A note on Toeplitz type operators on weighted Hardy spaces

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Abstract

This paper describes some properties of two classes of operators, namely weighted Toeplitz operators and compressions of k^{th}-order slant weighted Toeplitz operators ($k \geq 2$ is an integer). A necessary and sufficient condition is obtained for an operator to be an analytic weighted Toeplitz operator. It is also proved that the Coburn alternative holds for analytic weighted Toeplitz operators.

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1 Preliminaries

The study of Toeplitz operators originated with the work of Toeplitz [12] in 1911. Together with Hankel operators, they form two of the most important classes of operators on Hardy spaces. The matrices of Toeplitz operators with respect to an orthonormal basis are constant along each diagonal parallel to the main one. Ever since their inception, Toeplitz operators and matrices have remained a field of extensive research and this field has witnessed impressive development during the last twenty years. Toeplitz matrices arise in plenty of applications, while Toeplitz operators are building blocks to many other classes of operators. Reference [2] provides a comprehensive study of Toeplitz operators.

Toeplitz operators and their various generalizations over different function spaces have been a topic of interest to many mathematicians. An important variant of Toeplitz operators is the slant Toeplitz operator, introduced by Ho [8] in 1995. The matrix of this operator with respect to orthonormal basis is obtained from the matrix of the corresponding Toeplitz operator by eliminating every alternate row. Slant Toeplitz operators have been found to have various important practical applications, for example, in wavelet analysis and solutions of differential equations (see [7],[13]).

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A noteworthy direction in the study of Toeplitz operators emerged with the introduction of weighted sequence spaces $L^2(\beta)$ and $H^2(\beta)$ by Kelley [9]. Shields [11] made a comprehensive study of the class of weighted shift operators. In the year 2005, Lauric [10] studied the structure of a class of weighted Toeplitz operators on $H^2(\beta)$ and obtained a description of the commutant of each operator in this class. Erstle [6] described Toeplitz operators on weighted Hardy spaces, while Zheng [15] discussed Toeplitz and Hankel operators on Bergman spaces. Also, Yousefi [14] and Zorboska [16] studied composition operators on weighted Hardy spaces. The product and commutativity of slant weighted Toeplitz operators on $L^2(\beta)$, of different orders, was discussed by us in [3], while isometric and Hilbert-Schmidt weighted Toeplitz operators on $H^2(\beta)$ were studied in [4].

This paper is a step forward in this direction of study. We discuss the normality and compactness of weighted Toeplitz operators. Also, analytic weighted Toeplitz operators are characterized via an operator equation. The Coburn alternative (see [1]), known for Toeplitz operators, is proved to be true for weighted Toeplitz operators induced by some specific symbols. We are also motivated to undertake the study of compressions of k^{th}-order slant weighted Toeplitz operators to $H^2(\beta)$. In addition to study of some structural properties, some spectral properties of these operators are also discussed. Our study is fruitful because for the particular instances of the sequence β, $H^2(\beta)$ yields various well known Hilbert spaces of analytic functions. In fact, if $\beta_n = 1$ for each $n \geq 0$, then $H^2(\beta)$ corresponds to the classical Hardy space. If $\beta_n = (n+1)^{\frac{1}{2}}$, then $H^2(\beta)$ coincides with Dirichlet space and if $\beta_n = (n+1)^{-\frac{1}{2}}$, we obtain Bergman space.

Let us begin with brief descriptions of the spaces under consideration. We refer to [11] to provide historical growth, detailed descriptions and applications of these spaces. The symbols $\mathbb{Z}$ and $\mathbb{C}$ denote respectively the sets of all integers and of all complex numbers. Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$ and such that

$$r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1 \text{ for } \ n \geq 0 \text{ and } \ r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1 \text{ for } \ n \leq 0, \tag{1.1}$$

for some $r > 0$.

This assumption on β is taken throughout the paper. Let $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, be the formal Laurent series (whether or not the series converges for any values of z). Define $\|f\|_\beta$ as $||f||_\beta^2 = \sum\limits_{n=-\infty}^{\infty} |a_n|^2 {\beta_n}^2$. The space $L^2(\beta)$ is $\{f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n |\ a_n \in \mathbb{C},\ ||f||_\beta^2 = \sum\limits_{n=-\infty}^{\infty} |a_n|^2 {\beta_n}^2 < \infty\}$ and $H^2(\beta)$ is the subspace of $L^2(\beta)$ consisting of $\{f(z) = \sum\limits_{n=0}^{\infty} a_n z^n |\ a_n \in \mathbb{C},\ ||f||_\beta^2 = \sum\limits_{n=0}^{\infty} |a_n|^2 {\beta_n}^2 < \infty\}$.

$(L^2(\beta), ||\cdot||_\beta)$ is a Hilbert space with the norm $\|\cdot\|_\beta$ induced by the inner product $\langle f, g\rangle = \sum\limits_{n=-\infty}^{\infty} a_n\, \overline{b}_n {\beta_n}^2$, for $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $g(z) = \sum\limits_{n=-\infty}^{\infty} b_n z^n$. The collection $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\beta)$. $(H^2(\beta), ||\cdot||_\beta)$ is a subspace of $L^2(\beta)$ with an orthonormal basis $\{e_n(z) = z^n/\beta_n\}_{n\geq 0}$.

The symbol $L^\infty(\beta)$ denotes the set of formal Laurent series $\{\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \mid \phi L^2(\beta) \subseteq L^2(\beta)$ and $\exists\ c > 0 : \|\phi f\|_\beta \le c\|f\|_\beta$, for each $f \in L^2(\beta)\}$. $L^\infty(\beta)$ is a Banach space with respect to $\|\cdot\|_\infty$ defined as $\|\phi\|_\infty = \inf\{c > 0 : \|\phi f\|_\beta \le c\|f\|_\beta$ for each $f \in L^2(\beta)\}$.

$H^\infty(\beta)$ denotes the set of formal power series ϕ such that $\phi H^2(\beta) \subseteq H^2(\beta)$.

The multiplication of two formal power series $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^n$ is given by $(fg)(z) = \sum_{n=0}^{\infty}(fg)_n z^n$, where $(fg)_n = \sum_{m=0}^{n} f_{n-m}g_m$ for each $n \ge 0$. The symbol $\mathcal{K}er(T)$ denotes the kernel of an operator T, while $\sigma_p(T)$ and $\sigma(T)$ denote respectively the point spectrum and the spectrum of T.

2 Weighted Toeplitz operator

A weighted Toeplitz operator T_ϕ^β [10] on $H^2(\beta)$, induced by the symbol $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$, is defined as $T_\phi^\beta = P^\beta M_\phi^\beta$, where P^β is the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$ and M_ϕ^β is the weighted Laurent operator on $L^2(\beta)$ induced by ϕ. T_ϕ^β satisfies for each $j \ge 0$, $T_\phi^\beta e_j = \frac{1}{\beta_j}\sum_{n=0}^{\infty} a_{n-j}\beta_n e_n$.

To begin with, we observe the following.

Proposition 2.1. *The mapping $\phi \mapsto T_\phi^\beta$ is linear and one-one.*

Proof. Linearity is obvious. To show that the mapping is one-one, assume that $T_\phi^\beta = T_\psi^\beta$, where $\phi, \psi \in L^\infty(\beta)$ are such that $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n$. Then, $\langle T_\phi^\beta e_j, e_i\rangle = \langle T_\psi^\beta e_j, e_i\rangle$ for each $i, j \ge 0$, which provides that $a_n = b_n$ for each $n \in \mathbb{Z}$. Hence $\phi = \psi$. This completes the proof. □

It is known that the only compact Toeplitz operator is the zero operator [1]. We discuss the compactness of a weighted Toeplitz operator and obtain the same.

Theorem 2.1. *The weighted Toeplitz operator T_ϕ^β, induced by $\phi \in L^\infty(\beta)$, is compact if and only if the inducing symbol $\phi = 0$.*

Proof. Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$ be such that T_ϕ^β is a compact operator. We know that compact operators map weakly convergent sequences to strongly convergent sequences. Also, for each $s \ge 0$, $\{e_{s+n}\} \to 0$ weakly as $n \to \infty$. Therefore for each $s, t \ge 0$,

$$|a_{t-s}|\frac{\beta_{t+n}}{\beta_{s+n}} = |\langle T_\phi^\beta e_{s+n}, e_{t+n}\rangle| \le ||T_\phi^\beta e_{s+n}||_\beta \to 0 \text{ as } n \to \infty. \quad (2.2.1)$$

For $t \geq s$, equation (2.2.1) together with (1.1) gives

$$|a_{t-s}| \leq |a_{t-s}|\frac{\beta_{t+n}}{\beta_{s+n}} \to 0 \text{ as } n \to \infty.$$

This yields that $a_n = 0$ for each $n \geq 0$. In case $t < s$, equations (2.2.1) and (1.1) provide that $|a_{t-s}|r^{s-t} \leq |a_{t-s}|\frac{\beta_{t+n}}{\beta_{s+n}} \to 0$ as $n \to \infty$. As a consequence, $a_n = 0$ for each $n < 0$. Thus $\phi = 0$. Converse follows evidently. □

Remark 2.1. *Since every Hilbert-Schmidt operator is compact, Theorem 2.1 also provides that no non-zero weighted Toeplitz operator on $H^2(\beta)$ can be Hilbert-Schmidt. This was proved independently in [4] as well.*

It is known [1] that an operator on the Hardy space H^2 is an analytic Toeplitz operator T_ϕ (ϕ is analytic) if and only if $T_\phi U = UT_\phi$, where U denotes the unilateral shift operator on H^2. In this direction, we obtain the following for the weighted Toeplitz operator T_ϕ^β.

Let $U^\beta (= M_z^\beta)$ be an operator on $H^2(\beta)$ given by $U^\beta e_n = \frac{\beta_{n+1}}{\beta_n} e_{n+1}$ for each $n \geq 0$. We find that there exist weighted Toeplitz operators which do not satisfy the operator equation $T_\phi^\beta U^\beta = U^\beta T_\phi^\beta$. For this, we have the following example.

Example 2.1. *Let $T = T_\phi^\beta$ be a weighted Toeplitz operator on $H^2(\beta)$ induced by the symbol $\phi(z) = z^{-1} \in L^\infty(\beta)$. Then, the structure of T_ϕ^β provides that $Te_0 = 0$ and $Te_n = \frac{\beta_{n-1}}{\beta_n} e_{n-1}$ for each $n \geq 1$. Now, $U^\beta T e_n = \begin{cases} 0 & \text{if } n = 0 \\ e_n & \text{if } n \geq 1 \end{cases}$, while $TU^\beta e_n = \frac{\beta_{n+1}}{\beta_n} Te_{n+1} = \frac{\beta_{n+1}}{\beta_n}(\frac{\beta_n}{\beta_{n+1}} e_n) = e_n$ for each $n \geq 0$. Hence, T doesn't satisfy $TU^\beta = U^\beta T$.*

However, if the inducing symbol ϕ of T_ϕ^β is analytic, we obtain a similar result as in case of Toeplitz operators. We say that a weighted Toeplitz operator T_ϕ^β on $H^2(\beta)$, induced by $\phi(z) = \sum\limits_{n=-\infty}^{\infty} \phi_n z^n \in L^\infty(\beta)$, is analytic if $\phi_n = 0$ for each $n < 0$, where $\phi_n =< \phi, e_n >$ is the n^{th}-Fourier coefficient of ϕ.

Now we proceed to obtain a characterization for a weighted Toeplitz operator to be analytic.

Proposition 2.2. *A weighted Toeplitz operator T_ϕ^β on $H^2(\beta)$ is analytic if and only if $T_\phi^\beta U^\beta = U^\beta T_\phi^\beta$.*

Proof. Let T_ϕ^β be an analytic weighted Toeplitz operator induced by $\phi(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in H^\infty(\beta)$. Then, using the definition of T_ϕ^β, it is easy to see that $\langle T_\phi^\beta e_{j+1}, e_{i+1} \rangle = \frac{\beta_{i+1}}{\beta j+1}\frac{\beta_j}{\beta_i}\langle T_\phi^\beta e_j, e_i \rangle$, for each $i, j \geq 0$. Also, $U^{\beta *} e_n = \begin{cases} 0 & \text{if } n = 0 \\ \frac{\beta_n}{\beta_{n-1}} e_{n-1} & \text{if } n \geq 1 \end{cases}$.

Hence, for each $i \geq 1, j \geq 0$, we have

$$\begin{aligned}\langle T_\phi^\beta U^\beta e_j, e_i \rangle &= \frac{\beta_{j+1}}{\beta_j}\frac{\beta_i}{\beta_{j+1}}\frac{\beta_j}{\beta_{i-1}}\langle T_\phi^\beta e_j, e_{i-1}\rangle \\ &= \langle T_\phi^\beta e_j, U^{\beta *} e_i\rangle = \langle U^\beta T_\phi^\beta e_j, e_i\rangle.\end{aligned}$$

Also, for $j \geq 0$, $\langle T_\phi^\beta U^\beta e_j, e_0\rangle = \frac{\beta_{j+1}}{\beta_j}\langle T_\phi^\beta e_{j+1}, e_0\rangle = \frac{\beta_{j+1}}{\beta_j}\frac{\beta_0}{\beta_{j+1}} a_{-j-1} = 0 = \langle T_\phi^\beta e_j, U^{\beta *} e_0\rangle = \langle U^\beta T_\phi^\beta e_j, e_0\rangle$.

As a consequence, $\langle T_\phi^\beta U^\beta e_j, e_i\rangle = \langle U^\beta T_\phi^\beta e_j, e_i\rangle$ for each $i, j \geq 0$. Hence, $T_\phi^\beta U^\beta = U^\beta T_\phi^\beta$.

Converse holds trivially as $T_\phi^\beta U^\beta = U^\beta T_\phi^\beta$ gives that $\langle T_\phi^\beta U^\beta e_j, e_0\rangle = \langle U^\beta T_\phi^\beta e_j, e_0\rangle$ for each $j \geq 0$. This provides that $a_n = 0$ for each $n < 0$. Hence the result. □

We now raise the following question: Is every bounded operator A on $H^2(\beta)$ satisfying $AU^\beta = U^\beta A$ an analytic weighted Toeplitz operator?

An affirmative answer to this question, along with Proposition 2.2, will provide us that the commutant of the weighted shift U^β on $H^2(\beta)$ is the set $\{T_\phi^\beta |\ \phi \in H^\infty(\beta)\}$.

Now we answer the above raised question in the next result as follows.

Theorem 2.2. *Every bounded operator A on $H^2(\beta)$ satisfying the operator equation $AU^\beta = U^\beta A$ is an analytic weighted Toeplitz operator.*

Proof. Let A be a bounded operator on $H^2(\beta)$ such that $AU^\beta = U^\beta A$. Let $\phi = Ae_0$. Then, $\phi \in H^2(\beta)$ and using that $AU^\beta = U^\beta A$, we obtain that for each $n \geq 0$, $\beta_n(Ae_n) = AU^{\beta^n} e_0 = U^{\beta^n} Ae_0 = U^{\beta^n}\phi = \beta_n(\phi e_n)$. Thus $Ae_n = \phi e_n$ for each $n \geq 0$. By linearity, we obtain that $Ap = \phi p$ for each polynomial $p \in H^2(\beta)$. Also, if $f = \sum_{n=0}^{\infty} f_n z^n \in H^2(\beta)$, then there exists a sequence $\{p_n\}$ of polynomials in $H^2(\beta)$ converging to f, say $p_n = \sum_{m=0}^{\infty} p_{n,m} z^m$ with $p_{n,m} = 0$ for $m > n$. This implies that $\{\phi p_n\} = \{Ap_n\} \to Af$.

Denote each ϕp_n as $\phi p_n = \sum_{m=0}^{\infty} (\phi p_n)_m z^m$, where $(\phi p_n)_m = \sum_{k=0}^{m} \phi_{m-k} p_{n,k}$ for each $m \geq 0$ and $Af = \sum_{m=0}^{\infty} (Af)_m z^m$. Hence, for each $t \geq 0$, we find that whatever $\epsilon > 0$ we may take, $\exists\ \ n_0 \geq 0$ satisfying $|(\phi p_n)_t - (Af)_t|^2 \beta_t^2 \leq \sum_{m=0}^{\infty} |(\phi p_n)_m - (Af)_m|^2 \beta_m^2 = ||\phi p_n - Af||_\beta{}^2 < \epsilon {\beta_t}^2$, for each $n \geq n_0$. This gives that the sequence of complex numbers $\{(\phi p_n)_t\} \to (Af)_t$ for each $t \geq 0$.

Using similar arguements as above, the convergence in norm $\{p_n\} \to f$, provides for each $\epsilon > 0$, a $m_0 \geq 0$ such that for each $n \geq m_0$,

$$|p_{n,t} - f_t|^2 < \frac{\epsilon}{||\phi||_\beta^2}$$

for each $t \geq 0$ and hence

$$\begin{aligned}|(\phi p_n)_t - (\phi f)_t|^2 &= |\sum_{m=0}^{t} \phi_{t-m} p_{n,m} - \sum_{m=0}^{t} \phi_{t-m} f_m|^2 \\ &\leq \sum_{m=0}^{t} |\phi_{t-m}|^2 |p_{n,m} - f_m|^2 \\ &< \frac{\epsilon}{||\phi||_\beta^2} \sum_{m=0}^{t} |\phi_{t-m}|^2 \\ &\leq \epsilon,\end{aligned}$$

where $\phi f = \sum_{m=0}^{\infty} (\phi f)_m z^m$ and for each $m \geq 0$, $(\phi f)_m = \sum_{k=0}^{m} \phi_{m-k} f_k$.

This helps to provide that $\{(\phi p_n)_t\} \to (\phi f)_t$ for each $t \geq 0$.

Hence $(Af)_t = (\phi f)_t$ for each $t \geq 0$ and therefore $Af = \phi f$ for each $f \in H^2(\beta)$. Thus we conclude that $A = T_\phi^\beta$ for some $\phi \in H^\infty(\beta)$. This completes the proof. □

An immediate consequence to this theorem is the following.

Corollary 2.1. *The commutant of the weighted shift $U^\beta (= M_z^\beta)$ acting on $H^2(\beta)$ is $\{T_\phi^\beta | \ \phi \in H^\infty(\beta)\}$.*

For a Toeplitz operator T_ϕ, where ϕ is a non-zero function in L^∞, it is known [1] that at least one of T_ϕ and T_ϕ^* is injective (The Coburn Alternative). In this direction, we obtain some symbols ϕ in $L^\infty(\beta)$ so that The Coburn Alternative holds for the weighted Toeplitz operators.

Remark 2.2. *A weighted Toeplitz operator T_ϕ^β, in general, may not be injective. For, consider $\phi(z) = z^{-1} \in L^\infty(\beta)$. Simple computations provide that $\mathcal{K}er(T_{z^{-1}}^\beta) = [e_0]$, where $[e_0]$ denotes the subspace of $H^2(\beta)$ spanned by e_o. In fact, if $n \geq 1$ and $\phi(z) = z^{-n} \in L^\infty(\beta)$, then $\mathcal{K}er(T_{z^{-n}}^\beta) = [e_0, e_1, \cdots, e_{n-1}]$.*

Proposition 2.3. *The kernel of an analytic weighted Toeplitz operator is either $\{0\}$ or infinite dimensional.*

Proof. Let T_ϕ^β be an analytic weighted Toeplitz operator and $0 \neq f \in \mathcal{K}er(T_\phi^\beta)$. Since $T_\phi^\beta U^\beta = U^\beta T_\phi^\beta$ for an analytic T_ϕ^β, we obtain that $z^n f \in \mathcal{K}er(T_\phi^\beta)$ for each $n \geq 1$. Also, the set $\{z^k f : 1 \leq k \leq n\}$ is linearly independent for each $n \geq 1$ (see [5]). Hence the claim. □

However, our next result rules out the possibility of the existence of a non-zero vector in the kernel of a non-zero analytic weighted Toeplitz operator.

Theorem 2.3. *Let $0 \neq \phi \in H^\infty(\beta)$. Then, $\mathcal{K}er(T_\phi^\beta) = \{0\}$.*

Proof. Let, if possible, $0 \neq f \in \mathcal{K}er(T_\phi^\beta)$, where $\phi(z) = \sum_{n=0}^{\infty} a_n z^n \in H^\infty(\beta)$ and $f(z) = \sum_{n=0}^{\infty} f_n z^n \in H^2(\beta)$. Then, $\langle T_\phi^\beta f, e_m \rangle = 0$ for each $m \geq 0$. This provides us that for each $k \geq 0$, $\sum_{i=0}^{k} f_i a_{k-i} = 0$. For $k = 0$, we obtain that $f_0 a_0 = 0$, which gives rise to three possible cases.

1. $f_0 \neq 0$ and $a_0 = 0$. On using the equation $\sum_{i=0}^{k} f_i a_{k-i} = 0$ for successive values of k $(= 1, 2, \cdots)$, a simple computation yields that $\phi = 0$. This is a contradiction.
2. $f_0 = 0$ and $a_0 \neq 0$. Again, proceeding along similar lines as in case (1), we get that $f = 0$, which is a contradiction.
3. $f_0 = 0$ and $a_0 = 0$.

We now discuss case (3). Here, the equations $\sum_{i=0}^{k} f_i a_{k-i} = 0$ for each $k \geq 0$ reduce to $\sum_{i=1}^{k} f_i a_{k-i+1} = 0$ for each $k \geq 1$. For $k = 1$, we get that $f_1 a_1 = 0$, which again provides three possibilities as above. The possibilities $f_1 \neq 0, a_1 = 0$ and $f_1 = 0, a_1 \neq 0$ give contradiction along the lines of proof done above and we will be left with the possibility $f_1 = 0 = a_1$. Now the equations $\sum_{i=1}^{k} f_i a_{k-i+1} = 0$ for each $k \geq 1$ reduce to the equations $\sum_{i=2}^{k} f_i a_{k-i+2} = 0$ for each $k \geq 2$. Working recursively on same steps as above, we obtain that $f_i = 0 = a_i$ for each $i \geq 0$. This yields that $f = 0 = \phi$. This contradicts our assumption.

Hence the result. □

We would like to add here that the adjoint of a weighted Toeplitz operator with analytic symbol may not satisfy the condition in Proposition 2.3. For example, $\mathcal{K}er(T_z^{\beta *}) = [e_0]$.

If $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$, then the symbol $\overline{\phi}$ is defined as $\overline{\phi}(z) = \sum_{n=-\infty}^{\infty} \overline{a}_n z^{-n}$. If β is semi-dual (i.e. $\beta_n = \beta_{-n}$ for each $n \geq 1$), then for $\phi \in L^\infty(\beta)$, $\overline{\phi} \in L^\infty(\beta)$ with $||\phi||_\infty = ||\overline{\phi}||_\infty$.

With almost same computations as in Theorem 2.3, we arrive at our next result.

Proposition 2.4. *Let $\phi \in L^\infty(\beta)$ be such that $0 \neq \overline{\phi} \in H^\infty(\beta)$. Then, $\mathcal{K}er(T_\phi^{\beta *}) = \{0\}$.*

Next, we consider a special case when $\phi \in L^\infty(\beta)$ is of the form $\phi(z) = a_{-i} z^{-i} + a_i z^i$, where $i \geq 1$. We find that the Coburn alternative can be extended for weighted Toeplitz operators in this case.

Theorem 2.4. *Let $\phi(z) = a_{-i} z^{-i} + a_i z^i \in L^\infty(\beta)$, where a_{-i}, a_i are both non-zero complex numbers. Then, at least one of T_ϕ^β and $T_\phi^{\beta *}$ is injective.*

Proof. Suppose that $T_\phi^\beta f = 0$ for some $0 \neq f = \sum_{n=0}^{\infty} f_n z^n \in H^2(\beta)$. Suppose also that $T_\phi^{\beta *} g = 0$, where $g = \sum_{n=0}^{\infty} g_n z^n \in H^2(\beta)$. We claim that $g = 0$.

Since $\langle T_\phi^\beta f, e_m \rangle = 0$ for each $m \geq 0$, we obtain that for each $n \geq 0$, $f_{(2n+1)i+p} = 0$ while $f_{2ni+p} = (-1)^n f_p (\frac{a_i}{a_{-i}})^n$, for $p = 0, 1, \cdots, i-1$. Also, $|f_0|^2 \sum_{n=0}^{\infty} |\frac{a_i}{a_{-i}}|^{2n} \beta_{2ni}^2 \leq ||f||_\beta^2 < \infty$. Making use of the fact that $\beta_n \geq 1$ for each $n \in \mathbb{Z}$ (equation (1.1)), the above inequality provides that $|\frac{a_i}{a_{-i}}|^2 < 1$. This in turn provides that $||g||_\beta^2 \geq |g_0|^2 \sum_{n=0}^{\infty} |\frac{a_{-i}}{a_i}|^{2n} \frac{1}{\beta_{2ni}^2} > |g_0|^2 \sum_{n=0}^{\infty} \frac{1}{r^{4ni}}$ (using equation (1.1)), which is a geometric series with common ratio ≥ 1 and hence divergent. Thus g has no other option but to be 0. □

Now, we discuss some symbols inducing normal weighted Toeplitz operators. We first consider the weighted Toeplitz operator T_ϕ^β induced by the Laurent polynomial $\phi(z) = a_{-1} z^{-1} + a_0 + a_1 z \in L^\infty(\beta)$, where $a_{-1}, a_0, a_1 \in \mathbb{C}$ and find the condition(s) for this operator to be normal.

Proposition 2.5. *A necessary condition for T_ϕ^β on $H^2(\beta)$, where $\phi(z) = a_{-1}z^{-1} + a_0 + a_1 z$, $a_1 \neq 0$, to be normal is that $|a_{-1}| \geq |a_1|$.*

Proof. Since T_ϕ^β is normal, therefore $||T_\phi^\beta e_0||_\beta = ||T_\phi^{\beta^*} e_0||_\beta$, which gives ${\beta_1}^2 = \frac{|a_{-1}|}{|a_1|}$. But $\beta_1 \geq 1$, which implies that $|a_{-1}| \geq |a_1|$. □

We find that the condition obtained in Proposition 2.5 is not sufficient for T_ϕ^β to be normal on each weighted Hardy space $H^2(\beta)$. For, consider $H^2(\beta)$ with $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ such that $\beta_n = 2$ for $n \in \mathbb{Z} - \{0\}$ and let $\phi(z) = z + \frac{1}{z}$. Then $|a_{-1}| = |a_1| = 1$, but T_ϕ^β is not normal since $||T_\phi^\beta e_0||_\beta = 2$ and $||T_\phi^{\beta *} e_0||_\beta = \frac{1}{2}$.

We improve Proposition 2.5 to obtain a characterization for normal weighted Toeplitz operators in the following form.

Theorem 2.5. *A necessary and sufficient condition for T_ϕ^β on $H^2(\beta)$, induced by the Laurent polynomial $\phi(z) = a_{-1}z^{-1} + a_0 + a_1 z,\ a_1 \neq 0$, to be normal is that $(\frac{|a_{-1}|}{|a_1|})^n = {\beta_n}^2$ for each $n \geq 1$.*

Proof. Let T_ϕ^β be normal. Then, using the equality $||T_\phi^\beta e_j||_\beta = ||T_\phi^{\beta^*} e_j||_\beta$ for $j \geq 0$ and using the principle of mathematical induction, we obtain the necessary condition. For sufficiency, using that $(\frac{|a_{-1}|}{|a_1|})^n = {\beta_n}^2$ for each $n \geq 1$, we obtain that $T_\phi^\beta T_\phi^{\beta *} e_j = T_\phi^{\beta *} T_\phi^\beta e_j$ for each $j \geq 0$. Hence the result. □

The next observation, which is also obtained by Avendano [1, Corollary 3.2.16, p. 107], is straightforward from here.

Corollary 2.2. *The Toeplitz operator T_ϕ, where $\phi(z) = z + \frac{1}{z}$, on Hardy space is always normal.*

Remark 2.3. *It is easy to see that each weighted Toeplitz operator T_ϕ^β, where $\phi(z) = c$ (constant), is a normal operator. For, if $f(z) = \sum_{n=0}^{\infty} b_n z^n \in H^2(\beta)$, then the structure of T_ϕ^β provides that $||T_\phi^\beta f||_\beta^2 = |c|^2 \sum_{n=0}^{\infty} |b_n|^2 {\beta_n}^2 = |c|^2 ||f||_\beta^2 = ||T_\phi^{\beta *} f||_\beta^2$.*

Theorem 2.6. *For a weighted Toeplitz operator T_ϕ^β on $H^2(\beta)$, we obtain the following.*

1. *If $\phi \in L^\infty(\beta)$ is of the form $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ (or $\phi(z) = \sum_{n=-\infty}^{0} a_n z^n$), then T_ϕ^β is normal if and only if $a_n = 0$ for each $n \geq 1$ (or $a_{-n} = 0$ for each $n \geq 1$).*
2. *If $\phi(z) = z^{n_0} + \frac{1}{z^{n_0}}, n_0 \geq 1$, then T_ϕ^β is normal if and only if $\beta_n = 1 \ \forall \ n \geq 0$.*

Proof. Consider the weighted Toeplitz operator T_ϕ^β on $H^2(\beta)$ induced by the symbol $\phi \in L^\infty(\beta)$. For the proof of (1), we assume that T_ϕ^β, where $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$, is normal. Then, $||T_\phi^\beta e_0||_\beta = ||T_\phi^{\beta *} e_0||_\beta$, which gives that $a_n = 0$ for each $n \geq 1$. Converse follows using Remark 2.3.

The case when $\phi(z) = \sum_{n=-\infty}^{0} a_n z^n$ can be proved similarly.

Next we prove (2). Let T_ϕ^β, where $\phi(z) = z^{n_0} + \frac{1}{z^{n_0}}, n_0 \geq 1$, be normal. Then, $||T_\phi^\beta e_j||_\beta^2 = ||T_\phi^{\beta *} e_j||_\beta^2$, for each $j \geq 0$. This provides that

$$\frac{1}{\beta_j^2} \sum_{n=0}^{\infty} |a_{n-j}|^2 {\beta_n}^2 = \beta_j^2 \sum_{n=0}^{\infty} |\overline{a}_{j-n}|^2 \frac{1}{{\beta_n}^2}. \tag{2.2.2}$$

In particular, for $j = 0$, equation (2.2.2) yields that $\beta_{n_0} = 1$. Hence, $\beta_n = 1 \ \forall \ 0 \leq n \leq n_0$. Further, for $j = m$, where $(l-1)n_0 \leq m < ln_0$ and $l \geq 1$ is an integer, we get that $\beta_{m+n_0} = 1$. This provides that $\beta_n = 1$ for each $ln_0 \leq n < (l+1)n_0$. Hence, $\beta_n = 1 \ \forall \ n \geq 0$.

Conversely, if $\beta_n = 1 \ \forall \ n \geq 0$, then T_ϕ^β, where $\phi(z) = z^{n_0} + \frac{1}{z^{n_0}}, n_0 \geq 1$, is self adjoint and hence normal. □

Example 2.2. *Some examples of normal and not normal weighted Toeplitz operators.*

1. *Consider $H^2(\beta)$ with β defined as $\beta_n = 2^{|n|}$ for $n \in \mathbb{Z}$. Then, the weighted Toeplitz operator T_ϕ^β on $H^2(\beta)$, induced by $\phi(z) = \frac{4}{z} + z$, is a normal operator (using Theorem 2.5).*

2. *Consider $H^2(\beta)$ where β such that $\beta_n = 1$ for each $n \in \mathbb{Z}$. Then, T^β_ϕ and T^β_ψ, where $\phi(z) = z + \frac{1}{z}$ and $\psi(z) = z - \frac{1}{z}$ are both normal. But $T^\beta_\phi + T^\beta_\psi = T^\beta_{\phi+\psi} = T^\beta_{2z}$ is not normal (using Theorem 2.6(1)). However, $T^\beta_\phi + T^\beta_\phi = T^\beta_{2\phi} = T^\beta_{2z+\frac{2}{z}}$ is a normal operator (using Theorem 2.5).*

3 Compression of k^{th}-order slant weighted Toeplitz operator

We refer to [3] and the references therein for the definition and detailed study of a k^{th}-order slant weighted Toeplitz operator $U^\beta_{k,\phi}$ ($k \geq 2$ is an integer) on $L^2(\beta)$. Let us recall the definition of the compression of this operator to $H^2(\beta)$.

For $\phi \in L^\infty(\beta)$, let $V^\beta_{k,\phi}$ denote the compression of a k^{th}-order slant weighted Toeplitz operator $U^\beta_{k,\phi}$ to $H^2(\beta)$. Then, $V^\beta_{k,\phi} : H^2(\beta) \to H^2(\beta)$ is given by $V^\beta_{k,\phi} = P^\beta U^\beta_{k,\phi}|_{H^2(\beta)} = W_k T^\beta_\phi$, where W_k is the operator on $L^2(\beta)$ given by

$$W_k e_n = \begin{cases} \frac{\beta_m}{\beta_{km}} e_m & \text{if } n = km \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

The effect of $V^\beta_{k,\phi}$ on the orthonormal basis $\{e_n(z)\}_{n\geq 0}$ of $H^2(\beta)$ is given by $V^\beta_{k,\phi} e_j = \frac{1}{\beta_j} \sum\limits_{n=0}^{\infty} a_{kn-j} \beta_n e_n$. We begin with the following.

Theorem 3.1. *Let $m \geq 2$ be an integer such that $m \neq k$. Then, $V^\beta_{k,\phi}$ is the compression of a m^{th}-order slant weighted Toeplitz operator to $H^2(\beta)$ if and only if $\phi = 0$.*

Proof. Let, if possible, $V^\beta_{k,\phi}$ be the compression of a m^{th}-order slant weighted Toeplitz operator to $H^2(\beta)$, where $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$. Along the lines of proof of Theorem 2.7 in [3], using the structure of $V^\beta_{k,\phi}$ recursively, we get that

$$a_{ki-j} = a_{k(i+k)-j-mk} \tag{3.1.1}$$

for each $i, j \geq 0$. This provides that $a_0 = a_{tk|k-m|}, a_1 = a_{tk|k-m|+1}, \cdots, a_{k|k-m|-1} = a_{tk|k-m|+k|k-m|-1}$. But $\phi \in L^\infty(\beta) \subseteq L^2(\beta)$, which yields that $a_n \to 0$ as $n \to \infty$ and this helps us to conclude that $a_0 = a_1 = \cdots, a_{k|k-m|-1} = 0$. Consequently, (3.1.1) helps to provide that $a_n = 0$ for each $n \in \mathbb{Z}$. Thus $\phi = 0$.

The converse is trivial. □

Theorem 3.2. *Let $\phi \in L^\infty(\beta)$. Then, the operator $W_k V^\beta_{k,\phi}$ is the compression of a k^{th}-order slant weighted Toeplitz operator to $H^2(\beta)$ if and only if $\phi = 0$.*

Proof. "If" part is straightforward. We prove the "only if" part. Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$ be such that $W_k V_{k,\phi}^\beta = V_{k,\psi}^\beta$ for some $\psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n \in L^\infty(\beta)$. Then for each $i \geq 0$, $W_k V_{k,\phi}^\beta e_i = V_{k,\psi}^\beta e_i$. This yields that $a_{k^2 n-i} = b_{kn-i}$ for each $n, i \geq 0$. As $a_n \to 0$ as $n \to \infty$, this helps to conclude that $\phi = 0$. □

It is interesting to obtain that if the sequence β is such that $\{\frac{\beta_{kn}}{\beta_n}\}_{n\geq 0}$ is bounded, then there does not exist any non-zero Hilbert-Schmidt $V_{k,\phi}^\beta$ on $H^2(\beta)$.

Theorem 3.3. *Let $\{\frac{\beta_{kn}}{\beta_n}\}_{n\geq 0}$ be bounded. Then $V_{k,\phi}^\beta$ is Hilbert-Schmidt if and only if $\phi = 0$.*

Proof. Nothing needs to be proved in the "if" part. We prove the "only if" part. Let $V_{k,\phi}^\beta$, where $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$, be a Hilbert-Schmidt operator. Then, $\sum_{j=0}^{\infty} \|V_{k,\phi}^\beta e_j\|_\beta^2 < \infty$. Consider now the sum $\sum_{j=0}^{\infty} ||V_{k,\phi}^\beta e_j\|_\beta^2 = \sum_{j=0}^{\infty} \frac{1}{{\beta_j}^2} \sum_{n=0}^{\infty} |a_{kn-j}|^2 \beta_n^2 = A_0 + A_1 + \cdots + A_{k-1} + B$, where $A_l = \sum_{j=1}^{\infty} |a_{kj-l}|^2 \big(\sum_{n=0}^{\infty} \frac{\beta_{n+j}^2}{\beta_{kn+l}^2} \big)$ for each $0 \leq l \leq k-1$ and $B = \sum_{i=0}^{\infty} |a_{-i}|^2 \big(\sum_{n=0}^{\infty} \frac{\beta_n^2}{\beta_{kn+i}^2} \big)$. The boundedness of $\{\frac{\beta_{kn}}{\beta_n}\}_{n\geq 0}$ together with equation (1.1) helps to ensure us that each term in parenthesis of A_l's $(l = 0, 1, \cdots, k-1)$ and B is a divergent series. But each sum $A_0, A_1, \cdots, A_{k-1}$ and B is finite. This leaves a_n's, $n \in \mathbb{Z}$ with no option other than being all zeroes. Hence $\phi = 0$. □

With the help of next example, we can verify that the assumption of boundedness of $\{\frac{\beta_{kn}}{\beta_n}\}_{n\geq 0}$ cannot be dropped in Theorem 3.3.

Example 3.1. *Consider the space $H^2(\beta)$ with β be defined as $\beta_n = 2^n$, for each $n \geq 0$. Clearly, $\{\frac{\beta_{kn}}{\beta_n}\}_{n\geq 0}$ is not bounded. Let $\phi = c$, where $0 \neq c \in \mathbb{C}$. Consider the operator $V_{k,\phi}^\beta$ with β and ϕ as defined above. Then, $\sum_{j=0}^{\infty} \|V_{k,\phi}^\beta e_j\|_\beta^2 = |c|^2 \sum_{n=0}^{\infty} \frac{\beta_n^2}{\beta_{kn}^2} = |c|^2 \sum_{n=0}^{\infty} \frac{1}{2^{2n(k-1)}}$, which is finite. Thus $V_{k,\phi}^\beta$ is a non-zero Hilbert-Schmidt operator on $H^2(\beta)$.*

Since every Hilbert-Schmidt operator is compact, therefore Example 3.1 ensures the existence of non-zero compressions of k^{th}-order slant weighted Toeplitz operators which are compact. However, again, the boundedness of $\{\frac{\beta_{kn}}{\beta_n}\}_{n\geq 0}$ restricts the compact operators of this class to the zero operator.

Theorem 3.4. *If the sequence $\{\frac{\beta_{kn}}{\beta_n}\}_{n\geq 0}$ is bounded, then the only compact $V_{k,\phi}^\beta$ is the zero operator.*

Proof. Let $V^{\beta}_{k,\phi}$ be a compact operator with $\phi \in L^{\infty}(\beta)$ such that a_n denotes the n^{th} Fourier coefficient of ϕ for each $n \in \mathbb{Z}$. Then working along the lines of computations of Theorem 2.2, we obtain that for each $s, t \geq 0$, $|a_{kt-s}|\frac{\beta_{t+n}}{\beta_{s+kn}} = |\langle V^{\beta}_{k,\phi}e_{s+kn}, e_{t+n}\rangle| \leq ||V^{\beta}_{k,\phi}e_{s+kn}||_{\beta} \to 0$ as $n \to \infty$. The assumption of boundedness of $\{\frac{\beta_{kn}}{\beta_n}\}_{n\geq 0}$ provides us a real number $M > 0$ such that $\frac{\beta_{kn}}{\beta_n} \leq M$ for each $n \geq 0$.

Let $s > t$. Then $s + kn > t + n$ and hence on using equation (1.1), we get that

$$|a_{kt-s}|\frac{r^s}{M} \leq |a_{kt-s}|\frac{\beta_{t+n}}{\beta_n}\frac{\beta_n}{\beta_{kn}}\frac{\beta_{kn}}{\beta_{kn+s}} \to 0 \text{ as } n \to \infty.$$

Hence, $a_{kt-s} = 0$ for $s > t \geq 0$. As a consequence, $a_n = 0$ for each $n \in \mathbb{Z}$. Thus $\phi = 0$. Hence the claim. □

$V^{\beta}_{k,\phi}$ satisfies the operator equation $V^{\beta}_{k,\phi} = V^{\beta *}V^{\beta}_{k,\phi}T^{\beta}_{z^k}$, where V^{β} denotes the weighted shift operator on $H^2(\beta)$ such that $V^{\beta}e_n = \frac{\beta_n}{\beta_{n+1}}e_{n+1}$ for each $n \geq 0$ and $T^{\beta}_{z^k}$ denotes the weighted Toeplitz operator on $H^2(\beta)$ induced by $z^k \in L^{\infty}(\beta)$.

The converse, however, is not true. We present here below an example of a bounded operator T on $H^2(\beta)$, which is not the compression of a k^{th}-order slant weighted Toeplitz operator but satisfies the above equation.

Example 3.2. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}} = \{2^{|n|}\}_{n\in\mathbb{Z}}$. Define a formal Laurent series $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, where $a_n = \frac{1}{n^2 2^n}$ if $n > 0$ and $a_n = 0$ otherwise. Then ϕ is bounded and analytic in the domain $\|M^{\beta^{-1}}_z\|^{-1} < |z| < \|M^{\beta}_z\|$ and hence $\phi \in L^{\infty}(\beta)$ [11, Theorem 10'(vii)(b)].*

Let T be an operator on $H^2(\beta)$ defined as $T = T^{\beta}_{\phi}W_k$. Then T is a bounded operator on $H^2(\beta)$ satisfying for each $n \geq 0$,

$$Te_n = \begin{cases} \frac{1}{\beta_{km}} \sum_{l=0}^{\infty} a_{l-m}\beta_l e_l & \text{if } n = km \text{ for some } m \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Using the structure of T, $T^{\beta}_{z^k}$ and that $V^{\beta *}e_n = \begin{cases} 0 & \text{if } n = 0 \\ \frac{\beta_{n-1}}{\beta_n}e_{n-1} & \text{if } n \geq 1 \end{cases}$, it is easy to see that T satisfies the operator equation $T = V^{\beta *}TT^{\beta}_{z^k}$. However, if we assume that $T = V^{\beta}_{k,\psi}$ for some $\psi(z) = \sum_{n\in\mathbb{Z}} d_n z^n \in L^{\infty}(\beta)$. Then, simple computations provide that $d_n = 0$ for all $n \in \mathbb{Z}$ except d_{kn}'s which are given as $d_{kn} = a_n$ if $n > 0$. Hence, $\psi(z) = \sum_{n\in\mathbb{Z}} d_n z^n = \sum_{n\geq 1} a_n z^{kn} = \phi(z^k) \notin L^{\infty}(\beta)$. This contradicts our assumption and hence T can't be the compression of a k^{th}-order slant weighted Toeplitz operator to $H^2(\beta)$.

Theorem 3.5. *$V^{\beta}_{k,\phi}$ on $H^2(\beta)$ cannot be a Fredholm operator.*

Proof. Let, if possible, $V^{\beta}_{k,\phi}$ be a Fredholm operator of index n. Since $V^{\beta}_{k,\phi} = V^{\beta *}V^{\beta}_{k,\phi}T^{\beta}_{z^k}$ and the index of the operator $V^{\beta *}V^{\beta}_{k,\phi}T^{\beta}_{z^k}$ is $n - k + 1$, we get that $k = 1$, which is not possible. Hence the claim. □

Since every invertible operator is Fredholm, it is now straightforward to state the following.

Corollary 3.1. *Let $\phi \in L^{\infty}(\beta)$. Then $0 \in \sigma(V^{\beta}_{k,\phi})$.*

If $\phi \in L^{\infty}(\beta)$ and the sequence $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded, then $\phi(z^k) \in L^{\infty}(\beta)$ (see [3]). This helps to provide the following.

Theorem 3.6. *Let the sequence $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ be bounded and $\phi \in L^{\infty}(\beta)$. Then, $\sigma_p(V^{\beta}_{k,\phi(z^k)}) = \{0\} \cup \sigma_p(V^{\beta}_{k,\phi})$.*

Proof. Let $0 \neq \lambda \in \sigma_p(V^{\beta}_{k,\phi(z^k)})$. Then there exists a non-zero vector $f \in H^2(\beta)$ such that $V^{\beta}_{k,\phi(z^k)}f = \lambda f$. Now, $V^{\beta}_{k,\phi(z^k)}f = P^{\beta}W_kM^{\beta}_{\phi(z^k)}f = P^{\beta}M^{\beta}_{\phi}W_kf$. Hence, $W_kf \neq 0$ and

$$\begin{aligned} V^{\beta}_{k,\phi}(W_kf) = P^{\beta}W_kM^{\beta}_{\phi}W_kf &= P^{\beta}W_kW_kM_{\phi(z^k)}f \\ &= W_kP^{\beta}W_kM_{\phi(z^k)}f \\ &= W_kV^{\beta}_{k,\phi(z^k)}f = \lambda(W_kf). \end{aligned}$$

Hence, $\lambda \in \sigma_p(V^{\beta}_{k,\phi})$. Conversely, let $0 \neq \mu \in \sigma_p(V^{\beta}_{k,\phi})$. Then there exists $0 \neq g \in H^2(\beta)$ such that $V^{\beta}_{k,\phi}g = \mu g$. Also, $V^{\beta}_{k,\phi}g = W_kT^{\beta}_{\phi}g$. Therefore, $T^{\beta}_{\phi}g \neq 0$ and $V^{\beta}_{k,\phi(z^k)}(T^{\beta}_{\phi}g) = P^{\beta}W_kM^{\beta}_{\phi(z^k)}(T^{\beta}_{\phi}g) = (P^{\beta}M^{\beta}_{\phi})(W_kT^{\beta}_{\phi}g) = \mu(T^{\beta}_{\phi}g)$. Hence, $\mu \in \sigma_p(V^{\beta}_{k,\phi(z^k)})$. Therefore, the sets of non-zero eigen values of $V^{\beta}_{k,\phi(z^k)}$ and $V^{\beta}_{k,\phi}$ are same.

Also, $V^{\beta}_{k,\phi(z^k)}e_1 = 0$. So, 0 always belong to $\sigma_p(V^{\beta}_{k,\phi(z^k)})$. Hence, the result. □

Making use of Corollary 3.1 and the fact that $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$ for any two operators A and B, we have the following.

Theorem 3.7. *Let $\phi \in L^{\infty}(\beta)$ be such that $\phi(z^k) \in L^{\infty}(\beta)$. Then, $\sigma(V^{\beta}_{k,\phi}) = \sigma(V^{\beta}_{k,\phi(z^k)})$.*

Proof. The proof follows as

$$\begin{aligned} \sigma(V^{\beta}_{k,\phi(z^k)}) = \sigma(V^{\beta}_{k,\phi(z^k)}) \cup \{0\} &= \sigma(T^{\beta}_{\phi}W_kP^{\beta}) \cup \{0\} \\ &= \sigma(P^{\beta}W_kT^{\beta}_{\phi}) \cup \{0\} \\ &= \sigma(V^{\beta}_{k,\phi}). \end{aligned}$$

□

We have been able to partially extend the Coburn alternative for weighted Toeplitz operators. We have obtained some symbols ϕ in $L^\infty(\beta)$, in Theorem 2.3, Proposition 2.4 and Theorem 2.4, such that atleast one of T_ϕ^β and $T_\phi^{\beta*}$ is injective. With observations while working on different symbols, an intuitive answer seems to be in affirmative for any general $\phi \in L^\infty(\beta)$. However, it is yet to be obtained.

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