain a beautiful identity $(n!)det(\Lambda(a, \ldots, a, b)) = a^n - (a-b)^n$.

Corollary 1.1. *For any positive integers* $b < a$,

$$(n!) \det \begin{bmatrix} \frac{b}{1} & \frac{b^2}{2!} & \frac{b^3}{3!} & \cdots & \frac{b^{n-1}}{(n-1)!} & \frac{b^n}{n!} \\ 1 & \frac{a}{1} & \frac{a^2}{2!} & \cdots & \frac{a^{n-2}}{(n-2)!} & \frac{a^{n-1}}{(n-1)!} \\ 0 & 1 & \frac{a}{1} & \cdots & \frac{a^{n-3}}{(n-3)!} & \frac{a^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{a}{1} & \frac{a^2}{2!} \\ 0 & 0 & 0 & \cdots & 1 & \frac{a}{1} \end{bmatrix} = a^n - (a-b)^n.$$

We consider another monomial ideal $\mathcal{J}_n$ in the polynomial ring R given by $\mathcal{J}_n = \langle x_j^{j+1},\ x_1 x_2 \cdots x_n : 1 \leq j \leq n \rangle$. We show that $\dim_k \left(\frac{R}{\mathcal{J}_n} \right) = n(n!)$. Thus, the integer sequence $\left\{ \dim_k \left(\frac{R}{\mathcal{J}_n} \right) \right\}_{n \geq 1}$ is the integer sequence ($A001563$) in OEIS [9], whose nth term is the number of permutations of $[n+1]$ in which $n+1$ is not fixed. Again, it is an interesting problem to construct an explicit bijection between these objects.

For positive integers a, b with $b < a$, we consider the monomial ideal $\mathcal{J}_n(a,b) = \langle x_j^{a+j-1},\ (x_1 x_2 \cdots x_n)^b : 1 \leq j \leq n \rangle$ in the polynomial ring R.

Theorem 1.2. *The number of standard monomials of* $\frac{R}{\mathcal{J}_n(a,b)}$ *is given by*

$$\dim_k \left(\frac{R}{\mathcal{J}_n(a,b)} \right) = a^{\overline{n}} - (a-b)^{\overline{n}},$$

where the nth rising power $t^{\overline{n}} = t(t+1) \cdots (t+n-1)$.

Clearly, $\mathcal{J}_n(2,1) = \mathcal{J}_n$ and $2^{\overline{n}} - 1^{\overline{n}} = (n+1)! - n! = n(n!)$, as desired.

In order to unify these results, we consider a monomial ideal $\mathcal{K}_n^{(s)}(a,b)$ in the polynomial ring R for $1 \leq s \leq n$ and positive integers a, b with $b < a$ given by $\mathcal{K}_n^{(s)}(a,b) = \langle x_j^{a+j-1},\ x_l^{a+s-1},\ (x_1x_2\cdots x_n)^b : 1 \leq j < s \leq l \leq n\rangle$.

Theorem 1.3. *The number of standard monomials of* $\frac{R}{\mathcal{K}_n^{(s)}(a,b)}$ *is given by*

$$\dim_k\left(\frac{R}{\mathcal{K}_n^{(s)}(a,b)}\right) = a^{\overline{s}}(a+s-1)^{n-s} - (a-b)^{\overline{s}}(a-b+s-1)^{n-s}.$$

Since $\mathcal{K}_n^{(1)}(a,b) = \mathcal{I}_n(a,b)$ and $\mathcal{K}_n^{(n)}(a,b) = \mathcal{J}_n(a,b)$, we need only to prove Theorem 1.3 as other results stated earlier are its particular cases. We give a proof of Theorem 1.3 in the last section.

Let $\mathcal{K}_n = \mathcal{K}_n^{(n-1)}(2,1)$. Then $\dim_k\left(\frac{R}{\mathcal{K}_n}\right) = n(n!) - (n-1)(n-1)!$. This shows that the integer sequence $\left\{\dim_k\left(\frac{R}{\mathcal{K}_n}\right)\right\}_{n\geq 1}$ is the sequence ($A001564$) in OEIS [9], whose nth term is the number of isolated fixed points in all permutations of $[n+1]$. Thus we conclude that the nth term in the integer sequences $(A045531), (A001563), (A001564)$ are given by enumeration of certain standard monomials.

2 Cellular resolution and Betti numbers

Cellular resolution of monomial ideals supported by labelled simplicial complexes or labelled polyhedral cell complexes were introduced in [1, 2]. For positive integers $b < a$ and $1 \leq s \leq n$, we consider the monomial ideal $\mathcal{K}_n^{(s)}(a,b)$ as defined in the introduction. We shall proceed to construct its cellular resolution.

Let Δ_{n-1} be the $(n-1)$-simplex on the vertex set $[n]$ and let $\widetilde{\Delta}_{n-1}$ be the simplicial complex obtain by join of the boundary complex $\partial\Delta_{n-1}$ of Δ_{n-1} with the barycentre (say, v) of the $n-1$ simplex Δ_{n-1}. Clearly, $\widetilde{\Delta}_{n-1}$ is an $n-1$-dimensional simplicial complex on the vertex set $[n] \cup \{v\} = \{1, 2, \ldots, n, v\}$. Also, an $i-1$-dimensional face F of $\widetilde{\Delta}_{n-1}$ is either of the form $F \subsetneq [n]$ with $|F| = i$ or $F = G \cup \{v\}$ for $G \subseteq [n]$ with $|G| = i-1$. Let $\mathcal{F}_{i-1} = \mathcal{F}_{i-1}(\widetilde{\Delta}_{n-1})$ be the set of $i-1$-dimensional faces of $\widetilde{\Delta}_{n-1}$ and $f_{i-1}(\widetilde{\Delta}_{n-1}) = |\mathcal{F}_{i-1}(\widetilde{\Delta}_{n-1})|$. For $1 \leq i \leq n-1$, we have $f_{i-1}(\widetilde{\Delta}_{n-1}) = \binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$. Also, $f_{n-1}(\widetilde{\Delta}_{n-1}) = \binom{n}{n-1} = n$.

A vertex $\{j\} \subseteq [n]$ of the simplicial complex $\widetilde{\Delta}_{n-1}$ can be naturally labelled with the monomial x_j^{a+j-1} or x_j^{a+s-1} according as $1 \leq j < s$ or $s \leq j \leq n$, respectively. The barycentre $\{v\}$ can be labelled with the monomial $\left(\prod_{l=1}^n x_l\right)^b$. The monomial label $\mathbf{x}^{\nu(F)} = \prod_{j=1}^n x_j^{\nu_j(F)}$ on any $i-1$-dimensional face F of $\widetilde{\Delta}_{n-1}$ is given by the LCM of the monomial labels on all the vertices of F. By convention, monomial label on the empty face $\emptyset$ is 1. For a subset $A \subsetneq [n]$, set $A_{<s} = \{j \in A : j < s\}$ and $A_{\geq s} = \{j \in A : j \geq s\}$. Clearly, if $F \subsetneq [n]$, then

$$\mathbf{x}^{\nu(F)} = \left(\prod_{j \in F_{<s}} x_j^{a+j-1}\right)\left(\prod_{l \in F_{\geq s}} x_l^{a+s-1}\right),$$

while if $F = G \cup \{v\}$ for $G \subsetneq [n]$, then

$$\mathbf{x}^{\nu(F)} = \left(\prod_{j \in G_{<s}} x_j^{a+j-1}\right)\left(\prod_{l \in G_{\geq s}} x_l^{a+s-1}\right)\left(\prod_{r \in [n]-G} x_r^{b}\right).$$

Equivalently, for $G \subsetneq [n]$, $\nu_j(G) = \nu_j(G \cup \{v\}) = \begin{cases} a-j+1 & \text{if } j \in G_{<s} \\ a-s+1 & \text{if } j \in G_{\geq s} \end{cases}$. Also, $\nu_j(G) = 0$ and $\nu_j(G \cup \{v\}) = b$ for $j \in [n] - G$. This shows that $\widetilde{\Delta}_{n-1}$ is a labelled simplicial complex and the monomial ideal generated by the vertex labels of $\widetilde{\Delta}_{n-1}$ is precisely the monomial ideal $\mathcal{K}_n^{(s)}(a,b)$. The free R-complex $\mathbb{F}_*(\widetilde{\Delta}_{n-1})$ associated to the labelled simplicial $\widetilde{\Delta}_{n-1}$ is given by

$$\mathbb{F}_*(\widetilde{\Delta}_{n-1}) : \ldots \longrightarrow \mathbb{F}_i \xrightarrow{\delta_i} \mathbb{F}_{i-1} \longrightarrow \ldots \longrightarrow \mathbb{F}_1 \longrightarrow \mathbb{F}_0 \longrightarrow 0, \tag{2.1}$$

where $\mathbb{F}_i = \bigoplus_{F \in \mathcal{F}_{i-1}(\widetilde{\Delta}_{n-1})} R[-\nu(F)]$ and δ_i is a differential (see [1, 2]).

Theorem 2.1. *The free complex $\mathbb{F}_*(\widetilde{\Delta}_{n-1})$ supported by $\widetilde{\Delta}_{n-1}$ is the minimal cellular resolution of $\mathcal{K}_n^{(s)}(a,b)$. Thus i^{th} Betti number $\beta_i(\mathcal{K}_n^{(s)}(a,b))$ of $\mathcal{K}_n^{(s)}(a,b)$ is given by*

$$\beta_i(\mathcal{K}_n^{(s)}(a,b)) = |\mathcal{F}_i(\widetilde{\Delta}_{n-1})| = f_i(\widetilde{\Delta}_{n-1}).$$

Proof. The complex $\mathbb{F}_*(\widetilde{\Delta}_{n-1})$ is a free resolution of $\frac{R}{\mathcal{K}_n^{(s)}(a,b)}$, provided the truncated subcomplex $(\widetilde{\Delta}_{n-1})_{\leq \mathbf{b}} = \{F \in \widetilde{\Delta}_{n-1} : \nu(F) \leq \mathbf{b}\}$ (for any $\mathbf{b} \in \mathbb{N}^n$) is acyclic [5](Proposition 4.5). Since, $(\widetilde{\Delta}_{n-1})_{\leq \mathbf{b}}$ is either $\{\emptyset\}$ or a simplex, and $\mathbf{x}^{\nu(F)} \neq \mathbf{x}^{\nu(F')}$ for every proper subface F' of any face F, it follows that $\mathbb{F}_*(\widetilde{\Delta}_{n-1})$ is a minimal free resolution. Hence, $\beta_i(\mathcal{K}_n^{(s)}(a,b)) = f_i(\widetilde{\Delta}_{n-1})$. □ □

It follows from the Theorem 2.1 that the ideals $I(\mathbf{u})^{[\mathbf{n}]}, \mathcal{J}_n$ and $\mathcal{K}_n^{(s)}(a,b)$ have the same Betti numbers. Also, using the (minimal) free resolution $\mathbb{F}_*(\widetilde{\Delta}_{n-1})$, we can easily compute the (fine) Hilbert series $H\left(\frac{R}{\mathcal{K}_n^{(s)}(a,b)},\mathbf{x}\right)$. In fact,

$$\begin{aligned}H\left(\frac{R}{\mathcal{K}_n^{(s)}(a,b)},\mathbf{x}\right) &= \sum_{i=0}^{n}(-1)^i H\left(\mathbb{F}_i,\mathbf{x}\right)\\ &= \frac{1}{(1-x_1)\dots(1-x_n)} \quad \sum_{i=0}^{n}(-1)^i\left(\sum_{F\in\mathcal{F}_{i-1}(\widetilde{\Delta}_{n-1})}\mathbf{x}^{\nu(F)}\right).\end{aligned}$$

3 Standard monomials

We have already mentioned that Theorems 1.1 and 1.2 are particular cases of Theorem 1.3. In this section, we give a proof of Theorem 1.3.

Proof. Since the quotient $\frac{R}{\mathcal{K}_n^{(s)}(a,b)}$ is an Artinian k-algebra, the number of standard monomials of $\frac{R}{\mathcal{K}_n^{(s)}(a,b)}$ is precisely the dimension $\dim_k\left(\frac{R}{\mathcal{K}_n^{(s)}(a,b)}\right)$. Also, the rational function $H(\frac{R}{\mathcal{K}_n^{(s)}(a,b)},\ \mathbf{x}) = \frac{Q(\mathbf{x})}{(1-x_1)\dots(1-x_n)}$ reduces to a finite sum of all standard monomials of $\frac{R}{\mathcal{K}_n^{(s)}(a,b)}$, where

$$Q(\mathbf{x}) = \sum_{i=0}^{n}(-1)^i \sum_{F\in\mathcal{F}_{i-1}(\widetilde{\Delta}_{n-1})}\mathbf{x}^{\nu(F)}.$$

Hence, on applying limits $\mathbf{x} = (x_1,\dots,x_n) \to \mathbf{1} = (1,\dots,1)$ in the rational function $H(\frac{R}{\mathcal{K}_n^{(s)}(a,b)},\ \mathbf{x})$, we obtain $\dim_k\left(\frac{R}{\mathcal{K}_n^{(s)}(a,b)}\right) = \lim_{\mathbf{x}\to\mathbf{1}}\left(\frac{Q(\mathbf{x})}{(1-x_1)\dots(1-x_n)}\right)$. Thus by L'Hospital rule, we get

$$\dim_k\left(\frac{R}{\mathcal{K}_n^{(s)}(a,b)}\right) = \frac{1}{(-1)^n}\ \frac{\partial^n Q(\mathbf{x})}{\partial x_1\partial x_2\dots\partial x_n}\bigg|_{\mathbf{x}=\mathbf{1}}.$$

The term corresponding to a face $F\in\mathcal{F}_{i-1}(\widetilde{\Delta}_{n-1})$ survives in the partial derivative $\frac{\partial^n Q(\mathbf{x})}{\partial x_1\partial x_2\dots\partial x_n}$ if and only if $F = G\cup\{v\}$ for $G\subsetneq[n]$. This shows that

$$\begin{aligned}\dim_k\left(\frac{R}{\mathcal{K}_n^{(s)}(a,b)}\right) &= \sum_{i=1}^{n}(-1)^{n-i}\left(\sum_{\substack{G\subsetneq[n];\\|G|=i-1}}\prod_{j\in G}\nu_j(G)\right)b^{n-i+1}\\ &= -\sum_{i=1}^{n}\left(\sum_{\substack{G\subsetneq[n];\\|G|=i-1}}\prod_{j\in G}\nu_j(G)\right)(-b)^{n-i+1}.\end{aligned}$$

Now on expanding the product $\prod_{j=1}^{n}(y_j - b)$ as sum of powers of $(-b)$, we get

$$\prod_{j=1}^{n}(y_j - b) = \left(\prod_{j=1}^{n} y_j\right) + \sum_{i=1}^{n}\left(\sum_{\substack{G\subsetneq[n];\\ |G|=i-1}}\left(\prod_{j\in G} y_j\right)\right)(-b)^{n-i+1}.$$

On putting $y_j = \begin{cases} a-j+1 & \text{if } 1 \leq j < s \\ a-s+1 & \text{if } s \leq j \leq n \end{cases}$ in the last identity, we have

$$(a-b)^{\overline{s}}(a+s-1-b)^{n-s} = a^{\overline{s}}(a+s-1)^{n-s} + \sum_{i=1}^{n}\left(\sum_{\substack{G\subsetneq[n];\\ |G|=i-1}}\prod_{j\in G}\nu_j(G)\right)(-b)^{n-i+1}.$$

Hence, $\dim_k\left(\frac{R}{\mathcal{K}_n^{(s)}(a,b)}\right) = (a)^{\overline{s}}(a+s-1)^{n-s} - (a-b)^{\overline{s}}(a+s-1-b)^{n-s}$. This completes the proof of Theorem 1.3. □ □

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