Analytical solution of Couette-Poiseuille flow between two cylinders filled with a variable permeability porous medium

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Abstract

In the present paper we have studied steady flow of a viscous incompressible fluid in an annular region between two coaxial translating cylindrical tubes filled by a porous medium of variable permeability. Analytical solution of the problem is obtained for three different cases of permeability variation by using Brinkman equation. We have obtained relevant quantities such as velocity, volume flow rate, average velocity and stress on the surface of cylinders. The average permeability of the porous medium is obtained in case of variable permeability. The effect of various parameters on the flow are discussed and obtained results are exhibited graphically.

Keywords: Porous Medium, Brinkman equation, Viscous flow, Variable Permeability, Permeability Parameter, Couette-Poiseuille flow.

1 Introduction

Flow through porous media has numerous applications in filtering process, chemical engineering, ground water flow, oil and gas refinery etc. We can find numerous articles on the fluid flow through porous channels of various shape. Kaviany (1985) studied laminar flow through a porous channel bounded by two parallel plates maintained at a constant and equal temperature. Parang and Keyhani (1987) studied the boundary effect in laminar mixed convection flow through an annular porous medium. Nakayama et al. (1988) were investigated non-Darcian fully developed flow and heat transfer in a porous channel bounded by two parallel walls subjected to uniform heat flux. Vafai and Kim (1989) found an exact solution for forced convection in a channel filled with porous media. Chikh et al. (1995) obtained an analytical solution for a fully developed, forced convection in a gap between two concentric cylinders. Kuznetsov (1998) investigated a analytical expression of heat transfer in Couette flow through a porous medium utilizing the Brinkman-Forchheimer-extended Darcy model. Nield and Kuznetsov (2000) investigated analytically the effects of variation of permeability and thermal conductivity on fully developed forced convection in a parallel plate channel and circular duct filled with a saturated porous medium. Haji-Sheikh and Vafai (2004) analysed flow and heat transfer in porous media embedded inside various shaped ducts. Mohammadein and El-Shaer (2004) studied combined free and forced convection flow of viscous incompressible fluid past a semi-infinite vertical plate embedded in a porous medium incorporating the variation of permeability and thermal conductivity. Hooman (2006) investigated perturbation solution for forced convection in a porous parallel plates analytically on the basis of a Brinkman-Forchheimer model. Hooman and Gurgenci (2007) investigated the forced convection inside a circular tube filled with saturated porous medium and with uniform heat flux at the wall on the basis of a Brinkman-Forchheimer model. Wang (2008a) found an analytical solution for forced convection in a semi-circular channel filled with a porous medium. Wang (2008b) found analytical solutions for the transient starting flow due to a sudden pressure gradient in cylindrical, rectangular, and parallel plate ducts filled with a Darcy-Brinkman porous medium. Wang (2010a) studied the fully developed flow and constant flux heat transfer in super-elliptic ducts filled with a porous medium. Wang (2010b) a perturbation analysis is carried out to the second order to give effective equations for Darcy-Brinkman flow through a porous channel with slightly corrugated walls. Wang (2011) studied the fully developed flow and heat transfer in a polygonal duct filled with a Darcy-Brinkman medium. Verma and Singh (2015) investigated steady flow of a conducting fluid in a circular channel filled with a saturated porous medium in the presence of transverse magnetic field and obtained exact solution by using Brinkma model. Wang (2016) studied pulsatile flow in ducts filled with a Darcy-Brinkman medium and obtained analytic solutions for the annular duct, the rectangular duct and the sector duct.

In all of the above mentioned work the porous medium is homogeneous but in real problems porous medium may be of variable permeability. The available literature related to the flow in variable permeability porous medium is very little. Some authors have been investigated problems of this type such as Govender (2006), Verma and Datta (2012a,b), Kim and Yuan (2005) and Verma and Dixit (2016) etc.

In the present problem we have study the Couette-Poiseuille flow of a viscous, incompressible fluid between the gap of two coaxial cylinders filled by a variable permeability medium when inner cylinder is stationary and outer cylinder is translating along the axis of cylinder with uniform velocity. Exact solutions are obtained for three different useful cases of permeability variation; (i)when permeability is uniform, i.e. $k = k_o$ (ii) when permeability variation is linear, i.e. $k = k_o r$ and (iii) when permeability variation is quadratic, i.e. $k = k_o r^2$.

2 Mathematical Formulation

We consider steady fully developed flow of viscous incompressible fluid in the annular region between two coaxial impermeable cylinders filled with a porous medium of variable permeability. Outer cylinder is translating with uniform velocity u_1 parallel to the common axis and inner cylinder is kept stationary. Flow is due to translation of cylinders and applied constant pressure gradient $\partial p^*/\partial z^*$ along the axis of cylinders. We use cylindrical polar coordinate system (r, θ, z^*). The flow is unidirectional in z^* direction along the axis of cylinder. Radius of outer and inner cylinder is R_2 and R_1, respectively. Flow within the annular porous region is governed by Brinkman equation (1947). Brinkman equation for the present problem in cylindrical polar coordinates is given by

$$\mu_e \left(\frac{d^2u^*}{dr^{*2}} + \frac{1}{r^*}\frac{du^*}{dr^*} \right) - \frac{\mu}{k}u^* = \frac{\partial p^*}{\partial z^*} \tag{2.1}$$

where u^* is the fluid velocity, μ_e is the effective viscosity, μ is the fluid viscosity and k is the permeability of porous medium. Here $k = k(r)$ is taken as a function of radial distance r. Authors have different opinion about effective viscosity. According to Liu and Masliyah (2005) effective viscosity depending on the type of porous medium it may be greater or smaller than fluid viscosity. Many authors, for example, Brinkman (1947) and chikh et al. (1995) assume $\mu_e = \mu$. This assumption is valid for the porous medium of high porosity. With this assumption, Brinkman Eq.(2.1) becomes

$$\frac{d^2u^*}{dr^{*2}} + \frac{1}{r^*}\frac{du^*}{dr^*} - \frac{u^*}{k} = \frac{1}{\mu}\frac{\partial p^*}{\partial z^*} \tag{2.2}$$

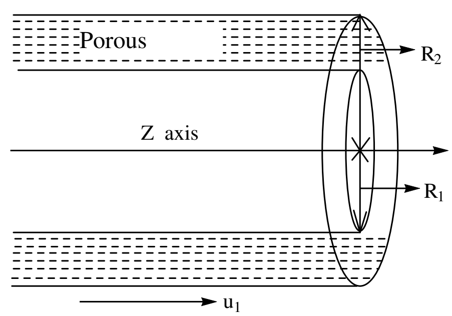

Fig. 1: Definition Sketch

Boundary conditions for the present flow are no slip condition on the surface of inner and outer cylinder. That are

$$u^* = 0 \quad \text{at} \quad r^* = R_1 \qquad \text{and} \qquad u^* = u_1 \quad \text{at} \quad r^* = R_2 \tag{2.3}$$

We introduce non dimensional quantities as follows

$$r = \frac{r^*}{R_1} \qquad \text{and} \qquad u = \frac{u^*}{U} \tag{2.4}$$

Boundary conditions (2.3) in terms of non dimensional quantities are

$$u(1) = 0 \qquad \text{and} \qquad u(q) = \frac{u_1}{U} = V \tag{2.5}$$

Here, U is some characteristic velocity of flow. It may be velocity of outer cylinder or average velocity of flow. V is non dimensional translating velocity of outer cylinder and $q = R_2/R_1$ is defined as gap parameter. Eq.(2.2) in non dimensional form is given by

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{R_1^2}{k}u = -P \tag{2.6}$$

$P = -\frac{R_1^2}{\mu\ U}\frac{\partial p^*}{\partial z^*}$ is non dimensional pressure gradient.

3 Solution

Now we assume that the permeability of the porous medium as a function of radial distance as $k(r) = k_o r^n$; n is a real number. Then equation (2.6) becomes

$$r^n\frac{d^2u}{dr^2} + r^{n-1}\frac{du}{dr} - \sigma^2 u = -r^n \tag{3.1}$$

where $\sigma^2 = \frac{R_1^2}{k_o}$ is the permeability variation parameter. Solution of above equation is very difficult to deal with the general value of n. Therefore here we consider three particular cases of permeability variation. (i) when $k = k_o$ (ii) when $k = k_o r$ and (iii) when $k = k_o r^2$. Here k_o is some characteristic permeability, that may be taken as permeability on the surface of inner cylinder. Now we discuss the flow for all the three cases of permeability variation.

3.1 Case-I

When $n = 0$ i.e. $k = k_o$, equation of motion (3.1) becomes

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} - \sigma^2 r\ u = -rP. \tag{3.2}$$

Eq.(3.2) is a modified Bessel's equation of order zero, it's general solution is

$$u(r) = a_1\ I_o(\sigma r) + a_2\ K_o(\sigma r) + \frac{P}{\sigma^2} \tag{3.3}$$

where I_o and K_o are the modified Bessel functions of zeroth order of first and second kind, respectively. Here a_1 and a_2 are constants of integration. Using boundary conditions (2.5). We get

$$a_1 = \frac{V\sigma^2 K_o(\sigma) + P(K_o(\sigma q) - K_o(\sigma))}{\sigma^2(I_o(\sigma q)K_o(\sigma) - I_o(\sigma)K_o(\sigma q))}, \quad a_2 = \frac{V\sigma^2 I_o(\sigma) + P(I_o(\sigma q) - I_o(\sigma))}{\sigma^2(I_o(\sigma)K_o(\sigma q) - I_o(\sigma q)K_o(\sigma))} \tag{3.4}$$

With this values of a_1 and a_2 eq. (3.3) gives us the dimensionless velocity of fluid at any point with in annular region when permeability of the medium is k_0. In limiting case, when $\sigma \to 0$ (i.e.when permeability of the medium is infinite) in eq.(3.3), we get velocity u_0 for clear fluid flow

$$u_0 = \lim_{\sigma \to 0}(u) = \frac{P(1-r^2)\log q + (P(q^2-1)+4V)\log r}{4\log q}. \tag{3.5}$$

When outer cylinder is also stationary i.e. when $V \to 0$. We obtain velocity profile for the classical poiseuille flow of clear fluid between two coaxial cylinders, which is

$$\lim_{V \to 0}(u_0) = \frac{P(1-r^2)\log q + P(q^2-1)\log r}{4\log q}. \tag{3.6}$$

3.1.1 Rate of volume flow

The dimensionless rate of volume flow through the cross-section of annular tube is given by

$$Q = 2\pi \int_1^q u(r)\ r\ dr. \tag{3.7}$$

Substituting u from eq. (3.3) and after integration we obtain

$$Q = 2\pi\left[\left(\frac{q^2-1}{2\sigma^2}\right) + a_1\left(\frac{qI_1(q\sigma)-I_1(\sigma)}{\sigma}\right) + a_2\left(\frac{K_1(\sigma)-qK_1(q\sigma)}{\sigma}\right)\right] \tag{3.8}$$

where I_1 and K_1 are the modified Bessel function of first and second kind of order one and constants a_1 and a_2 are given by eq.(3.4). In the evaluation of above integrals the following identity [Ref. Abramowitz and Stegun (1970)] has been used

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m \{z^\nu \pounds_\nu(z)\} = z^{\nu-m}\pounds_{\nu-m}(z) \tag{3.9}$$

with $m = 1$ and $\nu = 1$. $\pounds_\nu$ denotes I_ν and $e^{\nu\pi i}K_\nu$.

The dimensionless volume flow rate Q_o for clear fluid flow (when permeability is infinite) can be obtained by taking limit $\sigma \to 0$ in eq.(3.8). We get

$$Q_0 = \lim_{\sigma \to 0}(Q) = \frac{\pi}{8}\left[P(q^4-1) + 8q^2V - \frac{(q^2-1)(P(q^2-1)+4V)}{\log q}\right]. \tag{3.10}$$

When $V = 0$ i.e. outer cylinder is also stationary, we obtain volume flow rate for the classical poiseuille flow of clear fluid between two coaxial cylinders, which is

$$\lim_{V \to 0}(Q_0) = \frac{\pi P}{8}\left[(q^4-1) - \frac{(q^2-1)^2}{\log q}\right]. \tag{3.11}$$

3.1.2 Average velocity:

The dimensionless average velocity of the flow within the annular region when permeability of the region is uniform k_o, is given by

$$u_{avg} = \frac{Q}{\pi(q^2-1)}. \tag{3.12}$$

Substituting Q from the eq.(3.8) in the above equation, the average velocity of the flow is obtained as

$$u_{avg} = \frac{2}{(q^2-1)}\left[\left(\frac{q^2-1}{2\sigma^2}\right) + a_1\left(\frac{qI_1(q\sigma)-I_1(\sigma)}{\sigma}\right) + a_2\left(\frac{K_1(\sigma)-qK_1(q\sigma)}{\sigma}\right)\right]. \tag{3.13}$$

For clear fluid flow the average velocity of flow is obtained by eq.(3.13) by taking limit $\sigma \to 0$, which is

$$\lim_{\sigma\to 0}(u_{avg}) = \frac{1}{8}\left[P(q^2+1) + \frac{8q^2V}{(q^2-1)} - \frac{(P(q^2-1)+4V)}{\log q}\right]. \tag{3.14}$$

Average velocity for the classical Poiseuille flow is obtained by taking limit $V \to 0$ in the above equation (3.14), we get

$$\lim_{V\to\, 0}((u_{avg})_{\sigma=0}) = \frac{P}{8}\left[(q^2+1) - \frac{(q^2-1)}{\log q}\right]. \tag{3.15}$$

3.1.3 Shearing stress on the surface of cylinders:

The dimensionless shearing stress at any point is given by

$$\tau_{rz}(r) = \frac{du}{dr} \tag{3.16}$$

Substituting u from eq.(3.3) and differentiating the modified Bessel functions $I_o(\sigma r)$ and $K_o(\sigma r)$ with the use of identity $\frac{d}{dr}I_o(r) = I_1(r)$ and $\frac{d}{dr}K_o(r) = -K_1(r)$ [Ref. Abramowitz and Stegun (1970)], we obtain

$$\tau_{rz}(r) = a_1\sigma\ I_1(\sigma r) - a_2\ \sigma\ K_1(\sigma r) \tag{3.17}$$

where I_1 and $K_1$are modified Bessel function of order one. Stress on the surface of inner and outer cylinder is obtained by putting $r = 1$ and $r = q$ in eq.(3.17), respectively by using the appropriate sign. We get

$$\tau_{rz}(1) = -[a_1\sigma I_1(\sigma) - a_2\sigma K_1(\sigma)] \tag{3.18}$$

$$\tau_{rz}(q) = -[a_1\sigma I_1(\sigma q) - a_2\sigma\ K_1(\sigma q)] \tag{3.19}$$

where $a_1,\ a_2$ are given by eq.(3.4). Dimensionless shearing stress on the surface of inner and outer cylinders for the classical Couette-Poiseuille flow of clear fluid is obtained by taking limit $\sigma \to 0$) in eq.(3.18) and eq.(3.19), respectively. After taking limit we obtain

$$\tau_{rz}(1) = \frac{P\left(q^2-1\right)+4V-2P\log q}{4\log q} \tag{3.20}$$

$$\tau_{rz}(q) = \frac{P\left(q^2-1\right)+4V-2Pq^2\log q}{4q\log q} \tag{3.21}$$

The dimensionless shear stress on the surface of inner and outer cylinder for the classical Poiseuille flow of clear fluid is obtained by taking limit $V \to 0$ in eq. (3.20) and (3.21), respectively. Which provides us

$$\tau_{rz}(1) = \frac{P\left(q^2-1\right)-2P\log q}{4\log q} \tag{3.22}$$

$$\tau_{rz}(q) = \frac{P\left(q^2-1\right)-2Pq^2\log q}{4q\log q} \tag{3.23}$$

3.2 Case-II

Now we consider the case when $k = k_o r$, i.e. permeability of the porous medium is varying linearly with the radial distance. For this permeability eq.(3.1) becomes

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} - \sigma^2 u = -rP \tag{3.24}$$

General solution of eq.(3.24) is given by

$$u = b_1\ I_o(2\sigma\sqrt{r}) + b_2\ K_o(2\sigma\sqrt{r}) + \frac{P}{\sigma^4}\left(1+r\sigma^2\right) \tag{3.25}$$

where b_1 and b_2 are constants of integration which are obtained by using boundary conditions (2.5) in eq.(3.25). Thus we get constants b_1 and b_2 as

$$\begin{aligned} b_1 &= \frac{V\sigma^4K_0(2\sigma) - P\left(K_0(2\beta)\left(\sigma^2q+1\right)-\left(\sigma^2+1\right)K_0\left(2\sigma\sqrt{q}\right)\right)}{\sigma^4(K_0(2\sigma)I_0\left(2\sigma\sqrt{q}\right)-I_0(2\sigma)K_0\left(2\sigma\sqrt{q}\right))}, \\ b_2 &= \frac{V\sigma^4I_0(2\sigma) - P\left(I_0(2\sigma)\left(\sigma^2q+1\right)-\left(\sigma^2+1\right)I_0\left(2\sigma\sqrt{q}\right)\right)}{\sigma^4(I_0(2\sigma)K_0\left(2\sigma\sqrt{q}\right)-K_0(2\sigma)I_0\left(2\sigma\sqrt{q}\right))} \end{aligned} \tag{3.26}$$

The dimensionless velocity of fluid at any point within the annular porous region when permeability is $k_0 r$, is given by eq.(3.25). Velocity for the classical Couette-poiseuille flow of clear fluid is obtained by taking limit $\sigma \to 0$ (i.e. permeability of the medium is infinite) in eq.(3.25). Which is

$$u_0 = \lim_{\sigma \to 0}(u) = \frac{P(1-r^2)\log q + (P(q^2-1)+4V)\log r}{4\log q} \tag{3.27}$$

when $V = 0$ in the above equation, i.e. outer cylinder is also stationary, we obtain velocity for the classical poiseuille flow of clear fluid between two coaxial cylinders, which is

$$\lim_{V \to 0}(u_0) = \frac{P(1-r^2)\log q + P(q^2-1)\log r}{4\log q} \tag{3.28}$$

3.2.1 Rate of volume flow

The dimensionless rate of volume flow through the cross-section of annular region of cylindrical channel, when permeability of the porous channel is $k_o r$, is given by

$$Q = 2\pi \int_1^q u(r)\ r\ dr. \tag{3.29}$$

Substituting velocity u from eq.(3.25) in the above equation and integrating the resulting expression, we obtained

$$\begin{aligned} Q &= b_1 \frac{2\pi}{\sigma^2}\{\sigma q\sqrt{q}\ I_1(2\sqrt{q}\sigma) - \sigma I_1(2\sigma) - qI_2(2\sqrt{q}\sigma) + I_2(2\sigma)\} \\ &+ b_2\frac{2\pi}{\sigma^2}\{-\sigma q\sqrt{q}\ K_1(2\sqrt{q}\sigma) + \sigma K_1(2\sigma) - qK_2(2\sqrt{q}\sigma) + K_2(2\sigma)\} \\ &+ \frac{2\pi P}{\sigma^4}\{\frac{1}{2}(q^2-1) + \frac{\sigma^2}{3}(q^3-1)\} \end{aligned} \tag{3.30}$$

where constants $b_1,\ b_2$ are given by eq.(3.26). The dimensionless volume flow rate Q_o for clear fluid flow (when permeability is infinite) can be obtained by taking limit $\sigma \to 0$ in the above eq.(3.30). We get

$$Q_0 = \lim_{\sigma \to 0}(Q) = \frac{\pi}{8}\left[P(q^4-1) + 8q^2V - \frac{(q^2-1)(P(q^2-1)+4V)}{\log q}\right] \tag{3.31}$$

which is volume flow rate for Couette-poiseuille flow within clear annular region between two coaxial cylinders. The flow rate for classical clear Poiseuille flow through annular channel is obtained by taking $V = 0$ in the above expression for Q_0, which is

$$\lim_{V \to 0}(Q_0) = \frac{\pi P}{8}\left[(q^4-1) - \frac{(q^2-1)^2}{\log q}\right] \tag{3.32}$$

3.2.2 Average velocity:

The dimensionless average velocity of the flow is defined as

$$u_{avg} = \frac{Q}{\pi(q^2 - 1)}. \tag{3.33}$$

Substituting Q from the eq.(3.30) in the above equation we get average velocity of the flow through the porous annular region when permeability of the region is $k_o r$. For clear fluid flow average velocity of the flow is obtained by taking limit $\sigma = 0$ in eq.(3.33), that is

$$(u_{avg})_{\sigma=0} = \frac{1}{8}\left[P(q^2+1) + \frac{8q^2V}{(q^2-1)} - \frac{(P(q^2-1)+4V)}{\log q}\right] \tag{3.34}$$

when outer cylinder is stationary i.e. $V = 0$. Average velocity is

$$(u_{avg})_{\sigma=0} = \frac{P}{8}\left[(q^2+1) - \frac{(q^2-1)}{\log q}\right] \tag{3.35}$$

3.2.3 Average Permeability :

Darcy law in non dimensional form can be written as

$$u_{avg} = \frac{K}{{R_1}^2}P \tag{3.36}$$

Here, K is permeability of the porous channel and P is the nondimensional pressure gradient as defined in eq. (2.6). Using this law we can evaluate average permeability of the porous channel as given below

$$K_{avg} = \frac{u_{avg}{R_1}^2}{P} \tag{3.37}$$

where u_{avg} is the average velocity given by eq.(3.33).

3.2.4 Shearing stress on the surface of cylinders:

The dimensionless shearing stress at any point within the channel when permeability of the porous region vary according to law is $k_o r$ is given by

$$\tau_{rz}(r) = \frac{du}{dr}$$

where velocity u is given by eq. (3.25). With this velocity shear stress at any point is given by

$$\tau_{rz}(r) = \frac{P}{\sigma^2} + b_1\frac{\sigma\ I_1\left(2\sigma\sqrt{r}\right)}{\sqrt{r}} - b_2\frac{\sigma\ K_1\left(2\sigma\sqrt{r}\right)}{\sqrt{r}} \tag{3.38}$$

Stress on the surface of inner and outer cylinder is obtained by putting $r = 1$ and $r = q$, respectively in eq.(3.38) and using the appropriate sign. This give us

$$\tau_{rz}(1) = -[\frac{P}{\sigma^2} + b_1\sigma\ I_1\,(2\sigma) - b_2\ \sigma\ K_1\,(2\sigma)], \tag{3.39}$$

$$\tau_{rz}(q) = -[\frac{P}{\sigma^2} + b_1\frac{\sigma\ I_1\left(2\sigma\sqrt{q}\right)}{\sqrt{q}} - b_2\frac{\sigma\ K_1\left(2\sigma\sqrt{q}\right)}{\sqrt{q}}] \tag{3.40}$$

where $b_1,\ b_2$ are given by eq.(3.26). Dimensionless shearing stress on the surface of inner and outer cylinders for clear fluid flow, when permeability of the porous region is infinite, are obtained by taking limit $\sigma \to 0$ in eq.(3.39) and eq.(3.40), respectively. Then we get

$$\tau_{rz}(1) = \frac{P\left(q^2-1\right) + 4V - 2P\log q}{4\log q} \tag{3.41}$$

$$\tau_{rz}(q) = \frac{P\left(q^2-1\right) + 4V - 2Pq^2\log q}{4q\log q} \tag{3.42}$$

When outer cylinder is stationary i.e. $V = 0$. Shear stress on inner and outer cylinders are

$$\tau_{rz}(1) = \frac{P\left(q^2-1\right) - 2P\log q}{4\log q} \tag{3.43}$$

$$\tau_{rz}(q) = \frac{P\left(q^2-1\right) - 2Pq^2\log q}{4q\log q} \tag{3.44}$$

which is the classical result for clear Poiseuille flow.

3.3 Case-III

Now consider the case when $k = k_o r^2$, i.e. permeability of the porous medium is varying quadratically with the radial distance. For this permeability eq.(3.1) becomes

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - \sigma^2 u = -Pr^2. \tag{3.45}$$

Its general solution is

$$u = c_1\cosh(\sigma\log r) + c_2\sinh(\sigma\log r) - \frac{Pr^2}{4-\sigma^2} \tag{3.46}$$

Here c_1 and c_2 are the constants of integration. Using the boundary condition (2.5) in eq.(3.46) we find value of c_1 and c_2 and after substituting these values in in eq. (3.46), we get dimensionless velocity of the flow within annular porous region at any point as (3.47)

$$u = \frac{1}{(4-\sigma^2)}\left[P\cosh(\sigma\log r) + \frac{(V(4-\sigma^2)+Pq^2-P\cosh(\sigma\log q))\sinh(\sigma\log r)}{\sinh(\sigma\log q)} - r^2\right]$$

In the limiting case when $\sigma \to 0$ (i.e. when permeability of the porous medium is infinite) the eq.(3.47)provide us velocity u_0 for the clear fluid flow and is obtained as

$$u_0 = \lim_{\sigma\to 0}(u) = \frac{P(1-r^2)\log q + (P(q^2-1)+4V)\log r}{4\log q} \tag{3.48}$$

When outer cylinder is stationary i.e. $V = 0$. The above equation (3.48) give the velocity for classical Poiseuille flow within clear annular region. That is

$$\lim_{V\to 0}(u_0) = \frac{P(1-r^2)\log q + P(q^2-1)\log r}{4\log q}. \tag{3.49}$$

3.3.1 Rate of volume flow:

The dimensionless rate of volume flow through the cross-section of annular tube when permeability of the region is $k_o r^2$, is given by

$$Q = 2\pi\int_1^q u(r)\ r\ dr. \tag{3.50}$$

Substituting u from eq.(3.47) and after integration, we obtain

$$\begin{aligned} Q \; = \; & 2\pi\{\frac{P\left(1-q^4\right)}{4\left(4-\sigma^2\right)} - \frac{P\left(q^2(\sigma\sinh(\sigma\log q)-2\cosh(\sigma\log q))+2\right)}{\left(4-\sigma^2\right)^2} + \\ & \frac{\left(\sigma+q^2(2\sinh(\sigma\log q)-\sigma\cosh(\sigma\log q))\right)\left(Pq^2-P\cosh(\sigma\log q)+\left(4-\sigma^2\right)V\right)}{\left(4-\sigma^2\right)^2\sinh(\sigma\log q)}\} \end{aligned}$$

(3.51)

The dimensionless volume flow rate Q_o for clear fluid flow (when permeability is infinite) can be obtained by taking limit $\sigma \to 0$ in eq.(3.51), we get

$$Q_0 = \lim_{\sigma\to 0}(Q) = \frac{\pi}{8}\left[P(q^4-1) + 8q^2V - \frac{(q^2-1)(P(q^2-1)+4V)}{\log q}\right] \tag{3.52}$$

When outer cylinder is stationary i.e. $V = 0$. The above equation (3.52) give the volume flow rate for classical Poiseuille flow through clear annular region. That is

$$\lim_{V\to 0}(Q_0) = \frac{\pi P}{8}\left[(q^4-1) - \frac{(q^2-1)^2}{\log q}\right] \tag{3.53}$$

3.3.2 Average velocity:

The dimensionless average velocity of the flow is

$$u_{avg} = \frac{Q}{\pi(q^2-1)} \tag{3.54}$$

Substituting Q from the eq.(3.51) in the above equation we obtain the average velocity of the flow when permeability of the porous annular region is $k_0 r^2$. For clear fluid flow when $\sigma = 0$, the average velocity of flow is obtained by taking limit $\sigma \to 0$ in (3.54). This give us result

$$(u_{avg})_{\sigma\to 0} = \frac{1}{8}\left[P(q^2+1) + \frac{8q^2V}{(q^2-1)} - \frac{(P(q^2-1)+4V)}{\log q}\right] \tag{3.55}$$

The volume flow rate, when outer cylinder is stationary, is obtained by putting $V = 0$ is the above equation. which is

$$(u_{avg})_{\sigma=0} = \frac{P}{8}\left[(q^2+1) - \frac{(q^2-1)}{\log q}\right] \tag{3.56}$$

3.3.3 Average Permeability :

The average permeability of the porous channel can be evaluated as in the case-I by using Darcy law. Which is

$$K_{avg} = \frac{u_{avg}{R_1}^2}{P} \tag{3.57}$$

where u_{avg} is the average velocity given by eq.(3.54).

3.3.4 Shearing stress on the surface of cylinders:

The dimensionless shearing stress at any point within the channel when permeability is $k_0 r^2$ is obtained similarly as in case-I and II, which is

$$\begin{aligned} \tau_{rz}(r) \;=\; & \frac{\sigma\cosh(\sigma\log r)\left(Pq^2 - P\cosh(\sigma\log q) + \left(4-\sigma^2\right)V\right)}{(4-\sigma^2)\, r\sinh(\sigma\log q)} \\ & -\frac{2Pr}{4-\sigma^2} + \frac{\sigma P\sinh(\sigma\log r)}{(4-\sigma^2)\, r} \end{aligned} \tag{3.58}$$

Stress on the surface of inner and outer cylinder is obtained by putting $r = 1$ and $r = q$, respectively in eq.(3.58) and using the appropriate sign. We get

$$\tau_{rz}(1) = -[\frac{\sigma\left(Pq^2 - P\cosh(\sigma\log(q)) + \left(4-\sigma^2\right)V\right)}{(4-\sigma^2)\sinh(\sigma\log q)} - \frac{2P}{4-\sigma^2}] \tag{3.59}$$

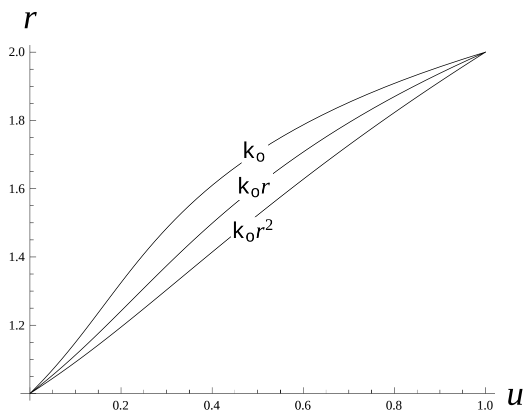

Fig. 2: Velocity profiles for different permeability variation when $\sigma = 3$, q=2, P=1 and V=1.

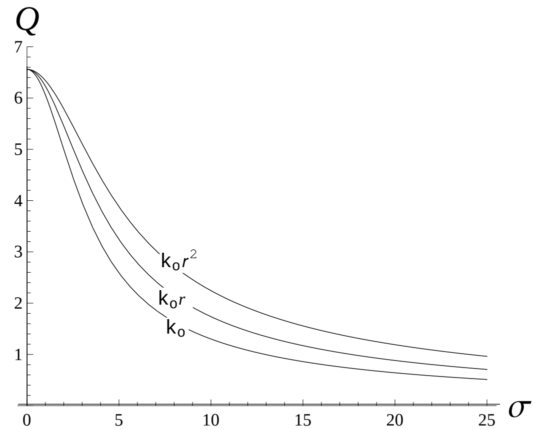

Fig. 3: Variation of volume flow rate with σ for different permeability variation when q=2, P=1 and V=1.

$$\tau_{rz}(q) = -\left[\frac{\sigma\cosh(\sigma\log q)\left(Pq^2 - P\cosh(\sigma\log q) + \left(4-\sigma^2\right)V\right)}{\left(4-\sigma^2\right)q\sinh(\sigma\log q)} - \frac{2Pq}{4-\sigma^2} + \frac{\sigma P\sinh(\sigma\log q)}{\left(4-\sigma^2\right)q}\right] \tag{3.60}$$

Dimensionless shearing stress on the surface of inner and outer cylinders for clear fluid flow are obtained by taking limit $\sigma \to 0$ in eq.(3.59)and eq.(3.60), respectively. We obtain

$$\tau_{rz}(1) = \frac{P\left(q^2-1\right) + 4V - 2P\log q}{4\log q} \tag{3.61}$$

$$\tau_{rz}(q) = \frac{P\left(q^2-1\right) + 4V - 2Pq^2\log q}{4q\log q} \tag{3.62}$$

When outer cylinder is stationary the shear stress on the surface of cylinders for clear flow is obtained by putting $V = 0$ in above eq. (3.62). That is

$$\tau_{rz}(1) = \frac{P\left(q^2-1\right) - 2P\log q}{4\log q} \tag{3.63}$$

$$\tau_{rz}(q) = \frac{P\left(q^2-1\right) - 2Pq^2\log q}{4q\log q} \tag{3.64}$$

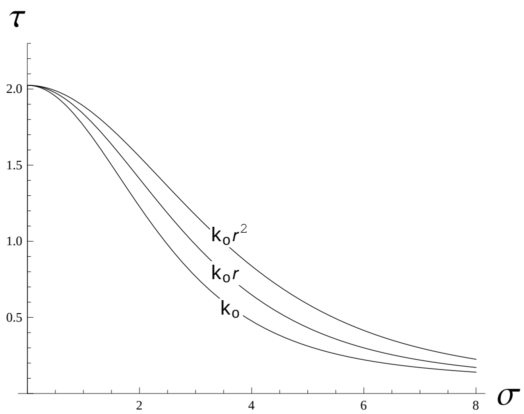

Fig. 4: Variation of shear stress on inner cylinder with σ for different permeability variation when q=2, P=1 and V=1.

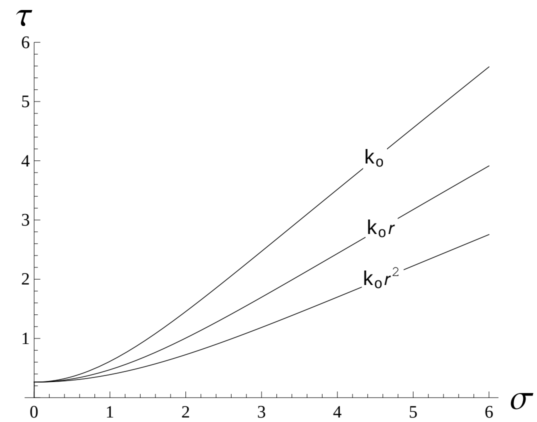

Fig. 5: Variation of shear stress on outer cylinder with σ for different permeability variation when q=2, P=1 and V=1.

4 Discussion

Figure (2) represent the velocity profile for the different cases of permeability variation $k = k_o$, $k = k_o r$ and $k = k_o r^2$ for fixed values of permeability parameter $\sigma = 3$, gap parameter $q = 2$, pressure gradient $P = 1$ and velocity of outer cylinder $V = 1$. We observe that velocity at any point within the annular region greater for permeability variation $k_o r^2$ than that of $k_o r$ and velocity in case of permeability $k_o r$ is greater than that of k_o. This is because permeability $k_o r^n$ at any point in the channel increases as n increases. Thus permeability variation has remarkable effect on the velocity profile.

Fig.(3) shows the variation of rate of volume flow Q with the parameter σ for different cases of permeability variation when q=2. It is seen that Q decreases as σ increases for all values of permeability variation. This is due to the fact that increase in σ is caused by decrease in permeability. It is noted that for uniform permeability k_o, Q is smaller than that of the case when permeability vary according to the law $k = k_o r$ and $k = k_o r^2$.

Fig.(4)represent the variation of shear stress on the surface of inner cylinder with the parameter σ for all the three cases of permeability variation. This variation is plotted for fixed value of gap parameter $q = 2$, $P = 1$ and $V = 1$. This figure reveal that shear stress on the inner cylinder decreases with the increase in value of σ (i.e. with decrease in permeability) and settled down to almost fixed value for large σ for all the three cases of permeability variation. We observe that the skin friction on the inner cylinder is smaller when permeability is uniform k_o than that of the case when permeability of the channel is variable.

Fig.(5)represent the variation of shearing stress on the surface of outer cylinder with the parameters σ for all the different cases of permeability variation for fixed values of q, P and V. This figure reveal that shearing stress on the outer cylinder increases with the increase of σ (i.e. decrease in permeability). This behaviour is stress is opposite to that of the stress on inner cylinder. This is because inner cylinder is stationary but outer is moving with fixed velocity. It is also notable that the skin friction on the outer cylinder is greater when permeability is uniform k_o than that of the case when permeability of the channel is variable.

5 Conclusion

A Couette - Poiseuille flow between two coaxial cylinders, filled by porous medium of variable permeability, is investigated. Analytical solution of the considered problem has been obtained by using Brinkman equation for three different cases of permeability variation (i) when permeability of the annular porous region is uniform (ii) when permeability variation is linear and (iii) when permeability variation is quadratic. Exact expressions for velocity, average velocity, volume flow rate, average permeability and skin friction on the surface of inner and outer cylinder are obtained and exhibited graphically. The effect of permeability variation on the flow has been discussed. In the limiting case when permeability of the porous region tend to infinity obtained results reduce to the classical results for Couette-Poiseuille flow of clear fluid. It is resolved from the discussion that the permeability variation has remarkable effect on the flow quantities. The obtained results are useful in the cases where permeability of the porous medium is variable.

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