# A new subclass of $\alpha$-starlike functions associated with Salagean derivative operator with positive coefficients

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**Abstract**

Making use of Salagean operator we introduce a new class of univalent functions with positive coefficients. Among the results presented in this paper include the coefficient bounds, distortion inequalities, extreme points and convolution property are studied.

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## 1 Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : z \in C \text{ and } |z| < 1\}$ and $S$ denote the subclass of $A$ that are univalent in $U$.

Salagean [5] introduced the following operator which is popularly known as the Salagean derivative operator :

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

and in general,

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \quad \left(n \in N_0 = N \cup \{0\}\right).$$

We easily find from (1.1) that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \qquad (f \in A; n \in N_0)$$

In 1999, Kanas and Wisniowaska [2] (see also [1]) studied the class of $\alpha$-uniformly convex analytic functions denoted by $\alpha - UCV$, $0 \leq \alpha < \infty$ so that $f \in \alpha - UCV$, if and only if

$$(1.2) \qquad Re\left\{1 + (z - \zeta)\frac{f''(z)}{f'(z)}\right\} \geq 0, \quad |\zeta| \leq \alpha, \quad (z \in U).$$

For real $\phi$ we may let $\zeta = -\alpha z e^{i\phi}$. Then condition (1.2) can be written as

$$(1.3) \qquad Re\left\{1 + \left(1 + \alpha e^{i\phi}\right)\frac{zf''(z)}{f'(z)}\right\} \geq 0,$$

and $\alpha - UCV(\beta)$ denote the subclass of S, if and only if

$$(1.4) \qquad Re\left\{1 + \left(1 + \alpha e^{i\phi}\right)\frac{zf''(z)}{f'(z)}\right\} \geq \beta, \qquad (0 \leq \beta < 1).$$

Further the class $\alpha - S_p(\beta)$ denotes the subclass of S, if and only if

$$(1.5) \qquad Re\left\{\left(1 + \alpha e^{i\phi}\right)\frac{zf'(z)}{f(z)} - \alpha e^{i\phi}\right\} \geq \beta, \qquad (0 \leq \beta < 1).$$

Further, let V be the subclass of S consisting of functions of the form

$$(1.6) \qquad f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k,$$

then $D^n f(z) = z + \sum_{k=2}^{\infty} k^n |a_k| z^k$.

Also for $1 < \gamma \leq \frac{4+\alpha}{3}$ and $z \in U$, we define

$$(1.7) \qquad \alpha - S^*_{p,n}(\gamma) = \left\{f \in S : Re\left\{\left(1 + \alpha e^{i\phi}\right)\frac{D^{n+1} f(z)}{D^n f(z)} - \alpha e^{i\phi}\right\} < \gamma\right\}.$$

**Remark** : If we put $n = 0$ we get the class $\alpha - S^*_p(\gamma)$ and if we put $n = 1$ we get the class $\alpha - UCV^*(\gamma)$ studied by S. Porwal and K.K.Dixit [4].

Let

$$(1.8) \qquad \alpha - PS^*_{p,n}(\gamma) \equiv \alpha - S^*_{p,n}(\gamma) \cap V.$$

In particular, when $\alpha = 0, n = 1$ we obtain $\alpha - S^*_{p,n}(\gamma) \equiv L(\gamma)$, when $\alpha = 0, n = 0$ we obtain $\alpha - S^*_{p,n}(\gamma) \equiv M(\gamma)$, when $\alpha = 0, n = 1$ we obtain $\alpha - PS^*_{p,n}(\gamma) \equiv U(\gamma)$ and when $\alpha = 0, n = 0$ we obtain $\alpha - PS^*_{p,n}(\gamma) \equiv V(\gamma)$. These classes $L(\gamma)$, $M(\gamma)$, $U(\gamma)$ and $V(\gamma)$ have been extensively studied by Uralegaddi et al. [6].

Several authors such as ([1],[2],[3]) studied the classes of $\alpha$-uniformly convex and $\alpha$-starlike functions only. In the present paper, using Salagean derivative operator, an attempt has been made to have unified study of above mentioned classes of functions with positive coefficients only.

## 2 Coefficient inequalities

The following theorem lays the foundation of our systematic study of the class $\alpha - S^*_{p,n}(\gamma)$ defined in the preceding section.

**Theorem 2.1.** *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be in $S$ if*

$$\sum_{k=2}^{\infty} k^n [k + k\alpha - \alpha - \gamma] |a_k| \leq \gamma - 1, \tag{2.1}$$

*then $f \in \alpha - S^*_{p,n}(\gamma)$.*

*Proof.* Let $\sum_{k=2}^{\infty} k^n [k + k\alpha - \alpha - \gamma] |a_k| \leq \gamma - 1$. It suffices to show that

$$\left| \frac{\left(1 + \alpha e^{i\phi}\right) \frac{D^{n+1} f(z)}{D^n f(z)} - \alpha e^{i\phi} - 1}{\left(1 + \alpha e^{i\phi}\right) \frac{D^{n+1} f(z)}{D^n f(z)} - \alpha e^{i\phi} - (2\gamma - 1)} \right| < 1, \qquad (z \in U). \tag{2.2}$$

We have

L.H.S. of (2.2)

$$= \left| \frac{\left(1 + \alpha e^{i\phi}\right) \frac{D^{n+1} f(z)}{D^n f(z)} - \alpha e^{i\phi} - 1}{\left(1 + \alpha e^{i\phi}\right) \frac{D^{n+1} f(z)}{D^n f(z)} - \alpha e^{i\phi} - (2\gamma - 1)} \right|$$

$$= \left| \frac{\left(1 + \alpha e^{i\phi}\right) \left[D^{n+1} f(z) - D^n f(z)\right]}{\left(1 + \alpha e^{i\phi}\right) D^{n+1} f(z) - \alpha e^{i\phi} D^n f(z) - (2\gamma - 1) D^n f(z)} \right|$$

$$= \left| \frac{\left(1 + \alpha e^{i\phi}\right) \left[\sum_{k=2}^{\infty} k^{n+1} a_k z^k - \sum_{k=2}^{\infty} k^n a_k z^k\right]}{2(\gamma - 1)\left[z + \sum_{k=2}^{\infty} k^n a_k z^k\right] - \left(1 + \alpha e^{i\phi}\right)\left[\sum_{k=2}^{\infty} k^{n+1} a_k z^k - \sum_{k=2}^{\infty} k^n a_k z^k\right]} \right|$$

$$= \left| \frac{\left(1 + \alpha e^{i\phi}\right) \sum_{k=2}^{\infty} k^n (k - 1) a_k z^k}{2(\gamma - 1) z - \sum_{k=2}^{\infty} \left[2(\gamma - 1) k^n + \left(1 + \alpha e^{i\phi}\right) k^n (k - 1)\right] a_k z^k} \right|$$

$$\leq \frac{(1 + \alpha) \sum_{k=2}^{\infty} k^n (k - 1) |a_k|}{2(\gamma - 1) - \sum_{k=2}^{\infty} k^n \left[(1 + \alpha)(k - 1) - 2(\gamma - 1)\right] |a_k|}.$$

The last expression is bounded above by 1 by hypothesis

$$\sum_{k=2}^{\infty} k^n \left[(1+\alpha)(k-1)+(1+\alpha)(k-1)-2(\gamma-1)\right]|a_k| \leq 2(\gamma-1)$$

or

$$\sum_{k=2}^{\infty} k^n \left[k+k\alpha-\alpha-\gamma\right]|a_k| \leq (\gamma-1).$$

□

**Theorem 2.2.** *A function $f(z)=z+\sum_{k=2}^{\infty}|a_k|z^k$ is in $\alpha-S^*_{p,n}(\gamma)$ if and only if*

$$\sum_{k=2}^{\infty} k^n \left[k+k\alpha-\alpha-\gamma\right]|a_k| \leq \gamma-1. \tag{2.3}$$

The result is Sharp.

*Proof.* The "if part" , follows from Theorem 2.1. To prove the "only if" part, let $f \in \alpha-S^*_{p,n}(\gamma)$, then by the definition (1.8), we have

$$Re\left\{\left(1+\alpha e^{i\phi}\right)\frac{D^{n+1}f(z)}{D^n f(z)}-\alpha e^{i\phi}\right\}<\gamma, \qquad (z\in U),$$

which is equivalent to

$$Re\left\{\left(1+\alpha e^{i\phi}\right)\frac{1+\sum_{k=2}^{\infty}k^{n+1}|a_k|z^{k-1}}{1+\sum_{k=2}^{\infty}k^{n}|a_k|z^{k-1}}-\alpha e^{i\phi}\right\}<\gamma.$$

The above condition must hold for all values of $z, |z|=r<1$. upon choosing the values of z on the positive real axis, where $0\leq z=r<1$ and $Re\left(-\alpha e^{i\phi}\right)\geq -|\alpha e^{i\phi}|=-\alpha$, the above inequality reduces to

$$(1+\alpha)+\sum_{k=2}^{\infty}(1+\alpha)k^{n+1}|a_k|r^{k-1}-\alpha-\sum_{k=2}^{\infty}\alpha k^n|a_k|r^{k-1}\leq \gamma+\sum_{k=2}^{\infty}\gamma k^n|a_k|r^{k-1}.$$

Letting $r\to 1$, we have $\sum_{k=2}^{\infty}k^n\left[k+k\alpha-\alpha-\gamma\right]|a_k|\leq\gamma-1$ and the proof of the Theorem 2.2 is complete.

Finally , we note that the assertion (2.3) of Theorem 2.2 is sharp, the extremal function being

$$f(z)=z+\frac{\gamma-1}{k^n\left[k+k\alpha-\alpha-\gamma\right]}z^k.$$

□

**Remark** : If we put $n = 1$ we get the Theorem 2.3 and if $n = 0$ we get the Theorem 2.4 of [4]

**Corollary 2.1.** *Let the function $f(z)$ defined by (1.6) belong to the class $\alpha - S^*_{p,n}(\gamma)$. Then*

$$|a_k| \leq \frac{\gamma - 1}{k^n [k + k\alpha - \alpha - \gamma]}. \tag{2.4}$$

## 3 Distortion Inequalities

In this section, we shall prove distortion theorem for the functions belonging to the class $\alpha - PS^*_{p,n}(\gamma)$ which yields the covering results for this class.

**Theorem 3.1.** *Let $f \in \alpha - PS^*_{p,n}(\gamma)$ then for $|z| = r < 1$, we have*

$$|f(z)| \leq r + \frac{\gamma - 1}{2^n [2 + \alpha - \gamma]} r^2 \tag{3.1}$$

*and*

$$|f(z)| \geq r - \frac{\gamma - 1}{2^n [2 + \alpha - \gamma]} r^2 \tag{3.2}$$

*with equality for $f(z) = z + \frac{\gamma-1}{2^n[2+\alpha-\gamma]} z^2$.*

*Proof.* Since $f(z) \in \alpha - PS^*_{p,n}(\gamma)$, we apply Theorem 2.2

$$2^n [2 + \alpha - \gamma] \sum_{k=2}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} k^n [k + k\alpha - \alpha - \gamma] |a_k| \leq \gamma - 1.$$

Thus we have

$$\sum_{k=2}^{\infty} |a_k| \leq \frac{\gamma - 1}{2^n [2 + \alpha - \gamma]}. \tag{3.3}$$

From (1.6) and (3.3) we obtain

$$|f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} |a_k|$$

$$\leq r + \frac{\gamma - 1}{2^n [2 + \alpha - \gamma]} r^2$$

and

$$|f(z)| \geq r - \frac{\gamma - 1}{2^n [2 + \alpha - \gamma]} r^2.$$

This completes the proof of the Theorem 3.1. □

**Theorem 3.2.** *Let the function* $f(z) \in \alpha - PS^*_{p,n}(\gamma)$ *then* $|z| = r < 1$, *we have*

$$|f'(z)| \leq 1 + \frac{\gamma - 1}{2^{n-1}\left[2 + \alpha - \gamma\right]} r \tag{3.4}$$

*and*

$$|f'(z)| \geq 1 - \frac{\gamma - 1}{2^{n-1}\left[2 + \alpha - \gamma\right]} r. \tag{3.5}$$

*The equalities in (3.4) and (3.5) are attained for the function* $f(z)$ *given by* $|f(z)| = z + \frac{\gamma-1}{2^n[2+\alpha-\gamma]} z^2$.

*Proof.* We have

$$|f'(z)| \leq 1 + \sum_{k=2}^{\infty} k|a_k||z|^{k-1} \leq 1 + r\sum_{k=2}^{\infty} k|a_k|.$$

Since $f(z) \in \alpha - PS^*_{p,n}(\gamma)$, we have

$$2^{n-1}\left[2 + \alpha - \gamma\right] \sum_{k=2}^{\infty} k|a_k| \leq \sum_{k=2}^{\infty} k^n \left[k + k\alpha - \alpha - \gamma\right] |a_k| \leq \gamma - 1.$$

Thus we have

$$\sum_{k=2}^{\infty} k|a_k| \leq \frac{\gamma - 1}{2^n\left[2 + \alpha - \gamma\right]},$$

hence

$$|f'(z)| \leq 1 + \frac{\gamma - 1}{2^{n-1}\left[2 + \alpha - \gamma\right]} r.$$

□

## 4 Extreme Points

**Theorem 4.1.** *Let* $f_1(z) = z$ *and* $f_k(z) = z + \frac{\gamma-1}{k^n[k+k\alpha-\alpha-\gamma]} z^k, \quad (k = 2, 3, 4, ...)$ *then* $f \in \alpha - PS^*_{p,n}(\gamma)$, *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z),$$

*where* $\lambda_k \geq 0$ *and* $\sum_{k=1}^{\infty} \lambda_k = 1$.

*Proof.* Suppose that

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z)=z+\sum_{k=2}^{\infty}\lambda_k\frac{\gamma-1}{k^n\left[k+k\alpha-\alpha-\gamma\right]}z^k.$$

Then from Theorem 2.2, we have

$$\sum_{k=2}^{\infty}k^n\left[k+k\alpha-\alpha-\gamma\right]\lambda_k\frac{\gamma-1}{k^n\left[k+k\alpha-\alpha-\gamma\right]}=\sum_{k=2}^{\infty}\left(\gamma-1\right)\lambda_k$$

$$=\left(1-\lambda_1\right)\left(\gamma-1\right)\leq\left(\gamma-1\right).$$

Thus, in view of Theorem 2.2, we find that $f(z)\in\alpha-PS^*_{p,n}\left(\gamma\right)$.

Conversely, suppose that $f(z)\in\alpha-PS^*_{p,n}\left(\gamma\right)$. Then, since

$$a_k\leq\frac{\gamma-1}{k^n\left[k+k\alpha-\alpha-\gamma\right]},$$

we may set

$$\lambda_k=\frac{k^n\left[k+k\alpha-\alpha-\gamma\right]}{\gamma-1}a_k$$

and

$$\lambda_1=1-\sum_{k=2}^{\infty}\lambda_k.$$

Thus, clearly, we have

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z).$$

This completes the proof of theorem. □

**Corollary 4.1.** *The extreme points of the class* $\alpha-PS^*_{p,n}\left(\gamma\right)$ *are given by*

$$f_1(z)=z$$

*and*

$$f_k(z)=z+\frac{\gamma-1}{k^n\left[k+k\alpha-\alpha-\gamma\right]}z^k,\qquad\left(k\geq2\right).\tag{4.1}$$

**Theorem 4.2.** *The class* $\alpha-PS^*_{p,n}\left(\gamma\right)$ *is a convex set .*

*Proof.* Suppose that each of the functions $f_i(z)$, $(i = 1, 2)$ given by

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{k,i} z^k, \qquad (a_{k,i} \geq 0)$$

is in the class $\alpha - PS^*_{p,n}(\gamma)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z) = \eta f_1(z) + (1 - \eta) f_2(z), \qquad (0 \leq \eta < 1)$$

is also in the class $\alpha - PS^*_{p,n}(\gamma)$. Since

$$g(z) = \eta \left( z + \sum_{k=2}^{\infty} a_{k,1} z^k \right) + (1 - \eta) \left( z + \sum_{k=2}^{\infty} a_{k,2} z^k \right)$$

$$= z + \sum_{k=2}^{\infty} \left[ \eta a_{k,1} + (1 - \eta) a_{k,2} \right] z^k$$

with the aid of Theorem 2.2, we have

$$\sum_{k=2}^{\infty} k^n \left[ k + k\alpha - \alpha - \gamma \right] \left[ \eta a_{k,1} + (1 - \eta) a_{k,2} \right]$$

$$= \eta \sum_{k=2}^{\infty} k^n \left[ k + k\alpha - \alpha - \gamma \right] a_{k,1} + (1 - \eta) \sum_{k=2}^{\infty} k^n \left[ k + k\alpha - \alpha - \gamma \right] a_{k,2}$$

$$\leq \eta (\gamma - 1) + (1 - \eta) (\gamma - 1) = (\gamma - 1).$$

Which completes the proof of the theorem. □

## 5 Theorems involving Hadamard Product

Let $f(z)$ be define by (1.6), and let

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \qquad (b_k \geq 0). \tag{5.1}$$

The Hadamard Product of $f(z)$ and $g(z)$ is defined by

$$f * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{5.2}$$

The following result presents an interesting property of Hadamard Product .

**Theorem 5.1.** *Let the function* $f_1(z), f_2(z), ..., f_p(z)$ *be defined by*

$$f_m(z) = z + \sum_{k=2}^{\infty} C_{k,m} z^k, \qquad (C_{k,m} \geq 0) \tag{5.3}$$

*be in the class* $\alpha - PS^*_{p,n}(\gamma)$, $m = 1, 2, ..., p$ *respectively and* $0 \leq \alpha < 1$.
*Then*

$$f_1 * f_2 * ... * f_p \in \alpha - PS^*_{p,n}(\gamma), \quad \textit{where} \quad \gamma = max\{\gamma_m, \quad m = 1, 2, ..., p\}. \tag{5.4}$$

*Proof.* Since $f_m(z) \in \alpha - PS^*_{p,n}(\gamma)$, $m = 1, 2, ..., p$ by using Theorem 2.2 we have

$$\sum_{k=2}^{\infty} k^n [k + k\alpha - \alpha - \gamma] |C_{k,m}| \leq (\gamma_m - 1) \tag{5.5}$$

and

$$\sum_{k=2}^{\infty} C_{k,m} \leq \frac{\gamma_m - 1}{k^n [k + k\alpha - \alpha - \gamma]}, \tag{5.6}$$

for each $m = 1, 2, ..., p$.
Using (5.5) for any $m_0$ and (5.6) for the rest, we have

$$\sum_{k=2}^{\infty} k^n [k + k\alpha - \alpha - \gamma] \prod_{m=1}^{p} C_{k,m} \leq \prod_{m=1}^{p} (\gamma_m - 1) \left[\frac{1}{k^n [k + k\alpha - \alpha - \gamma]}\right]^{p-1}$$

$$\leq \prod_{m=1}^{p} (\gamma_m - 1) \qquad \leq (\gamma - 1)^p$$

$$\leq (\gamma - 1) \quad since \quad \gamma_m > 1 \ for \ m = 1, 2, 3, ..., p.$$

Consequently, we have the assertion (5.4) with the aid of Theorem 2.2 .Theorem 5.1 yields. □

**Theorem 5.2.** *Let the function* $f(z)$ *be defined by (1.6) and* $g(z)$ *defined by (5.1) be in the class* $\alpha - PS^*_{p,n}(\gamma_1)$ *and* $\alpha - PS^*_{p,n}(\gamma_2)$ *respectively. Then the hadamard Product*

$$(f * g)(z) \in \alpha - PS^*_{p,n}(\gamma), \quad \textit{where} \quad \gamma = max\{\gamma_1, \gamma_2\} \quad \textit{and} \quad 0 \leq \alpha < 1. \tag{5.7}$$

*Proof.* Since $f(z) \in \alpha - PS^*_{p,n}(\gamma_1)$ and $g(z) \in \alpha - PS^*_{p,n}(\gamma_2)$ in view of Theorem 2.2 we have

$$\sum_{k=2}^{\infty} k^n [k + k\alpha - \alpha - \gamma] a_k b_k \leq \sum_{k=2}^{\infty} k^n [k + k\alpha - \alpha - \gamma] |a_k| \frac{(\gamma_2 - 1)}{k^n [k + k\alpha - \alpha - \gamma]}$$

$$\leq \frac{(\gamma_2 - 1)}{k^n [k + k\alpha - \alpha - \gamma]} \sum_{k=2}^{\infty} k^n [k + k\alpha - \alpha - \gamma] |a_k| \leq \frac{(\gamma_1 - 1)(\gamma_2 - 1)}{k^n [k + k\alpha - \alpha - \gamma]}$$

$$\leq (\gamma - 1)^2 = \left(\gamma^2 - 2\gamma + 2\right) - 1.$$

(Since $1 < \gamma \leq \frac{4+\alpha}{3}$, therefore $1 < \left(\gamma^2 - 2\gamma + 2\right) \leq \frac{4+\alpha}{3}$ )

Hence by the Theorem 2.2, the Hadamard Product $(f * g)(z) \in \alpha - PS^*_{p,n}\left(\gamma^2 - 2\gamma + 2\right)$ with $\gamma$ is given by (5.7). □

## References

[1] Kanas, S. and Srivastava, H.M. (2000). Linear operators associated with k-uniformly convex functions, Integral Transforms Spec. Funct. 9,121-132.

[2] Kanas, S. and Wisniowska, A.(1999). Conic regions and k-uniform convexity, J. Comput. Appl. Math. 105, 327-336.

[3] Kanas, S. and Wisniowska, A.(1999). Conic regions and k-starlike functions, Rev. Roumaine Math. Pure Appl. 45, 647-657.

[4] Porwal, S. and Dixit, K.K.(2010). An application of certain convolution operator involving hypergeometric function, J. Rajasthan Acad. Phy. Sci., vol. 9, no. 2, 173-186.

[5] Salagean, G.S. (1983). Subclasses of univalent functions, in Complex Analysis, vol. 1013 of Lecture Notes in math,Springer, Berlin, Germany, pp.362-372.

[6] Uralegaddi, B. A. Ganigi, M. D. and Sarangi, S. M. (1994).Univalent functions with positive coefficients, Tamkang Journal of Mathematics, vol. 25, no. 3, pp. 225-230.