Some Integer Sequences and Standard Monomials

Ajay Kumar\textsuperscript{1} & Chanchal Kumar\textsuperscript{2}

\textsuperscript{1}DAV University, Jalandhar, Punjab-144012, India
\textsuperscript{2}IISEI Mohali, Knowledge City, Sector 81, SAS Nagar, Punjab-140306, India.

ajaychhabra.msc@gmail.com \& chanchal@iisermohali.ac.in

Abstract

For some classes of monomial ideals $J$ in the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field $k$, the number $\dim_k (R/J)$ of standard monomials of the Artinian quotient $R/J$ has a nice combinatorial interpretation. In this paper, we have studied many monomial ideals in $R$ and obtain explicit formula for enumerating their standard monomials.

Subject class [2010]: 05E40, 13D02

Keywords: Cellular resolution; Betti number; standard monomials.

1 Introduction

Let $\mathfrak{S}_n$ be the group of all permutations of $[n] = \{1, 2, \ldots, n\}$. For each permutation $\sigma \in \mathfrak{S}_n$, let $x^{\sigma} = \prod_{i=1}^{n} x_{\sigma(i)}$ be the associated monomial in the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field $k$. The monomial ideal $I_{\mathfrak{S}_n} = I(1, 2, \ldots, n) = \langle x^{\sigma} : \sigma \in \mathfrak{S}_n \rangle$ in $R$ generated by monomial associated to all permutations $\sigma \in \mathfrak{S}_n$ is called a permutohedron ideal. The Alexander dual $I_{\mathfrak{S}_n}^{[n]}$ of $I_{\mathfrak{S}_n}$ with respect to $n = (n, \ldots, n) \in \mathbb{N}^n$ is given by

$$I_{\mathfrak{S}_n}^{[n]} = \left\langle \left( \prod_{j \in A} x_j \right)^{n - |A| + 1} : \emptyset \neq A \subseteq [n] \right\rangle.$$

Further, standard monomials of $R/I_{\mathfrak{S}_n}^{[n]}$ correspond bijectively to the ordinary parking functions of length $n$, and hence $\dim_k \left( \frac{R}{I_{\mathfrak{S}_n}^{[n]}} \right) = (n + 1)^{n-1}$. For more on Alexander duals, we refer to [5, 6].

For a sequence $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ with $\lambda_1 \geq \ldots \geq \lambda_n \geq 1$, the monomial ideal $I_{\lambda} = \left\langle \left( \prod_{j \in A} x_j \right)^{\lambda_{n-|A|+1}} : \emptyset \neq A \subseteq [n] \right\rangle$ has standard monomial basis corresponding to $\lambda$-parking functions of length $n$. We recall that a sequence $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is called a $\lambda$-parking function of length $n$ if the non-decreasing rearrangement $q_1 \leq q_2 \leq \ldots \leq q_n$
function $\Lambda(\lambda_1, \ldots, \lambda_n)$ is a matrix with $m_{ij} = \left\{ \begin{array}{ll} \frac{n!}{(j-i+1)!} & i \leq j + 1, \\ 0 & i > j + 1. \end{array} \right.$ For $\lambda$-parking functions and Steck determinant formula, we refer to [3, 7, 8].

More generally, for any finite (oriented) graph $G$, Postnikov and Shapiro [8] associated a monomial ideal $M_G$ in $R$ so that the Artinian $k$-algebra $R / M_G$ has the standard monomial basis corresponding to the $G$-parking functions. The number $\dim_k \left( \frac{R}{M_G} \right)$ of $G$-parking functions equals the number of (oriented) spanning trees of $G$, and hence $\dim_k \left( \frac{R}{M_G} \right) = \det(L_G)$, where $L_G$ is an $n \times n$ matrix called truncated Laplace matrix of $G$.

The monomial ideal $J_n = \langle x_i^n, (x_i x_j)^{n-(j-i)} : 1 \leq l \leq n; 1 \leq i < j \leq n \rangle$ in the polynomial ring $R$ has been studied in [4] and it is shown that the integer sequence $\left\{ \dim_k \left( \frac{R}{J_n} \right) \right\}_{n \geq 1}$ is same as the sequence (A000262) in OEIS [9], whose $n^{th}$ term is the number of partitions of $[n] = \{1, 2, \ldots, n\}$ into sets of lists. Thus

\[
\dim_k \left( \frac{R}{J_n} \right) = \det([a_{ij}]_{1 \leq i,j \leq n}); \quad \text{where} \quad a_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } i > j, \\ i & \text{if } i = j, \\ i-j & \text{if } i < j. \end{array} \right.
\]

In this paper, we study monomial ideals whose standard monomials have nice combinatorial interpretations. Let $I(\mathbf{u})$ be the multipermutohedron ideal corresponding to $\mathbf{u} = (1, \ldots, 1, n) \in \mathbb{N}^n$. The minimal generators of $I(\mathbf{u})$ are precisely the monomials $x_1^n x_2 \cdots x_n$, $x_1 x_2^n \cdots x_n$, $\ldots$, $x_1 x_2 \cdots x_n^n$. Also, the Alexander dual $I(\mathbf{u})^{[n]}$ of $I(\mathbf{u})$ with respect to $\mathbf{n} = (n, \ldots, n)$ is given by

\[
I(\mathbf{u})^{[n]} = \langle x_j^n, x_1 x_2 \cdots x_n : 1 \leq j \leq n \rangle.
\]

Using the cellular resolution of $I(\mathbf{u})^{[n]}$, we show that

\[
\dim_k \left( \frac{R}{I(\mathbf{u})^{[n]}} \right) = n^n - (n - 1)^n.
\]

It is now clear that the integer sequence $\left\{ \dim_k \left( \frac{R}{I(\mathbf{u})^{[n]}} \right) \right\}_{n \geq 1}$ coincides with the integer sequence (A045531) in OEIS [9], whose $n^{th}$ term is the number of functions $\phi : [n] \rightarrow [n]$, where \( q_i \) is a number of monomials $x_i^n$, $x_1 x_2 \cdots x_n$, $x_1 x_2^n \cdots x_n$, $\ldots$, $x_1 x_2 \cdots x_n^n$.
with \( \phi(1) = 1 \) (such functions are called *sticky functions* of \([n]\)). It is an interesting problem to construct an explicit bijection between the set of standard monomials of \( R_{\mathcal{I}(\mathbf{u})}^n \) and the set of sticky functions of \([n]\).

More generally, for positive integers \( a, b \) with \( b < a \), we consider the monomial ideal \( \mathcal{I}_n(a, b) = \langle x_j^a, (x_1x_2 \cdots x_n)^b : 1 \leq j \leq n \rangle \) in \( R \). Clearly, \( \mathcal{I}_n(n, 1) = I(\mathbf{u})^n \).

**Theorem 1.1.** The number of standard monomials of \( R_{\mathcal{I}_n(a, b)}^n \) is given by

\[
\dim_k \left( \frac{R}{\mathcal{I}_n(a, b)} \right) = a^n - (a - b)^n.
\]

Since \( \mathcal{I}_n(a, b) = I_\lambda \) for \( \lambda = (a, \ldots, a, b) \in \mathbb{N}^n \), the standard monomials of \( \mathcal{I}_n(a, b) \) correspond to the \( \lambda \)-parking functions of length \( n \). Hence by Steck determinant formula, we obtain a beautiful identity \((n!) \det(\Lambda(a, \ldots, a, b)) = a^n - (a - b)^n \).

**Corollary 1.1.** For any positive integers \( b < a \),

\[
(n!) \det \begin{bmatrix}
 b & b^2 & b^3 & \cdots & b^{n-1} & b^n \\
 1 & a & a^2 & \cdots & a^{n-2} & a^{n-1} \\
 0 & 1 & a & \cdots & a^{n-3} & a^{n-2} \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & a & a^2 \\
 0 & 0 & 0 & \cdots & 1 & 1
\end{bmatrix} = a^n - (a - b)^n.
\]

We consider another monomial ideal \( \mathcal{J}_n \) in the polynomial ring \( R \) given by \( \mathcal{J}_n = \langle x_j^{a+1}, x_1x_2 \cdots x_n : 1 \leq j \leq n \rangle \). We show that \( \dim_k \left( \frac{R}{\mathcal{J}_n} \right) = n(n!) \). Thus, the integer sequence \( \{\dim_k \left( \frac{R}{\mathcal{J}_n} \right) \}_{n \geq 1} \) is the integer sequence \((A001563)\) in OEIS \([9]\), whose \( n \)th term is the number of permutations of \([n + 1]\) in which \( n + 1 \) is not fixed. Again, it is an interesting problem to construct an explicit bijection between these objects.

For positive integers \( a, b \) with \( b < a \), we consider the monomial ideal \( \mathcal{J}_n(a, b) = \langle x_j^{a+b-1}, (x_1x_2 \cdots x_n)^b : 1 \leq j \leq n \rangle \) in the polynomial ring \( R \).

**Theorem 1.2.** The number of standard monomials of \( R_{\mathcal{J}_n(a, b)}^n \) is given by

\[
\dim_k \left( \frac{R}{\mathcal{J}_n(a, b)} \right) = a^n - (a - b)^n,
\]

where the \( n \)th rising power \( t^n = t(t + 1) \cdots (t + n - 1) \).
Theorem 1.3. The number of standard monomials of $n$-complex on the vertex set $\{\nu_i \}_{i=1}^n$ is given by

$$\dim_k \left( \frac{R}{K_n^{(s)}(a,b)} \right) = a^s (a + s - 1)^{n-s} - (a - b)^s (a - b + s - 1)^{n-s}.$$ 

Since $K_n^{(1)}(a,b) = J_n(a,b)$ and $K_n^{(n)}(a,b) = J_n(a,b)$, we need only to prove Theorem 1.3 as other results stated earlier are its particular cases. We give a proof of Theorem 1.3 in the last section.

Let $K_n = K_n^{(n-1)}(2,1)$. Then $\dim_k \left( \frac{R}{K_n} \right) = n(n!) - (n-1)(n-1)!$. This shows that the integer sequence $\{\dim_k \left( \frac{R}{K_n} \right) \}_{n \geq 1}$ is the sequence (A001564) in OEIS [9], whose $n$th term is the number of isolated fixed points in all permutations of $[n+1]$. Thus we conclude that the $n$th term in the integer sequences (A045531), (A001563), (A001564) are given by enumeration of certain standard monomials.

2 Cellular resolution and Betti numbers

Cellular resolution of monomial ideals supported by labelled simplicial complexes or labelled polyhedral cell complexes were introduced in [1, 2]. For positive integers $b < a$ and $1 \leq s < n$, we consider the monomial ideal $K_n^{(s)}(a,b)$ as defined in the introduction. We shall proceed to construct its cellular resolution.

Let $\Delta_{n-1}$ be the $(n-1)$-simplex on the vertex set $[n]$ and let $\widetilde{\Delta}_{n-1}$ be the simplicial complex obtain by join of the boundary complex $\partial \Delta_{n-1}$ of $\Delta_{n-1}$ with the barycentre (say, $v$) of the $n-1$ simplex $\Delta_{n-1}$. Clearly, $\widetilde{\Delta}_{n-1}$ is an $n-1$-dimensional simplicial complex on the vertex set $[n] \cup \{v\} = \{1, 2, \ldots, n, v\}$. Also, an $i-1$-dimensional face $F$ of $\widetilde{\Delta}_{n-1}$ is either of the form $F \subsetneq [n]$ with $|F| = i$ or $F = G \cup \{v\}$ for $G \subsetneq [n]$ with $|G| = i - 1$. Let $\mathcal{F}_{i-1} = \mathcal{F}_{i-1}(\widetilde{\Delta}_{n-1})$ be the set of $i-1$-dimensional faces of $\Delta_{n-1}$ and $f_{i-1}(\widetilde{\Delta}_{n-1}) = |\mathcal{F}_{i-1}(\widetilde{\Delta}_{n-1})|$. For $1 \leq i \leq n-1$, we have $f_{i-1}(\Delta_{n-1}) = \binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$. Also, $f_{n-1}(\widetilde{\Delta}_{n-1}) = \binom{n}{n-1} = n$.

A vertex $\{j\} \subsetneq [n]$ of the simplicial complex $\widetilde{\Delta}_{n-1}$ can be naturally labelled with the monomial $x_j^{a+j-1}$ or $x_j^{a+s-1}$ according as $1 \leq j < s$ or $s \leq j \leq n$, respectively. The barycentre $\{v\}$ can be labelled with the monomial $(\prod_{i=1}^n x_i)^b$. The monomial label $x^\mu(F) = \prod_{j=1}^n x_j^{\nu_j(F)}$ on any $i-1$-dimensional face $F$ of $\widetilde{\Delta}_{n-1}$ is given by the LCM of the monomial labels on all the vertices of $F$. By convention, monomial label on the empty
Proof. The complex $\mathbb{F}_*(\tilde{\Delta}_{n-1})$ is a free resolution of $\frac{R}{\mathcal{K}_n^{(s)}(a,b)}$, provided the truncated subcomplex $(\tilde{\Delta}_{n-1})_{\leq b} = \{ F \in \tilde{\Delta}_{n-1} : \nu(F) \leq b \}$ (for any $b \in \mathbb{N}^n$) is acyclic [5](Proposition 4.5). Since, $(\tilde{\Delta}_{n-1})_{\leq b}$ is either $\emptyset$ or a simplex, and $x^{\nu(F)} \neq x^{\nu(F')} \neq x^{\nu(F'')}$ for every proper subface $F'$ of any face $F$, it follows that $\mathbb{F}_*(\tilde{\Delta}_{n-1})$ is a minimal free resolution. Hence, $\beta_i(\mathcal{K}_n^{(s)}(a,b)) = f_i(\tilde{\Delta}_{n-1})$. \hfill \Box

It follows from the Theorem 2.1 that the ideals $I((a)_{\lfloor [n]}$, $\mathcal{J}_n$ and $\mathcal{K}_n^{(s)}(a,b)$ have the same Betti numbers. Also, using the (minimal) free resolution $\mathbb{F}_*(\tilde{\Delta}_{n-1})$, we can easily compute
the (fine) Hilbert series \( H\left( \frac{R}{\mathcal{K}^{(s)}_n(a,b)}, x \right) \). In fact,

\[
H\left( \frac{R}{\mathcal{K}^{(s)}_n(a,b)}, x \right) = \sum_{i=0}^{n} (-1)^i H\left( F_i, x \right) = \frac{1}{(1 - x_1) \cdots (1 - x_n)} \sum_{i=0}^{n} (-1)^i \sum_{F \in F_i} x^{\nu(F)}.
\]

3 Standard monomials

We have already mentioned that Theorems 1.1 and 1.2 are particular cases of Theorem 1.3. In this section, we give a proof of Theorem 1.3.

**Proof.** Since the quotient \( \frac{R}{\mathcal{K}^{(s)}_n(a,b)} \) is an Artinian \( k \)-algebra, the number of standard monomials of \( \frac{R}{\mathcal{K}^{(s)}_n(a,b)} \) is precisely the dimension \( \dim_k \left( \frac{R}{\mathcal{K}^{(s)}_n(a,b)} \right) \). Also, the rational function

\[
H\left( \frac{R}{\mathcal{K}^{(s)}_n(a,b)}, x \right) = \frac{Q(x)}{(1 - x_1) \cdots (1 - x_n)} \text{ reduces to a finite sum of all standard monomials of } \frac{R}{\mathcal{K}^{(s)}_n(a,b)},
\]

where

\[
Q(x) = \sum_{i=0}^{n} (-1)^i \sum_{F \in F_i} x^{\nu(F)}.
\]

Hence, on applying limits \( x = (x_1, \ldots, x_n) \to 1 = (1, \ldots, 1) \) in the rational function \( H\left( \frac{R}{\mathcal{K}^{(s)}_n(a,b)}, x \right) \), we obtain \( \dim_k \left( \frac{R}{\mathcal{K}^{(s)}_n(a,b)} \right) = \lim_{x \to 1} \left( \frac{Q(x)}{(1 - x_1) \cdots (1 - x_n)} \right) \). Thus by L’Hospital rule, we get

\[
\dim_k \left( \frac{R}{\mathcal{K}^{(s)}_n(a,b)} \right) = \frac{1}{(-1)^n} \frac{\partial^n Q(x)}{\partial x_1 \partial x_2 \cdots \partial x_n} \bigg|_{x=1}.
\]

The term corresponding to a face \( F \in F_i \subset (\bar{\Delta}_{n-1}) \) survives in the partial derivative \( \frac{\partial^n Q(x)}{\partial x_1 \partial x_2 \cdots \partial x_n} \) if and only if \( F = G \cup \{v\} \) for \( G \subseteq [n] \). This shows that

\[
\dim_k \left( \frac{R}{\mathcal{K}^{(s)}_n(a,b)} \right) = \sum_{i=1}^{n} (-1)^{n-i} \left( \sum_{G \subseteq [n]; |G| = i-1} \prod_{j \in G} \nu_j(G) \right) b^{n-i+1}
\]

\[
= -\sum_{i=1}^{n} \left( \sum_{G \subseteq [n]; |G| = i-1} \prod_{j \in G} \nu_j(G) \right) (-b)^{n-i+1}.
\]
Now on expanding the product $\prod_{j=1}^n (y_j - b)$ as sum of powers of $(-b)$, we get

$$\prod_{j=1}^n (y_j - b) = \left( \prod_{j=1}^n y_j \right) + \sum_{i=1}^n \left( \sum_{G \subseteq [n]; |G| = i-1} \left( \prod_{j \in G} y_j \right) \right)(-b)^{n-i+1}. $$

On putting $y_j = \begin{cases} a - j + 1 & \text{if } 1 \leq j < s \\ a - s + 1 & \text{if } s \leq j \leq n \end{cases}$ in the last identity, we have

$$(a - b)^{s}(a + s - 1 - b)^{n-s} = a^{s}(a + s - 1)^{n-s} + \sum_{i=1}^n \left( \sum_{G \subseteq [n]; |G| = i-1} \prod_{j \in G} \nu_j(G) \right)(-b)^{n-i+1}. $$

Hence, $\dim_k \left( \frac{R}{\mathcal{I}^s(a,b)} \right) = (a)^{s}(a + s - 1)^{n-s} - (a - b)^{s}(a + s - 1 - b)^{n-s}$. This completes the proof of Theorem 1.3. $\square$

References


   http://www.research.att.com/~njas/sequences/