A Hybrid Computational Approach for Steady Flow of Walter's B Fluid in a Vertical Channel with Porous Wall

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Abstract

The key objective of the present study is to propose a hybrid computational technique, namely q-homogony analysis transform method (q-HATM) to investigate the steady flow of a Walter B fluid through a vertical channel with a porous medium. The q-HATM is employed to solve the nonlinear equation occurring from this kind of flow. The velocity profile and physical significance of various parameters are shown graphically. The numerical results demonstrate that the suggested technique is simple, well organized and can be used to handle more complex nonlinear differential equations arising in fluid mechanics.

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Keywords: Walter's B fluid; Vertical channel; Porous medium; q-homotopy analysis transform method.

1 Introduction

The flow of Newtonian and non-Newtonian fluids via a porous medium has gained interest of research workers due to its wide use in scientific and technological fields, namely transpiration cooling, gaseous diffusion, boundary layer control, for adding reactants and preventing corrosion and reducing drag. In a study Berman [1] has examined the 2 D Newtonian fluid flow through a porous channel. Furthermore, Choi *et al.* [2] and Cox [3] have investigated this problem by considering variations in the problem. This flow problem is modeled by differential equations. The steady flow of Walter B fluid in a vertical channel in a porous media has been studied by employing various techniques such as homotopy analysis method [4], homotopy perturbation method [5], homotopy Sumudu transform method [6], etc. These methodologies have their own limitations such as huge computational calculation of He's polynomials etc. Ghorbani *et al.* [7, 8] introduced the He's polynomials where nonlinear terms were split into a series of polynomials which are

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calculated from HPM. It has been noted that He's polynomials are attuned with Adomian's polynomials, yet it is showed that the He's polynomials are easier to compute, and are very user friendly. To overcome these difficulties, we propose a hybrid computational technique namely q-homotopy analysis transform method (q-HATM) [9] to investigate steady flow of Walter B fluid in a vertical channel via porous media. The q-HATM is a combination of q-homotopy analysis method (q-HAM) [10, 11] and standard Laplace transform technique. The q-HAM is a modification of homotopy analysis method (HAM) [12, 13, 14]. Very recently analytical techniques have also been combined with Laplace transform algorithm to simulate mathematical models of real world problems such as advection problem [15], nonlinear Blasius equation [16], nonlinear differential difference equation occurring in nanotechnology [17], fractional model of Fornberg-Whitham equation [18], KdV equation arising in warm plasma [19], convection-diffusion problems [20], etc.

Sajid *et al.* [21] have studied the stagnation point flow of Walters' B fluid using Hybrid Homotopy Analysis Method. Hussain and Ullah [22] have analyzed the steady boundary layer flow of a Walter's B fluid due to a stretching cylinder with temperature dependent variable viscosity. The heat transfer analysis is also considered. They found that the velocity profile decreases with increase in Re. Ahmad *et al.* [23] have investigated axisymmetric stagnation-point flow of second grade fluid over a lubricated surface with power law non-Newtonian fluid. They found that velocity increases by increasing the slip and visco-elastic parameters. Temperature and thermal boundary layer thickness are decreasing functions of Prandtl number. An increase in visco-elastic parameter decreases the temperature distribution. The effect of Eckert number is to increase the temperature distribution in the flow region. Majeed *et al.*[24] have investigated two-dimensional hydromagnetic flow and heat transfer of Walter's B fluid towards a stagnation point region over a stretching cylinder.

In this paper, an efficient computational approach based on q-homotopy analysis transform method (q-HATM) is proposed to solve the steady flow of a Walter B fluid in a vertical channel with porous wall.

The main aim of this work is to extend the use of the q-HATM to derive the analytic and approximate solutions to the steady flow of a Walter B fluid in a vertical channel with a porous wall. The walls of a channel moving in a hot fluid are constructed of a porous type material along which fluid is inserted to form a cowling layer of cooler fluid close the wall. The q-HATM gives the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions.

2 Mathematical Formulation of the Problem

In the present work, we investigate the steady of two dimensional flow of Walter's B fluid with porous medium in a vertical channel. A non-Newtonian fluid is inserted via a vertical permeable plate at $y = d$ having uniform velocity u. The non-Newtonian fluid hits some other vertical incompressible plate at $y = 0$. Because of the action of gravity across the z-axis, the fluid flows away along the opening of the plates. We have taken the distance linking the plates, d, which is very compact as equated to the dimensions of the plates; due to this notion, edge impacts can be neglected . The equations of the non-Newtonian fluid flow are presented as follows:

The equation of continuity [4, 5]:

$$\nabla \cdot V = 0, \tag{2.1}$$

The equation of motion

$$\rho(V \cdot \nabla V) = \nabla T + \rho\, g, \tag{2.2}$$

In the above equations ρ and g denote the density and acceleration respectively. The following hypothesis is made to solve differential equations: (a) The fluid flow is stable and laminar; (b) The fluid flow is isochoric flow; (c) The acceleration is considered to be equal to body force per unit mass; (d) The physical properties of the fluid remain uniform throughout the fluid; (e) The impact of viscous dissipation is disregarded.

We make use of formalism described by Joneidi *et al.* [4] for the convected differentiation of the rate of strain tensor, substituting Cauchy stress tensor, we write

$$\rho(V \cdot \nabla V) = -\nabla p + \rho g + \eta_0 \nabla^2 V - 2k_0 V \cdot \nabla\nabla^2 V + k_0 \nabla^2 (V \cdot \nabla V), \tag{2.3}$$

On employing some variables and modification, the equation can be expressed as

$$f'''' + \mathrm{Re}\,(f''' f - f' f'') + \mathrm{Re}\, S(f f''''' - f' f'''') = 0. \tag{2.4}$$

with the corresponding boundary conditions:

$$f(0) = 0, \qquad f(2.1) = 1, \qquad f'(0) = 0, \qquad f'(1) = 0, \tag{2.5}$$

3 q-HATM Algorithm

In order to explain the q-HATM algorithm, we assume the following standard form of 3^{rd} order non-homogenous nonlinear ODE

$$f''' + a_1(x) f'' + a_2(x) f' + a_3(x) f = h(f), \tag{3.1}$$

along with the initial conditions

$$f(0) = \alpha_1, \quad f'(0) = \alpha_2, \quad f''(0) = \alpha_3. \tag{3.2}$$

Applying the Laplace transform operator on Eq. (3.1), we get

$$L\,[f] - \frac{\alpha_1}{s} - \frac{\alpha_2}{s^2} - \frac{\alpha_3}{s^3} - \frac{1}{s^3} L[h(f)] + \frac{1}{s^3} L[a_1(x) f'' + a_2(x) f' + a_3(x) f] = 0. \tag{3.3}$$

where $\alpha_1, \alpha_2, \alpha_3$ are constants.

We define the nonlinear operator in the following manner

$$\begin{aligned} N[\phi(x;q)] = L\,[\phi(x;q)] - \frac{\alpha_1}{s} - \frac{\alpha_2}{s^2} - \frac{\alpha_3}{s^3} - \frac{1}{s^3} L[h(\phi(x;q))] \\ + \frac{1}{s^3} L[a_1(x)\phi''(x;q) + a_2(x)\phi'(x;q) + a_3(x)\phi(x;q)], \end{aligned} \tag{3.4}$$

where $0 \leq q \leq 1/n$ and $\phi(x;q)$ is a real function of x and q. In view of q-HAM, we develop the homotopy as follows

$$(1 - nq)L[\phi(x;q) - f_0(x)] = \hbar\, qN\,[f(x)], \tag{3.5}$$

where L indicates the Laplace transform, $q \in [0,\, 1/n]$ denotes the embedding parameter, $\hbar \neq 0$ represents an auxiliary parameter and $f_0(x)$ is an initial guess of $f(x)$.

When the embedding parameter $q = 0$ and $q = 1/n$, it gives

$$\phi(x;0) = f_0(x), \qquad \phi(x;1) = f(x), \tag{3.6}$$

respectively. So we notice that as q increases form 0 to $1/n$, the solution $\phi(x;q)$ varies from the initial guess $f_0(x)$ to the solution $f(x)$. Expressing $\phi(x;q)$ in Taylor series about q, we get

$$\phi(x;q) = f_0(x) + \sum_{m=1}^{\infty} f_m(x)\,(1/n)^m, \tag{3.7}$$

where

$$f_m(x) = \frac{1}{m\,!}\frac{\partial^m \phi(x\,;q)}{\partial q^m}\bigg|_{q=0}. \tag{3.8}$$

On selecting the initial guess and the parameter $\hbar$ suitably, the series (3.7) converges at $q = 1/n$; then we have

$$f(x) = f_0(x) + \sum_{m=1}^{\infty} f_m(x), \tag{3.9}$$

which must be one of the solutions of the original 3^{rd} order nonlinear differential equation. Making use of the Eq. (3.8), the governing equation can be deduced from Eq. (3.5).

Define the vectors

$$\overrightarrow{f}_m = \{f_0(x), f_1(x), \ldots, f_m(x)\}. \tag{3.10}$$

On differentiating the zeroth-order deformation Eq. (3.5) m-times w.r.t. q and then dividing them by $m!$ and finally setting $q = 0$, we get the following equation:

$$L\,[f_m(x) - k_m f_{m-1}(x)] = \hbar \Re_m(\overrightarrow{f}_{m-1}). \tag{3.11}$$

The inverse of Laplace operator, yields

$$f_m(x) = k_m f_{m-1}(x) + \hbar\, L^{-1}[\Re_m(\overrightarrow{f}_{m-1})], \tag{3.12}$$

where

$$\Re_m(\overrightarrow{f}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N\,[\phi(x;q)]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{3.13}$$

and

$$k_m = \begin{cases} 0, & m \leq 1, \\ n, & m > 1. \end{cases} \tag{3.14}$$

4 Solution of the Problem

Here we put up the q-HATM in use to find the solution of Eq. (2.4). On applying the Laplace transform on Eq. (2.4), we have

$$L[f] - \frac{\lambda_1}{s^3} - \frac{\lambda_2}{s^4} + \frac{1}{s^4} L[\mathrm{Re}\,(ff''' - f'f'') + \mathrm{Re}\, S(ff''''' - f'f'''')] = 0, \tag{4.1}$$

where $\lambda_1 = f''(0)$ and $\lambda_2 = f'''(0)$.

The nonlinear operator is

$$\begin{aligned} L[\phi(\eta;q)] - \frac{\lambda_1}{s^3} - \frac{\lambda_2}{s^4} + \frac{1}{s^4} L\big[\mathrm{Re}\,(\phi(\eta;q)\phi'''(\eta;q) - \phi'(\eta;q)\phi''(\eta;q)) \\ + \mathrm{Re}\, S(\phi(\eta;q)\phi'''''(\eta;q) - \phi'(\eta;q)\phi''''(\eta;q))\big] = 0, \end{aligned} \tag{4.2}$$

and thus

$$\begin{aligned} R_m(\overrightarrow{f}_{m-1}) &= L[f_{m-1}] - (1 - k_m)\left(\frac{\lambda_1}{s^3} + \frac{\lambda_2}{s^4}\right) \\ &+ \frac{1}{s^4} L\left[\mathrm{Re}\left(\sum_{r=0}^{m-1} f_{m-1-r}{f'''}_r - \sum_{r=0}^{m-1} {f'}_{m-1-r}{f''}_r\right)\right. \\ &\left. + \mathrm{Re}\, S\left(\sum_{r=0}^{m-1} f_{m-1-r}{f'''''}_r - \sum_{r=0}^{m-1} {f'}_{m-1-r}{f''''}_r\right)\right]. \end{aligned} \tag{4.3}$$

The m^{th}-order deformation equation is given by

$$L[f_m(\eta) - k_m f_{m-1}(\eta)] = \hbar R_m(\overrightarrow{f}_{m-1}). \tag{4.4}$$

Applying the inverse Laplace transform, we have

$$f_m(\eta) = k_m[f_{m-1}(\eta)] + \hbar L^{-1}[R_m(\overrightarrow{f}_{m-1})]. \tag{4.5}$$

Using the initial approximation $f_0(\eta) = \frac{1}{2}\lambda_1\eta^2 + \frac{1}{6}\lambda_2\eta^3$ and the iterative approach (4.5), we obtain the various components of the series solution:

$$f_1(\eta) = -\hbar\left(\frac{\mathrm{Re}\,\lambda_2^2}{2520}\eta^7 + \frac{\mathrm{Re}\,\lambda_1\lambda_2}{360}\eta^6 + \frac{\mathrm{Re}\,\lambda_1^2}{120}\eta^5\right), \tag{4.6}$$

where $\lambda_1 = f''(0)$ and $\lambda_2 = f'''(0)$ to be found from the boundary conditions. The solutions of the Eq. (2.4) be as follows:

$$f(\eta) = f_0(\eta) + f_1(\eta) + f_2(\eta) + \cdots. \tag{4.7}$$

5 Results and Discussion

In this article, we have used the q-HATM to analyze the steady flow of a Walter B fluid through a vertical channel in the presence of porous media which is available in section 4, the normal and tangential velocities of the steady flow of a Walter's B fluid are attained. The proposed technique provides more realistic series solutions that converge very fast in problems of physical importance.

Figures 1 and 2 depict the $f(\eta)$ and $f'(\eta)$ derived for the distinct Reynolds numbers at $S = 0.1$. Figure 1 illustrates the normal velocity component for various values of Reynolds numbers at $S = 0.1$, as an increase in the value of Reynolds number, the normal velocity rises. Figure 2 depicts tangential velocity component for various values of Reynolds numbers at $S = 0.1$, as an increase in the value of Reynolds number, the tangential velocity rises. It is clear from this figure that the tangential velocity profile behavior is different from roughly $\eta = 0.5$ to the $\eta = 1$. Therefore, from this figure, the enhancing of the cross flow Reynolds number increases the maximum of tangential velocity and moves it away from the porous plate.

Figures 3 and 4 illustrate the $f(\eta)$ and $f'(\eta)$ derived for the distinct elastic numbers at $\mathrm{Re} = 0.9$. Figure 3 illustrates the normal velocity component for different values of elastic numbers when $\mathrm{Re} = 0.9$. It has been observed that an enhancing of the elastic number, the normal velocity rises. Figure 4 depicts the impact of elastic number on tangential velocity descriptions when Reynolds numbers $\mathrm{Re} = 0.9$. It has been noticed from this figure that the tangential velocity profile behavior is different from roughly $\eta = 0.5$ to the $\eta = 1$. Therefore, from this figure, an enhancing of an elastic numbers when $\mathrm{Re} = 0.9$, an increases the maximum of tangential velocity and moves it away from the porous plate.

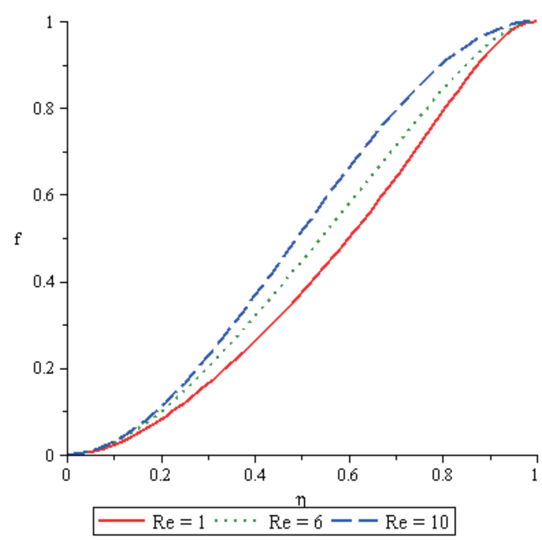

Fig. 1: Plot of $f(\eta)$ v/s η at $S = 0.1$ for different values of Reynolds number (Re).

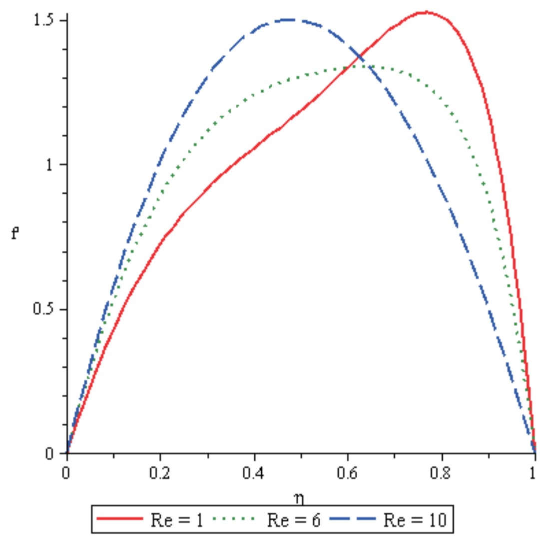

Fig. 2: Plot of $f'(\eta)$ v/s η at $S = 0.1$ for different values of Reynolds number (Re).

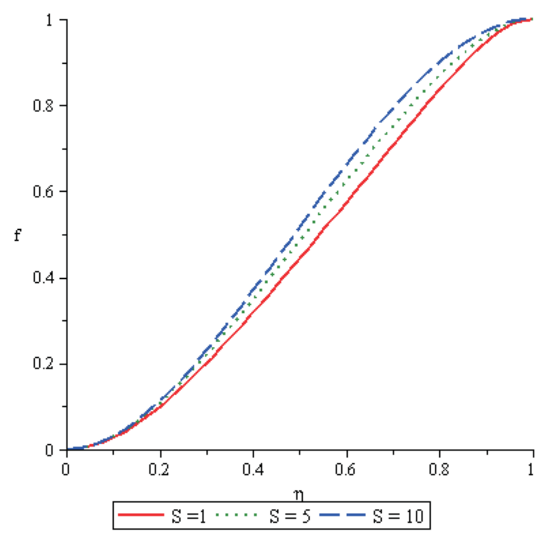

Fig. 3: Plot of $f(\eta)$ v/s η at Re $= 0.9$ for different values of elastic number (S).

6 Conclusion

In the present article, the q-HATM method is used to investigate the steady flow of a Walter B fluid through a vertical channel in the presence of porous media. The numerical results demonstrate that this approach gives very good approximations to the solution of the Eq. (5) with very high accuracy. In a nutshell, the q-HATM may be assumed as an excellent improvement in existing methods and might find the very waste importance and

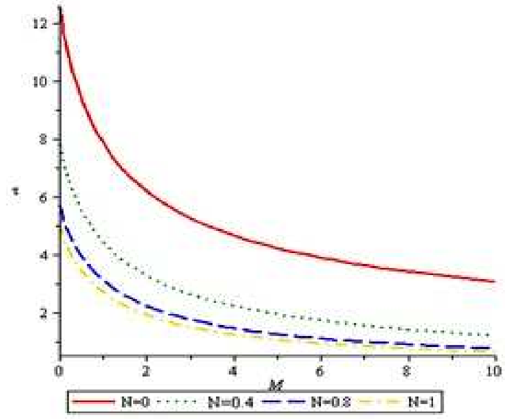

Fig. 4: Plot of $f'(\eta)$ v/s η at Re $= 0.9$ for different values of elastic number (S).

uses in science and engineering.

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