Modified Weighted $(0, 1, 3)$-Interpolation

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Abstract

The aim of this paper is to give the existence, uniqueness, and explicit representation of the weighted $(0, 1, 3)$-interpolation polynomials on the roots all classical orthogonal polynomials.

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1 Introduction

On the suggestion of P.Turán, J. Balázs [1] initiated the study of weighted (0, 2) interpolation, which means the determination of the polynomial $S_n(x)$ of minimum possible degree satisfying the conditions:

$$S_n(x_{j,n}) = \alpha_{j,n}\ , \quad (w(x)S_n(x))''(x_{j,n}) = \beta_{j,n}, \quad j = 1(1)n \tag{1.1}$$

where $\alpha_{j,n}, \beta_{j,n}$ are arbitrarily given real numbers, $w(x) \in C^2(a, b)$ is a weight function and $\{x_{j,n}\}_{j=1}^{n}$ is the set of nodal points. By taking $x_j's$ as the zeros of the nth Ultraspherical polynomial $P_n^{\alpha}(x), (\alpha > -1)$ and the weight function as $(1-x^2)^{\frac{\alpha+1}{2}}$ he proved that generally there does not exist any polynomial of degree $\leq 2n - 1$ satisfying the conditions (1.1). However, by taking an additional condition:

$$S_n(0) = \sum_{j=1}^{n} \alpha_{j,n}\ell_{j,n}^2(0) \tag{1.2}$$

where 0 is not a nodal point and $\ell'_{j,n}s$ are the fundamental polynomials of Lagrange interpolation, he proved that there exists a unique interpolatory polynomial $S_n(x)$ of degree $\leq 2n$ (n even) satisfying the conditions (1.1) and (1.2). If n is odd, the uniqueness fails to exist. After which many authors [5, 6, 12, 13, 14] have taken up similar problem on

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different set of nodes lying in the interval $[a,b]$ which may be finite, semi-finite or infinite and obtained a uniquely determined interpolatory polynomial of minimal possible degree satisfying the conditions (1.1) and (1.2) in the case when 0 is not a nodal point.

Mathur and Saxena [8] studied the case of weighted (0,1,3)-interpolation on infinite interval by taking the nodes as the zeros of nth Hermite polynomial $H_n(x)$ and showed that if n is even, there exists a unique polynomial $R_n(x)$ of degree $\leq 3n$ satisfying the conditions for $j = 0, 1, \cdots, n$:

$$R_n(x_{j,n}) = a_{j,n}\ ,\ R'_n(x_{j,n}) = b_{j,n}\ ,\left(w^2R_n\right)'''(x_{j,n}) = c_{j,n} \tag{1.3}$$

and

$$R_n(0) = \sum_{j=1}^{n}\{(1+3x_{j,n}^2)a_{j,n} - b_{j,n}x_{j,n}\}\ell_{j,n}^3(0) \tag{1.4}$$

where 0 is not a nodal point. They also obtained the explicit representation of the fundamental polynomials and proved a convergence theorem. In another paper, Krebsz [7] considered the weighted $(0,1,3)$ interpolation in the cases when the nodal points are the zeros of the classical orthogonal polynomials say $p_n(x)$ and in the case when $p_n(0) \neq 0$, obtained an uniquely determined polynomial $R_n(x)$ of degree $\leq 3n$ satisfying the conditions (1.3) and (1.4). In the case when $p_n(0) = 0$, he showed that there are infinitely many polynomials of degree $\leq 3n$ satisfying the conditions (1.3) and (1.4).

In weighted lacunary interpolation and Pál type interpolation processes, results have been obtained under the special condition of the type (1.4), which appears to be artificial. Also, in almost every lacunary interpolation it has been proved that, when 0 is a nodal point then either the interpolatory polynomial of minimum possible degree does not exist or if it exists, they are infinitely many.

In this paper we have considered the modified weighted $(0,1,3)$ - interpolation by taking the nodal points as the roots of the classical orthogonal polynomials $p_n(x)$, in both the cases when $p_n(0) \neq 0$ or $p_n(0) = 0$. Explicit representation of the interpolatory polynomial has been obtained by replacing the artificial looking condition (1.4) in each of the cases. Precisely we have shown that there exists a unique interpolatory polynomial $R_n(x)$ of degree $\leq 3n$ satisfying the conditions, for $i = 1, 2, \ldots, n$

$$R_n(x_{i,n}) = y_{i,n}, R'_n(x_{i,n}) = y'_{i,n}, (w^2R_n)'''(x_{i,n}) = y'''_{i,n}, \tag{1.5}$$

$$\begin{cases} R_n(0) = y_{0,n}, & if\ \ p_n(0) \neq 0 \\ (or) & \\ R''_n(0) = y''_{0,n}, & if\ \ p_n(0) = 0 \end{cases} \tag{1.6}$$

where $\{x_{k,n}, k = 1, 2, \cdots, n, n \in N\}$ is a given system of the nodal points in finite or infinite interval (a,b), $w \in C^3(a,b)$ is a weight function and $\{y_{0,n}, y''_{0,n}, y_{k,n}, y'''_{k,n}, k = 1, \cdots, n, n \in N\}$ are arbitrarily given real numbers.

Other authors [2, 3, 9, 11, 15] have also considered similar modifications in Lacunary and Pál type interpolations.

2 Explicit Representation (when $p_n(0) = 0$)

The interpolatory polynomial $R_n(x)$ of degree $\leq 3n$ satisfying the conditions (1.5) and (1.4), when $p_n(0) = 0$, can be uniquely represented as

$$R_n(x) = \sum_{k=1}^{n} y_{k,n} A_{k,n}(x) + \sum_{k=1}^{n} y'_{k,n} B_{k,n}(x) + \sum_{k=1}^{n} y'''_{k,n} C_{k,n}(x) + y''_{0,n} D_{0,n}(x) \tag{2.1}$$

where $A_{k,n}(x), B_{k,n}(x), C_{k,n}(x), D_{0,n}(x)$ are the fundamental polynomials each of degree $\leq 3n$ satisfying the conditions:

$$\begin{cases} D_{0,n}(x_{j,n}) = 0, D'_{0,n}(x_{j,n}) = 0, \\ (w^2 D_{0,n})'''(x_{j,n}) = 0, D''_{0,n}(0) = 1. \end{cases} \tag{2.2}$$

For $j, k = 1, 2, \ldots, n$

$$\begin{cases} C_{k,n}(x_{j,n}) = 0, C'_{k,n}(x_{j,n}) = 0, \\ (w^2 C_{k,n})'''(x_{j,n}) = \delta_{kj}, C''_{k,n}(0) = 0, \end{cases} \tag{2.3}$$

$$\begin{cases} B_{k,n}(x_{j,n}) = 0, B'_{k,n}(x_{j,n}) = \delta_{kj}, \\ (w^2 B_{k,n})'''(x_{j,n}) = 0, B''_{k,n}(0) = 0 \end{cases} \tag{2.4}$$

and

$$\begin{cases} A_{k,n}(x_{j,n}) = \delta_{kj}, A'_{k,n}(x_{j,n}) = 0, \\ (w^2 A_{k,n})'''(x_{j,n}) = 0, A''_{k,n}(0) = 0 \end{cases} \tag{2.5}$$

where $w(x) \in C^3[a,b]$ is a weight function such that

$$\left\{ w^2(x) p_n^3(x) \right\}'''_{x_{j,n}} = 0, \quad j = 1, 2, \cdots, n. \tag{2.6}$$

The explicit form of these polynomials is given in the following

Lemma 2.1. *For $p_n(0) = 0$, the interpolatory polynomials $D_{0,n}(x)$, $\{C_{k,n}(x)\}_{k=1}^n$, $\{B_{k,n}(x)\}_{k=1}^n$ and $\{A_{k,n}(x)\}_{k=1}^n$ each of degree $\leq 3n$ satisfying the conditions* (2.2), (2.3), (2.4) *and* (2.5) *respectively are given by*

$$D_{0,n}(x) = \frac{p_n^2(x)}{2p'_n(0)^2} \tag{2.7}$$

$$C_{k,n}(x) = \frac{p_n^2(x)}{6w^2(x_{k,n})[p'_n(x_{k,n})]^2} \int_0^x \ell_k(x)dx, \tag{2.8}$$

$$B_{k,n}(x) = (x - x_{k,n})\ell^3_{k,n}(x) + \frac{p_n^2(x)}{[p_n'(x_{k,n})]^2}\int_0^x \frac{[\gamma_{k,n}(x - x_{k,n}) + \ell'_{k,n}(x_{k,n})]\ell_{k,n}(x) - \ell'_{k,n}(x)}{(x - x_{k,n})}dx \tag{2.9}$$

where

$$\gamma_{k,n} = -\frac{w''(x_{k,n})}{w(x_{k,n})} + [\ell'_{k,n}(x_{k,n})]^2 - \frac{1}{2}\ell''_{k,n}(x_{k,n})$$

and

$$\ell_{k,n}(x) = \frac{p_n(x)}{(x - x_{k,n})p_n'(x_{k,n})}. \tag{2.10}$$

Also,

$$A_{k,n}(x) = \ell^3_{k,n}(x) - 3\ell'_{k,n}(x_{k,n})B_{k,n}(x) + \tag{2.11}$$

$$+\frac{p_n^2(x)}{[p_n'(x_{k,n}]^2}\int_0^x \frac{[\alpha_{k,n}(x - x_{k,n})^2 + \beta_{k,n}(x - x_{k,n}) + \ell'_{k,n}(x_{k,n})]\ell_{k,n}(x) - \ell'_{k,n}(x)}{(x - x_{k,n})^2}dx$$

where

$$\alpha_{k,n} = -\frac{1}{3}\frac{w'''(x_{k,n})}{w(x_{k,n})} - 2\ell'_{k,n}(x_{k,n})\frac{w''(x_{k,n})}{w(x_{k,n})} + 3\left(\ell'_{k,n}(x_{k,n})\right)^3 - \frac{3}{2}\ell'_{k,n}(x_{k,n})\ell''_{k,n}(x_{k,n})$$

$$\beta_{k,n} = \ell''_{k,n}(x_{k,n}) - [\ell'_{k,n}(x_{k,n})]^2. \tag{2.12}$$

Proof. Obviously, $D_{0,n}(x)$ given by (2.7) is a polynomial of degree $\leq 3n$ with $D_{0,n}(x_{j,n}) = D'_{0,n}(x_{j,n}) = 0, j = 1, 2, \cdots, n$, $[w^2(x)D_{0,n}(x)]'''_{x_{j,n}} = 0$, due to (2.6) and $D''_{0,n}(0) = 1$. Thus $D_{0,n}(x)$ given by (2.7) satisfy all the conditions (2.2).

Since $\ell_{k,n}(x)$, given by (2.10), is a polynomial of degree $\leq n-1$ hence $C_{k,n}(x)$, given by (2.8), is a polynomial of degree $\leq 3n$ such that $C_{k,n}(x_{j,n}) = C'_{k,n}(x_{j,n}) = 0, j = 1, 2, \cdots, n$ and $C''_{k,n}(0) = 0$. On multiplying (2.8) by $w^2(x)$ and differentiating thrice, then due to (2.6), we get $[w^2(x)C_{k,n}(x)]'''_{x_{j,n}} = \delta_{jk}$. Thus $C_{k,n}(x)$ given by (2.8) satisfy all the conditions (2.3).

Since,

$$\lim_{x\to x_{k,n}}\left[\gamma_{k,n}(x - x_{k,n}) + \ell'_{k,n}(x_{k,n})]\ell_{k,n}(x) - \ell'_{k,n}(x)\right] = 0$$

hence $\{B_{k,n}(x)\}_{k=1}^n$ given by (2.9) is a polynomial of degree $\leq 3n$ with $B_{k,n}(x_{j,n}) = 0$ and $B'_{k,n}(x_{j,n}) = \delta_{jk}, j, k = 1, 2, \cdots, n$ and $B''_{k,n}(0) = 0$. On multiplying (2.9) by $w^2(x)$ and differentiating twice, we have $[w^2(x)B_{k,n}(x)]'''_{x=x_{j,n}} = 0, j = 1, 2, \cdots, n$. Thus $B_{jk}(x)$, given by (2.9) satisfies all the conditions (2.4). Also, since

$$\lim_{x\to x_{k,n}}\left[\{\alpha_{k,n}(x - x_{k,n})^2 + \beta_{k,n}(x - x_{k,n}) + \ell'_{k,n}(x_{k,n})\}\ell_{k,n}(x) - \ell'_{k,n}(x)\right] = 0$$

and due to (2.12)

$$\lim_{x\to x_{k,n}}\Big[\{2\alpha_{k,n}(x-x_{k,n})+\beta_{k,n}\}\ell_{k,n}(x)$$
$$+\{\alpha_{k,n}(x-x_{k,n})^2+\beta_{k,n}(x-x_{k,n})+\ell'_{k,n}(x_{k,n})\}\ell'_{k,n}(x)-\ell''_{k,n}(x)\Big]=0$$

hence $\{A_{k,n}(x)\}_{k=1}^{n}$ given by (2.11) is a polynomial of degree $\leq 3n$ with $A'_{k,n}(x_{j,n}) = \delta_{jk}, j,k=1,2,\cdots,n$, $A'_{k,n}(x_{j,n})=0$ and $A''_{k,n}(0)=0$. On multiplying (2.11) by $w^2(x)$ and differentiating thrice, we have $[w^2(x)A_{k,n}(x)]'''_{x=x_{j,n}}=0, j=1,2,\cdots,n$. Thus $A_{jk}(x)$, given by (2.11) satisfies all the conditions (2.5), which completes the proof of the theorem. □

3 Explicit Representation (when $p_n(0)\neq 0$)

Consider

$$A^*_{0,n}(x)=\frac{p_n^2(x)}{p_n^2(0)}$$

and for $k=1,2,\cdots,n$

$$A^*_{k,n}(x)=A_{k,n}(x)-\ell^3_{k,n}(0)A^*_{0,n}(x),$$
$$B^*_{k,n}(x)=B_{k,n})(x)+b_{k,n}A^*_{0,n}(x),$$
$$C^*_{k,n}(x)=C_{k,n}(x),$$

where $\{A_{k,n}(x)\}_{k=1}^{n}, \{B_{k,n}(x)\}_{k=1}^{n}$ and $\{C_{k,n}(x)\}_{k=1}^{n}$ are given by (2.11), (2.9) and (2.8) respectively. Each of these polynomial is of degree $\leq 3n$ and satisfy the conditions for $j,k=1,2,\ldots,n$

$$\text{(3.1)}\qquad \begin{cases} C^*_{k,n}(x_{j,n})=0, & (C^*_{k,n})'(x_{j,n})=0,\\ (w^2C^*_{k,n})'{}'(x_{j,n})=\delta_{kj}, & C^*_{k,n}(0)=0\end{cases}$$

$$\text{(3.2)}\qquad \begin{cases} B^*_{k,n}(x_{j,n})=0, & (B^*_{k,n})'(x_{j,n})=\delta_{kj},\\ (w^2B^*_{k,n})'''(x_{j,n})=0, & B^*_{k,n}(0)=0\end{cases}$$

and for $k=0,1,2,\ldots,n,\ j=1,2,\ldots,n.$

$$\text{(3.3)}\qquad \begin{cases} A^*_{k,n}(x_{j,n})=\delta_{kj}; & (A^*_{k,n})'(x_{j,n})=0,\\ (w^2A^*_{k,n})'''(x_{j,n})=0, & A_{k,n}(0)=0.\end{cases}$$

Then

$$\text{(3.4)}\qquad R_n(x)=\sum_{k=0}^{n}y_{k,n}A^*_{k,n}(x)+\sum_{k=1}^{n}y'_{k,n}B^*_{k,n}(x)+\sum_{k=1}^{n}y''_{k,n}C^*_{k,n}(x)$$

is a uniquely determined interpolatory polynomial of degree $\leq 3n$ satisfying the conditions (1.5) and (1.6), when $p_n(0)\neq 0$.

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