Weaving K-Fusion Frames in Hilbert Spaces

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Abstract

Motivated by a new concept of weaving frames in separable Hilbert spaces by Bemrose, Casazza, Gröchenig, Lammers and Lynch [Weaving Frames, Oper. Matrices, 10 (4) (2016), 1093–1116], we study weaving properties of K-fusion frames in Hilbert space. We present necessary and sufficient conditions for weaving K-fusion frames in separable Hilbert spaces. A Paley-Wiener type perturbation result for weaving K-fusion frames is given.

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1 Introduction

In 1952, Duffin and Schaffer [12] introduced the concept of a frame in separable Hilbert spaces. A countable sequence $\{x_k\}_{k\in\mathbb{I}} \subset \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$ if there exist constants $0 < \alpha_o \leq \beta_o < \infty$ such that

$$\alpha_o\|x\|^2 \leq \sum_{k\in\mathbb{I}} |\langle x, x_k\rangle|^2 \leq \beta_o\|x\|^2 \text{ for all } x \in \mathcal{H}.$$

The numbers α_o and β_o are called *lower* and *upper frame bounds*, respectively. If it is possible to choose $\alpha_o = \beta_o$, then we say that the frame $\{x_k\}_{k\in\mathbb{I}}$ is *tight*. Following three operators are associated with a frame $\{x_k\}_{k=1}^{\infty}$ for $\mathcal{H}$:

$$\textit{pre-frame operator} \quad T : \ell^2(\mathbb{I}) \to \mathcal{H}, \quad T\{c_k\}_{k=1}^{\infty} = \sum_{k\in\mathbb{I}} c_k x_k, \ \{c_k\}_{k\in\mathbb{I}} \in \ell^2(\mathbb{I}),$$

$$\textit{analysis operator} \ \ (\textit{adjoint of } T) \quad T^* : \mathcal{H} \to \ell^2(\mathbb{I}), \quad T^*x = \{\langle f, x_k\rangle\}_{k\in\mathbb{I}}, \ \ x \in \mathcal{H},$$

$$\textit{frame operator} \quad \Lambda = TT^* : \mathcal{H} \to \mathcal{H}, \quad \Lambda x = \sum_{k\in\mathbb{I}} \langle x, x_k\rangle x_k, \ x \in \mathcal{H}.$$

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Λ is a bounded linear positive and invertible operator on $\mathcal{H}$. Thus, each $x \in \mathcal{H}$ has the expansion

$$x = \Lambda\Lambda^{-1}x = \sum_{k\in\mathbb{I}} \langle \Lambda^{-1}x, x_k\rangle x_k = \sum_{k\in\mathbb{I}} \langle x, \Lambda^{-1}x_k\rangle x_k.$$

The scalars $\{\langle \Lambda^{-1}x, x_k\rangle\}_{k\in\mathbb{I}}$ are called *frame coefficients* of the vector $x \in \mathcal{H}$. The representation of x in the reconstruction formula need not be unique. Thus, frames allow each element in the space to be written as a linear combination of frame elements, where linear independence of frame elements is not required. Nowadays frames have potential applications in applied mathematics, see [4, 6, 16, 19] and references therein.

Casazza and Kutyniok [5] introduced the notion of *frames of subspaces* or *fusion frames* in separable Hilbert spaces which provide a suitable mathematical framework to design and analyze applications related to distributed processing. Fusion frames provide a natural mathematical framework for two-stage data processing. A fusion frame is a frame like collection of subspaces in a Hilbert space. Very recently, Bemrose et al. [1] introduced a new concept of "weaving frames" (Definition 2.5) in separable Hilbert spaces which is motivated by a problem regarding distributed signal processing. Găvruta in [15] introduced and studied K-frames in Hilbert spaces to study atomic systems with respect to a bounded linear operator K on Hilbert spaces. There are many differences between standard frames and K-frames, so we study weaving K-fusion frames in Hilbert spaces.

Outline: Section 2 contains the necessary background about fusion frames, K-frames and weaving frames in separable Hilbert spaces to make the paper self-contained. In Section 3, we present necessary and sufficient conditions for weaving K-fusion frames in separable Hilbert spaces. In one of the directions of applications of perturbation theory in frames, we give a Paley-Wiener type perturbation result for weaving K-fusion frames.

2 Preliminaries

$\mathbb{I}$ denotes a countable indexing set and $\mathbb{N}$ is the set of positive integers. The family of all bounded linear operators from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X},\mathcal{Y})$. If $\mathcal{X} = \mathcal{Y}$, then we write $\mathcal{B}(\mathcal{X},\mathcal{Y}) = \mathcal{B}(\mathcal{X})$. $\text{ran}(K)$ denote the range of an operator $K \in \mathcal{B}(\mathcal{X},\mathcal{Y})$. Let $\mathcal{H}$ be a separable complex Hilbert spaces and let $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ be a sequence of closed subspaces of $\mathcal{H}$. $\mathcal{H}_i$ $(i \in \mathbb{I})$ are called *atomic spaces*. The Hilbert-adjoint operator of $U \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is denoted by U^*.

2.1 Fusion frames in Hilbert spaces:

Let $\{v_i\}_{i\in\mathbb{I}}$ be a countable family of weights, i.e., v_i >0 for all $i \in \mathbb{I}$.

Definition 2.1. *[5] A family of closed subspaces $\{W_i\}_{i\in\mathbb{I}}$ of an infinite dimensional separable Hilbert space $\mathcal{H}$ is a* fusion frame *with respect to $\{v_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$, if there exist constants $0 < C \leq D < \infty$ such that*

$$C\|x\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 ||\pi_{W_i}(x)||^2 \leq D\|x\|^2 \text{ for all } x \in \mathcal{H}. \tag{2.1}$$

The positive constants C and D are called *lower* and *upper fusion frame bounds*, respectively. The inequality in (2.1) is called the *fusion frame inequality.* The family $\{W_i\}_{i\in\mathbb{I}}$ is called a C-*tight fusion frame* with respect to $\{v_i\}_{i\in\mathbb{I}}$ if it is possible to choose $C = D$, a *parseval fusion frame* with respect to $\{v_i\}_{i\in\mathbb{I}}$ if it is possible to choose $C = D = 1$ and an *orthonormal basis of subspaces* if $\mathcal{H}=\bigoplus_{i\in\mathbb{I}} W_i$. Also, fusion frame with respect to $\{v_i\}_{i\in\mathbb{I}}$ is called *v-uniform*, if $v{:=}v_i{=}v_j$ for all $i, j \in \mathbb{I}$. Now, if only upper inequality holds in (2.1) then we say $\{W_i\}_{i\in\mathbb{I}}$ a *Bessel fusion sequence* with respect to $\{v_i\}_{i\in\mathbb{I}}$ with *Bessel bound* D.

For a family of subspaces $\{W_i\}_{i\in\mathbb{I}}$ of $\mathcal{H}$, its associated space denoted by

$$\Big(\sum_{i\in\mathbb{I}}\bigoplus W_i\Big)_{\ell^2} = \ell^2(\mathcal{H},\mathbb{I}) = \Big\{\{x_i\}_{i\in\mathbb{I}} | x_i \in W_i \text{ and } \sum_{i\in\mathbb{I}} ||x_i||^2 < \infty\Big\}$$

with inner product given by

$$\langle \{x_i\}_{i\in\mathbb{I}}, \{y_i\}_{i\in\mathbb{I}}\rangle = \sum_{i\in\mathbb{I}} \langle x_i, y_i\rangle$$

is a Hilbert space.

2.2 K-frames in Hilbert spaces:

Feichtinger and Werther [13] introduced a family of analysis and synthesis systems with frame-like properties for closed subspaces of $\mathcal{H}$ and called it an *atomic system* (or *local atoms*). The motivation for the atomic system is based on examples arising in sampling theory, see [14]. One of the important properties of the atomic system is that it can generate a proper subspace even when they do not belong to them.

Definition 2.2. *[13] Let $\mathcal{H}_o$ be a closed subspace of $\mathcal{H}$. A countable sequence $\{x_k\}_{k=1}^{\infty} \subset \mathcal{H}$ is called a* family of local atoms *(or* atomic system*) for $\mathcal{H}_o$, if*

1. *$\{x_k\}_{k=1}^{\infty}$ is Bessel sequence in $\mathcal{H}$.*

2. *there exists a sequence of linear functionals $\{c_k\}$ and a real number $C_o > 0$ such that*

$$\sum_{k=1}^{\infty} |c_k(x)|^2 \leq C_o\|x\|^2 \text{ for all } x \in \mathcal{H}_o.$$

3. *$x = \sum_{k=1}^{\infty} c_k(x)x_k$ for all $x \in \mathcal{H}_o$.*

Remark 2.1. *One may observe that the linear functionals $\{c_k\}$ need to be defined on the subspace $\mathcal{H}_o$ only. We say that c_k are* associated functionals *of the local atoms $\{x_k\}$. The constant C_o is called the* atomic bound. *Furthermore, the partial sum $\sum_{k=1}^{N} c_k(x)x_k$ of the series in* (3) *can converge to x from "outside" of $\mathcal{H}_o$. Perturbation of local atoms were studied by Deepshikha and Vashisht [8]. Găvruta in [15] introduced and studied K-frames in Hilbert spaces to study atomic systems with respect to a bounded linear operator K on Hilbert spaces.*

Definition 2.3. *[15] Let K be a bounded linear operator on $\mathcal{H}$. A sequence $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H}$ is called a* K-frame *for $\mathcal{H}$, if there exist constants $A, B > 0$ such that*

$$A\|K^*x\|^2 \leq \sum_{k=1}^{\infty} |\langle x, x_k\rangle|^2 \leq B\|x\|^2 \text{ for all } x \in \mathcal{H}.$$

A and B are called *lower* and *upper K-frame bounds*, respectively. If K is the identity operator on $\mathcal{H}$, then K-frames are the standard frames. K-frames are more general than standard frames in the sense that the lower frame bound only holds for the elements in the range of K^*. Since a K-frame $\{x_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ is a Bessel sequence, we can define the frame operator associated with $\{x_k\}_{k=1}^{\infty}$. The frame operator of a K-frame is not invertible on $\mathcal{H}$ in general, but it is invertible on a subspace ran(K), where the range ran$(K) \subset \mathcal{H}$ is closed. Xiao et al. [7] introduced the concept of K-fusion frame in separable Hilbert spaces by combining the concepts of fusion frame and K-frame in Hilbert spaces. Their main contribution in the paper [7] includes the construction of K-fusion frames from fusion Bessel sequences.

Definition 2.4. *[7] Let $K \in \mathcal{B}(\mathcal{H})$ and let $\{v_i\}_{i\in\mathbb{I}}$ be a countable family of weights, i.e., v_i >0 for all $i \in \mathbb{I}$. A family of closed subspaces $\{W_i\}_{i\in\mathbb{I}}$ of an infinite dimensional separable Hilbert space $\mathcal{H}$ is a* K-fusion frame *with respect to $\{v_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$, if there exist scalars $0 < a_o \leq b_o < \infty$ such that*

$$a_o\|K^*x\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2||\pi_{W_i}(x)||^2 \leq b_o\|x\|^2 \text{ for all } x \in \mathcal{H}. \tag{2.2}$$

To conclude the section we give a key-theorem which can be found in [11].

Theorem 2.1. *[11] Let $L_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$, $L_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H})$ be two bounded operators. The following statements are equivalent:*

1. *$ran(L_1) \subset ranR(L_2)$.*
2. *$L_1L_1^* \leq \lambda^2 L_2L_2^*$ for some $\lambda \geq 0$.*
3. *there exists a $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $L_1 = L_2U$ and $\|U\|^2 = \inf\{\mu : L_1L_1^* \leq \mu^2 L_2L_2^*\}$.*

2.3 Background on Weaving frames:

For a fixed $m \in \mathbb{N}$, we write

$$[m] = \{1, 2, ..., m\}$$

Definition 2.5. *[1] A family of frames $\Big\{\{\phi_{ij}\}_{i\in\mathbb{I}} : j \in [m]\Big\}$ for $\mathcal{H}$ is said to be* woven *if there are universal constants A and B such that for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$ the family $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds A and B, respectively.*

Definition 2.6. *[1] A family of frames* $\Big\{\{\phi_{ij}\}_{i\in\mathbb{I}} : j \in [m]\Big\}$ *for* $\mathcal{H}$ *is (weakly)* woven *if for every partition* $\{\sigma_j\}_{j\in[m]}$ *of* $\mathbb{I}$*, the family* $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ *is a frame for* $\mathcal{H}$*.*

Bemrose et al. proved in [1] that this weaker form of weaving (given in Definition 2.6) is equivalent to weaving. They also considered the case of weaving Riesz bases and proved an abstract characterization of when two Riesz bases are woven. A geometric characterization of woven Riesz bases in terms of distance between subspaces of a Hilbert space $\mathcal{H}$ is given in [1]. Casazza and Lynch in [2] reviewed fundamental properties of weaving frames. They also discussed a weaving equivalent of an unconditional basis. Casazza, Freeman and Lynch [3] extended the concept of weaving Hilbert space frames to the Banach space setting. They proved that for any two approximate Schauder frames for a Banach space, every weaving is an approximate Schauder frame if and only if there exists a uniform constant $C \geq 1$ such that every weaving is a C-approximate Schauder frame. Deepshikha and Vashisht studied weaving properties of an infinite family of frames in separable Hilbert spaces in [9] . They proved in [9] that if an infinite family of frames woven into Riesz sequences, then it can be woven into Riesz bases. Deepshikha and Vashisht [10] studied vector-valued weaving frames in Hilbert spaces. Weaving frames in Hilbert spaces with respect to measure spaces can be found in [17, 18].

3 The Main Results

We begin by defining the weaving K-fusion frames in separable Hilbert spaces.

Definition 3.1. *Let* $K \in \mathcal{B}(\mathcal{H})$*. Two* K*-fusion frames* $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ *and* $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$ *for* $\mathcal{H}$ *are said to be woven if there are universal constants* $0 < A \leq B < \infty$ *so that for every subset* σ *of* $\mathbb{I}$*, the family* $\{(\mathcal{H}_i, v_i)\}_{i\in\sigma} \bigcup \{(\mathcal{W}_i, w_i)\}_{i\in\sigma^c}$ *is a* K*-fusion frame for* $\mathcal{H}$ *with lower and upper* K*-fusion frame bounds* A *and* B*, respectively.*

As in the case of standard weaving frames, see [1], every family of K-fusion frame has a universal K-fusion Bessel bound.

Proposition 3.1. *Let* $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ *and* $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$ *be* K*-fusion Bessel sequences in* $\mathcal{H}$ *with* K*-fusion Bessel bound* B_1 *and* B_2*, respectively. Then, for any subset* σ *of* $\mathbb{I}$*, the family* $\{(\mathcal{H}_i, v_i)\}_{i\in\sigma} \cup \{(\mathcal{W}_i, w_i)\}_{i\in\sigma^c}$ *is a* K*-fusion Bessel sequence with* K*-fusion Bessel bound* $B_1 + B_2$*. Indeed, let* σ *be any subset of* $\mathbb{I}$*. Then for all* $x \in \mathcal{H}$*, we have*

$$\sum_{i\in\sigma} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 + \sum_{i\in\sigma^c} w_i^2\|\pi_{\mathcal{W}_i}(x)\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 + \sum_{i\in\mathbb{I}} w_i^2\|\pi_{\mathcal{W}_i}(x)\|^2 \leq \Big(B_1 + B_2\Big)\|x\|^2.$$

This gives the required universal upper K*-fusion frame bound for* $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ *and* $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$*.*

The following theorem provides a necessary and sufficient condition for weaving K-fusion frames in terms of an operator. This is inspired by [15].

Theorem 3.1. *Suppose that* $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ *and* $\{\mathcal{W}_i\}_{i\in\mathbb{I}}$ *are two families of closed subspaces of* $\mathcal{H}$ *and let* $\{v_i\}_{i\in\mathbb{I}}$ *and* $\{w_i\}_{i\in\mathbb{I}}$ *be two families of positive real numbers, called weights. Then the following conditions are equivalent.*

1. *$\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$ are woven K-fusion frames for $\mathcal{H}$.*
2. *There exists $\alpha > 0$ such that for any subset σ of $\mathbb{I}$ there exists a bounded linear operator $U_\sigma : \Big(\sum\limits_{i\in\mathbb{I}} \oplus \mathcal{Z}_i\Big)_{\ell^2} \to \mathcal{H}$ (where, $\mathcal{Z}_i = \mathcal{H}_i$ for $i \in \sigma$ and $\mathcal{Z}_i = \mathcal{W}_i$ for $i \in \sigma^c$) such that*
$$U_\sigma\Big(\{y_i\}_{i\in\mathbb{I}}\Big) = \sum_{i\in\sigma} v_i\pi_{\mathcal{H}_i}(y_i) + \sum_{i\in\sigma^c} w_i\pi_{\mathcal{W}_i}(y_i),$$
where $\{y_i\}_{i\in\mathbb{I}} \in (\sum\limits_{i\in\mathbb{I}} \oplus\mathcal{Z}_i)_{\ell^2}$ and $\alpha KK^* \le U_\sigma U_\sigma^*$.

Proof. $(i) \Longrightarrow (ii)$: Suppose A is a universal lower K-fusion frame bound for $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$. For any subset σ of $\mathbb{I}$, let T_σ be the pre-frame operator associated with $\{(\mathcal{H}_i, v_i)\}_{i\in\sigma} \cup \{(\mathcal{W}_i, w_i)\}_{i\in\sigma^c}$.

Choose $\alpha := A$ and $U_\sigma := T_\sigma$. Then, for $\{y_i\}_{i\in\mathbb{I}} \in (\sum\limits_{i\in\mathbb{I}} \oplus\mathcal{Z}_i)_{\ell^2}$, we have
$$U_\sigma\Big(\{y_i\}_{i\in\mathbb{I}}\Big) = T_\sigma\Big(\{y_i\}_{i\in\mathbb{I}}\Big) = \sum_{i\in\sigma} v_i y_i + \sum_{i\in\sigma^c} w_i y_i = \sum_{i\in\sigma} v_i\pi_{\mathcal{H}_i}(y_i) + \sum_{i\in\sigma^c} w_i\pi_{\mathcal{W}_i}(y_i).$$
Furthermore, for all $x \in \mathcal{H}$ we have
$$\alpha\langle KK^*x, x\rangle = A\|K^*x\|^2 \le \sum_{i\in\sigma} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 + \sum_{i\in\sigma^c} w_i^2\|\pi_{\mathcal{W}_i}(x)\|^2 = \|T_\sigma^*(x)\|^2 = \|U_\sigma^*(x)\|^2.$$
This gives $\alpha KK^* \le U_\sigma U_\sigma^*$.

$(ii) \Longrightarrow (i)$: Let σ be any subset of $\mathbb{I}$. For any $\{y_i\}_{i\in\mathbb{I}} \in (\sum\limits_{i\in\mathbb{I}} \oplus\mathcal{Z}_i)_{\ell^2}$ (where, $\mathcal{Z}_i = \mathcal{H}_i$ for $i \in \sigma$ and $\mathcal{Z}_i = \mathcal{W}_i$ for $i \in \sigma^c$) and $x \in \mathcal{H}$, we compute:
$$\begin{aligned}\Big\langle U_\sigma\Big(\{y_i\}_{i\in\mathbb{I}}\Big), x\Big\rangle &= \Big\langle \sum_{i\in\sigma} v_i\pi_{\mathcal{H}_i}(y_i) + \sum_{i\in\sigma^c} w_i\pi_{\mathcal{W}_i}(y_i), x\Big\rangle \\ &= \sum_{i\in\sigma}\Big\langle v_i\pi_{\mathcal{H}_i}(y_i), x\Big\rangle + \sum_{i\in\sigma^c}\Big\langle w_i\pi_{\mathcal{W}_i}(y_i), x\Big\rangle \\ &= \sum_{i\in\sigma}\Big\langle y_i, v_i\pi_{\mathcal{H}_i}(x)\Big\rangle + \sum_{i\in\sigma^c}\Big\langle y_i, w_i\pi_{\mathcal{W}_i}(x)\Big\rangle \\ &= \Big\langle \{y_i\}_{i\in\mathbb{I}}, \{v_i\pi_{\mathcal{H}_i}(x)\}_{i\in\sigma} \cup \{w_i\pi_{\mathcal{W}_i}(x)\}_{i\in\sigma^c}\Big\rangle,\end{aligned}$$
This gives $U_\sigma^*(x) = \{v_i\pi_{\mathcal{H}_i}(x)\}_{i\in\sigma} \cup \{w_i\pi_{\mathcal{W}_i}(x)\}_{i\in\sigma^c}$. Since $\alpha KK^* \le U_\sigma U_\sigma^*$, we have
$$\begin{aligned}\alpha\|K^*x\|^2 &= \alpha\langle KK^*x, x\rangle \\ &\le \langle U_\sigma U_\sigma^* x, x\rangle \\ &= \|U_\sigma^* x\|^2 \\ &= \sum_{i\in\sigma} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 + \sum_{i\in\sigma^c} w_i^2\|\pi_{\mathcal{W}_i}(x)\|^2,\ x \in \mathcal{H}.\end{aligned}$$

This gives a universal lower K-fusion frame bound for the family $\{(\mathcal{H}_i, v_i)\}_{i\in\sigma}\cup\{(\mathcal{W}_i, w_i)\}_{i\in\sigma^c}$.

Next we show that $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ satisfies upper K-fusion frame inequality. For any $x \in \mathcal{H}$, we have

$$\sum_{i\in\mathbb{I}} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 = \|U_{\mathbb{I}}^* x\|^2 \leq \|U_{\mathbb{I}}^*\|^2\|x\|^2.$$

Similarly, $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$ satisfies upper K-fusion frame inequality. Thus, by applying Proposition 3.1 we can obtain a universal upper K-fusion frame bound. Hence $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$ are woven K-fusion frames for $\mathcal{H}$. □

The following theorem gives an application of Theorem 3.1.

Let $\mathbb{I} = \mathbb{N}$, $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of a complex separable Hilbert space $\mathcal{H}$ and K be the orthogonal projection from $\mathcal{H}$ onto $\overline{\text{span}}\{e_j\}_{j=2}^{\infty}$.

1. For each $j \in \mathbb{N}$, let $\mathcal{H}_j = \text{span}\{e_j\}$, $\mathcal{W}_j = \text{span}\{e_j, e_{j+1}\}$, and $v_j = w_j = 1$. For any subset σ of $\mathbb{N}$, define a linear operator $U_\sigma : \Big(\sum\limits_{i\in\mathbb{N}} \oplus\mathcal{Z}_i\Big)_{\ell^2} \to \mathcal{H}$ (where, $\mathcal{Z}_i = \mathcal{H}_i$ for $i \in \sigma$ and $\mathcal{Z}_i = \mathcal{W}_i$ for $i \in \sigma^c$) by

$$U_\sigma\Big(\{y_i\}_{i\in\mathbb{N}}\Big) = \sum_{i\in\sigma} \pi_{\mathcal{H}_i}(y_i) + \sum_{i\in\sigma^c} \pi_{\mathcal{W}_i}(y_i)$$

Then U_σ is bounded. Indeed for any $\{y_i\}_{i\in\mathbb{N}} \in (\sum\limits_{i\in\mathbb{N}} \oplus\mathcal{Z}_i)_{\ell^2}$ where $y_j = \alpha_j e_j$ for $j \in \sigma$ and $y_j = \alpha_j e_j + \beta_j e_{j+1}$ for $j \in \sigma^c$ $(j \in \mathbb{N})$, we have

$$\begin{aligned}
&\|U_\sigma\Big(\{y_j\}_{j\in\mathbb{N}}\Big)\|^2 \\
&= \Big\|\sum_{j\in\sigma} \pi_{\mathcal{H}_j}(y_j) + \sum_{j\in\sigma^c} \pi_{\mathcal{W}_j}(y_j)\Big\|^2 \\
&= \Big\|\sum_{j\in\sigma} \pi_{\mathcal{H}_j}(\alpha_j e_j) + \sum_{j\in\sigma^c} \pi_{\mathcal{W}_j}(\alpha_j e_j + \beta_j e_{j+1})\Big\|^2 \\
&= \Big\|\sum_{j\in\mathbb{N}} \alpha_j e_j + \sum_{j\in\sigma^c} \beta_j e_{j+1}\Big\|^2 \\
&\leq 2\Big\|\sum_{j\in\mathbb{N}} \alpha_j e_j\Big\|^2 + 2\Big\|\sum_{j\in\sigma^c} \beta_j e_{j+1}\Big\|^2 \\
&\leq 2\Big(\sum_{j\in\mathbb{N}} |\alpha_j|^2 + \sum_{j\in\sigma^c} |\beta_j|^2\Big) \\
&= \|\{y_j\}_{j\in\mathbb{N}}\|^2.
\end{aligned}$$

Furthermore, since $U_\sigma^*(x) = \{\pi_{\mathcal{H}_i}(x)\}_{i\in\sigma} \cup \{\pi_{\mathcal{W}_i}(x)\}_{i\in\sigma^c}$, we compute

$$\begin{aligned}
\|U_\sigma^*(x)\|^2 &= \sum_{j\in\sigma} \|\pi_{\mathcal{H}_j}(x)\|^2 + \sum_{j\in\sigma^c} \|\pi_{\mathcal{W}_j}(x)\|^2 \\
&= \sum_{j\in\sigma} |\langle x, e_j\rangle|^2 + \sum_{j\in\sigma} \Big(|\langle x, e_j\rangle|^2 + |\langle x, e_j\rangle|^2\Big) \\
&\geq \sum_{j\in\mathbb{N}} |\langle x, e_j\rangle|^2 \\
&\geq \|K^*x\|^2 \text{ for all } x \in \mathcal{H}
\end{aligned}$$

This gives $KK^* \leq U_\sigma U_\sigma^*$. Hence, by Theorem 3.1, $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{N}}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{N}}$ are woven K-fusion frames for $\mathcal{H}$.

2. For each $j \in \mathbb{N}$, let

$$\mathcal{H}_j = \begin{cases} \text{span}\{e_2\}, & j = 1, \\ \{0\}, & j = 2, \\ span\{e_j\}, & j \geq 3. \end{cases} \quad \text{and} \quad \mathcal{W}_j = \begin{cases} \{0\}, & j = 1, \\ \text{span}\{e_j\}, & j \geq 2. \end{cases}$$

and $v_j = w_j = 1$. Then, $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{N}}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{N}}$ are K-fusion frames for $\mathcal{H}$.
Choose $\sigma = \mathbb{N} \setminus \{1\}$. Assume that there exists a bounded linear operator

$$U_\sigma : \Big(\sum_{i\in\mathbb{N}} \oplus \mathcal{Z}_i\Big)_{\ell^2} \to \mathcal{H} \quad (\text{where, } \mathcal{Z}_i = \mathcal{H}_i \text{ for } i \in \sigma \text{ and } \mathcal{Z}_i = \mathcal{W}_i \text{ for } i \in \sigma^c)$$

such that

$$\begin{aligned}
U_\sigma\Big(\{y_i\}_{i\in\mathbb{N}}\Big) &= \sum_{i\in\sigma} \pi_{\mathcal{H}_i}(y_i) + \sum_{i\in\sigma^c} \pi_{\mathcal{W}_i}(y_i) \\
\text{i.e.,} \quad U_\sigma\Big(\{y_i\}_{i\in\mathbb{N}}\Big) &= \sum_{i=2}^{\infty} \pi_{\mathcal{H}_i}(y_i) + \pi_{\mathcal{W}_1}(y_1).
\end{aligned}$$

Then, for any $\alpha > 0$, we have

$$\alpha\langle KK^*e_2, e_2\rangle = \alpha\langle K^*e_2, K^*e_2\rangle = \alpha\|e_2\|^2 \geq 0 = \|U_\sigma^*(e_2)\|^2 = \langle U_\sigma U_\sigma^* e_2, e_2\rangle.$$

Thus, by Theorem 3.1, $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{N}}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{N}}$ are not woven.

Next theorem provides a necessary and sufficient condition for the weaving of K-fusion frames.

Theorem 3.2. *Two K-fusion frames $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$ for $\mathcal{H}$ are woven if and only if there exists $\alpha > 0$ such that for any $\sigma \subseteq \mathbb{I}$ and for any $x \in \mathcal{H}$ there exists a sequence $\{y_i^\sigma\}_{i\in\mathbb{I}} \in (\sum_{i\in\mathbb{I}} \oplus\mathcal{H}_i')_{\ell^2}$ (where, $\mathcal{Z}_i = \mathcal{H}_i$ for $i \in \sigma$ and $\mathcal{Z}_i = \mathcal{W}_i$ for $i \in \sigma^c$) such that*

$$\Big(\sum_{i\in\mathbb{I}} \|y_i^\sigma\|^2\Big)^{\frac{1}{2}} \le \alpha\|x\| \quad \text{and} \quad Kx = \sum_{i\in\sigma} v_i\pi_{\mathcal{H}_i}(y_i^\sigma) + \sum_{i\in\sigma^c} w_i\pi_{\mathcal{W}_i}(y_i^\sigma). \tag{3.1}$$

Proof. Assume that, A, B are the universal K-fusion frame bounds for $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$. Let $\sigma \subseteq \mathbb{I}$ be arbitrary and T_σ be the pre-frame operator associated with $\{(\mathcal{H}_i, v_i)\}_{i\in\sigma} \cup \{(\mathcal{W}_i, w_i)\}_{i\in\sigma^c}$. Then

$$\begin{aligned} A\langle KK^*x, x\rangle &= A\|K^*x\|^2 \\ &\le \sum_{i\in\sigma} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 + \sum_{i\in\sigma^c} w_i^2\|\pi_{\mathcal{W}_i}(x)\|^2 \\ &= \|T_\sigma^*x\|^2. \end{aligned}$$

So, $AKK^* \le T_\sigma T_\sigma^*$. By Theorem 2.1, there exists a bounded linear operator $\Theta_\sigma : \mathcal{H} \to \Big(\sum_{i\in\mathbb{I}} \oplus\mathcal{Z}_i\Big)_{\ell^2}$ such that

$$\|\Theta_\sigma\|^2 \le \frac{1}{\sqrt{A}} \quad \text{and} \quad K = T_\sigma\Theta_\sigma.$$

Therefore, for any $x \in \mathcal{H}$ the sequence $\Theta_\sigma(x) = \{y_i^\sigma\}_{i\in\mathbb{I}} \in (\sum_{i\in\mathbb{I}} \oplus\mathcal{H}_i)_{\ell^2}$ is such that

$$\Big(\sum_{i\in\mathbb{I}} \|y_i^\sigma\|^2\Big)^{\frac{1}{2}} = \|\Theta_\sigma(x)\| \le \|\Theta_\sigma\|\|x\| \le \frac{1}{A^{\frac{1}{4}}}\|x\|,$$

and

$$Kx = T_\sigma\Theta_\sigma x = T_\sigma\Big(\{y_i^\sigma\}_{i\in\mathbb{I}}\Big) = \sum_{i\in\sigma} v_i\pi_{\mathcal{H}_i}(y_i^\sigma) + \sum_{i\in\sigma^c} w_i\pi_{\mathcal{W}_i}(y_i^\sigma).$$

Conversely, assume that 3.1 holds. Then, for any $y \in \mathcal{H}$ we compute,

$$\begin{aligned}
\|K^*y\|^2 &= \sup_{\|x\|=1} |\langle K^*y, x\rangle|^2 \\
&= \sup_{\|x\|=1} |\langle y, Kx\rangle|^2 \\
&= \sup_{\|x\|=1} \Big|\Big\langle y, \sum_{i\in\sigma} v_i\pi_{\mathcal{H}_i}(y_i^\sigma) + \sum_{i\in\sigma^c} w_i\pi_{\mathcal{W}_i}(y_i^\sigma)\Big\rangle\Big|^2 \\
&= \sup_{\|x\|=1} \Big|\sum_{i\in\sigma}\Big\langle v_i\pi_{\mathcal{H}_i}(y), y_i^\sigma\Big\rangle + \sum_{i\in\sigma^c}\Big\langle w_i\pi_{\mathcal{W}_i}(y), y_i^\sigma\Big\rangle\Big|^2 \\
&\leq \sup_{\|x\|=1} \Big\{\Big|\sum_{i\in\sigma}\Big\langle v_i\pi_{\mathcal{H}_i}(y), y_i^\sigma\Big\rangle\Big| + \Big|\sum_{i\in\sigma^c}\Big\langle w_i\pi_{\mathcal{W}_i}(y), y_i^\sigma\Big\rangle\Big|\Big\}^2 \\
&\leq \sup_{\|x\|=1} \Big\{\sum_{i\in\sigma} v_i\|\pi_{\mathcal{H}_i}(y)\|\|y_i^\sigma\| + \sum_{i\in\sigma^c} w_i\|\pi_{\mathcal{W}_i}(y)\|\|y_i^\sigma\|\Big\}^2 \\
&\leq \sup_{\|x\|=1} 2\Big\{\Big(\sum_{i\in\sigma} v_i\|\pi_{\mathcal{H}_i}(y)\|\|y_i^\sigma\|\Big)^2 + \Big(\sum_{i\in\sigma^c} w_i\|\pi_{\mathcal{W}_i}(y)\|\|y_i^\sigma\|\Big)^2\Big\} \\
&\leq 2\sup_{\|x\|=1} \Big\{\sum_{i\in\sigma} v_i^2\|\pi_{\mathcal{H}_i}(y)\|^2\|y_i^\sigma\|^2 + \sum_{i\in\sigma^c} w_i^2\|\pi_{\mathcal{W}_i}(y)\|^2\|y_i^\sigma\|^2\Big\} \\
&\leq 2\sup_{\|x\|=1} \Big\{\sum_{i\in\sigma} v_i^2\|\pi_{\mathcal{H}_i}(y)\|^2\alpha^2\|x\|^2 + \sum_{i\in\sigma^c} w_i^2\|\pi_{\mathcal{W}_i}(y)\|^2\alpha^2\|x\|^2\Big\} \\
&= 2\alpha^2\Big\{\sum_{i\in\sigma} v_i^2\|\pi_{\mathcal{H}_i}(y)\|^2 + \sum_{i\in\sigma^c} w_i^2\|\pi_{\mathcal{W}_i}(y)\|^2\Big\}.
\end{aligned}$$

Thus, $\dfrac{1}{2\alpha^2}$ is a universal lower K-fusion frame bound for the family $\{(\mathcal{H}_i, v_i)\}_{i\in\sigma}\cup\{(\mathcal{W}_i, w_i)\}_{i\in\sigma^c}$ and the universal upper K-fusion frame bound can be obtained by Proposition 3.1. Hence $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$ are woven K-fusion frames for $\mathcal{H}$. □

Bemrose et al. [1] proved sufficient conditions for weaving frames by means of perturbation theory and diagonal dominance. Casazza, Freeman and Lynch [3] gave two perturbation results for woven approximate Schauder frames in Banach spaces. Perturbation of K-frames can be found in [8]. Deepshikha and Vashisht proved perturbation results for infinitely woven frames in [9]. The following theorem presents a Paley-Wiener type peturbation for weaving K-fusion frames.

Theorem 3.3. *Let $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$ be K-fusion frames in $\mathcal{H}$ with bounds A_1, B_1 and A_2, B_2, respectively. Assume that there exist scalars $\mu \geq 0,\ 0 \leq \lambda < \frac{1}{2}$ such that $A_1(\frac{1}{2} - \lambda) - \mu > 0$ and*

$$\sum_{i\in\mathbb{I}} \|(v_i\pi_{\mathcal{H}_i} - w_i\pi_{\mathcal{W}_i})(x)\|^2 \leq \lambda\sum_{i\in\mathbb{I}} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 + \mu\|K^*x\|^2 \text{ for all } x \in \mathcal{H}.$$

Then, $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$ are woven K-fusion frames.

Proof. For any $\sigma \subseteq \mathbb{I}$, we compute

$$\begin{aligned}
(B_1 + B_2)\|x\|^2 &\geq \sum_{i\in\sigma} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 + \sum_{i\in\sigma^c} w_i^2\|\pi_{\mathcal{W}_i}(x)\|^2 \\
&\geq \sum_{i\in\sigma} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 + \frac{1}{2}\sum_{i\in\sigma^c} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 - \sum_{i\in\sigma^c}\|v_i\pi_{\mathcal{H}_i}(x) - w_i\pi_{\mathcal{W}_i}(x)\|^2 \\
&= \frac{1}{2}\sum_{i\in\mathbb{I}} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 + \frac{1}{2}\sum_{i\in\sigma} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 - \sum_{i\in\sigma^c}\|v_i\pi_{\mathcal{H}_i}(x) - w_i\pi_{\mathcal{W}_i}(x)\|^2 \\
&\geq \frac{1}{2}\sum_{i\in\mathbb{I}} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 - \sum_{i\in\mathbb{I}}\|v_i\pi_{\mathcal{H}_i}(x) - w_i\pi_{\mathcal{W}_i}(x)\|^2 \\
&\geq \frac{1}{2}\sum_{i\in\mathbb{I}} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 - \lambda\sum_{i\in\mathbb{I}} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 - \mu\|K^*x\|^2 \\
&= (\frac{1}{2} - \lambda)\sum_{i\in\mathbb{I}} v_i^2\|\pi_{\mathcal{H}_i}(x)\|^2 - \mu\|K^*x\|^2 \\
&\geq \Big((\frac{1}{2} - \lambda)A_1 - \mu\Big)\|K^*x\|^2 \quad \text{for all } x \in \mathcal{H}.
\end{aligned}$$

Hence $\{(\mathcal{H}_i, v_i)\}_{i\in\mathbb{I}}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in\mathbb{I}}$ are woven K-fusion frames with universal K-fusion frame bounds $(B_1 + B_2)$ and $\Big(A_1(\frac{1}{2} - \lambda) - \mu\Big)$. □

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