$\phi-$Recurrent generalized Sasakian-space-forms

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Abstract

The main purpose of the present paper is to introduce the notion of generalized $\phi-$recurrency of generalized Sasakian-space-forms. We studied generalized $\phi-$recurrent generalized Sasakian-space-forms, generalized concircular $\phi-$recurrent generalized Sasakian-space-forms and obtained a number of results. We also proved generalized Sasakian-space-forms satisfying the condition $S(X,\xi).R = 0$ is reduced to $\eta-$Einstein.

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1 Introduction

In differential geometry, the curvature of a Riemannian manifold (M, g) plays a fundamental role. A Riemannian manifold with constant sectional curvature c is called a real$-$space form and its curvature tensor is given by the equation

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \tag{1.1}$$

for any vector fields X, Y, Z on M. Models for these spaces are the Euclidean space $(c = 0)$, the sphere $(c > 0)$ and the Hyperbolic space $(c < 0)$.

A Sasakian manifold $M(\phi, \xi, \eta, g)$ is said to be a Sasakian space form if all the $\phi-$sectional curvatures $K(X \wedge \phi X)$ are equal to a constant c, where $K(X \wedge \phi X)$ denotes the sectional curvature of the section spanned by the unit vector field X, orthogonal to ξ and ϕX. In such a case, Riemannian curvature tensor of M is given by

$$
\begin{aligned}
(1.2) \qquad R(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\
&+\frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
&+\frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\
&-g(Y,Z)\eta(X)\xi\}.
\end{aligned}
$$

In 2004, P. Alegre, D. E. Blair and A. Carriazo [13] introduced the concept of generalized Sasakian space forms. The generalized Sasakian space form is defined as follows:

A generalized Sasakian-space-form is an almost contact metric manifold $M(\phi, \xi, \eta, g)$ whose curvature tensor is given by

$$
\begin{aligned}
(1.3) \qquad R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\
&+f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
&+f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\
&-g(Y,Z)\eta(X)\xi\},
\end{aligned}
$$

where f_1, f_2, f_3 are differentiable functions on M and X, Y, Z are vector fields on M. Sasakian-space-forms appear as natural examples of generalized Sasakian-space-forms, with constant functions $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, where c denotes constant $\phi-$sectional curvature. The generalized Sasakian-space-forms have been extensively studied by $[2, 3, 14, 15, 16, 21]$ and many others.

The notion of locally $\phi-$symmetric Sasakian manifold was introduced by T. Takahashi [17] in 1977. $\phi-$recurrent Sasakian manifold and generalized $\phi-$recurrent Sasakian manifold were studied by the author [5] and [16] respectively.

The notion of generalized $\phi-$recurrent Kenmotsu manifolds was introduced by A. Basari and C. Murathan [1] and also generalizing the notion of $\phi-$recurrency, the authors D. A. Patil, D. G. Prakasha and C. S. Bagewadi [5] introduced the notion of generalized $\phi-$recurrent Sasakian manifolds. Motivated by the above studies, we have studied of generalized $\phi-$recurrent generalized Sasakian-space-forms and obtained number of interesting results.

Thus motivated sufficiently, in this paper we study generalized $\phi-$recurrent generalized Sasakian-space-forms. Section 2 contains necessary details about generalized Sasakian-space-forms. Section 3 is devoted to the study of generalized $\phi-$recurrent generalized Sasakian-space-forms and it is shown that generalized $\phi-$recurrent generalized Sasakian-space-forms is an Einstein manifold and for generalized $\phi-$recurrent generalized Sasakian-space-forms, a relation between the $1-$forms α and β is established. Further it is shown that generalized $\phi-$recurrent generalized Sasakian-space-form is a manifold of constant curvature. In section 4, we obtained a relation between the associated $1-$forms α and β for a generalized $\phi-$recurrent and concircular $\phi-$recurrent generalized Sasakian-space-forms. In section 5, we study generalized Sasakian-space-forms satisfying the condition

$S(X,\xi).R = 0$, where S and R are the Ricci and Riemannian curvature tensors respectively. Here it is shown that the manifold under this condition is reduced to $\eta-$Einstein.

2 Preliminaries

An odd dimensional manifold $M^{2n+1}(n \geq 1)$ is said to admit an almost contact structure, sometimes called a $(\phi,\xi,\eta)-$structure, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a $1-$form η satisfying ([8], [9]) :

$$\eta(\xi) = 1, \tag{2.1}$$

$$\phi^2(X) = -X + \eta(X)\xi, \quad g(X,\xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y), \tag{2.4}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.5}$$

for any vector fields X, Y on M. In particular, in an almost contact metric manifold we also have

$$\phi\xi = 0, \qquad \eta \circ \phi = 0. \tag{2.6}$$

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$, where

$$d\eta(X,Y) = \Phi(X,Y) = g(X, \phi Y), \tag{2.7}$$

and Φ is called the fundamental $2-$form of M. If, in addition, ξ is a Killing vector field, then M is said to be a $K-$contact manifold. It is well-known that a contact metric manifold is a $K-$contact manifold if and only if

$$\nabla_X \xi = -\phi X, \tag{2.8}$$

for any vector field X on M. On the other hand, the almost contact metric structure of M is said to be normal if

$$[\phi,\phi](X,Y) = -2d\eta(X,Y)\xi,$$

for any X, Y on M, where $[\phi,\phi]$ denotes the Nijenhuis torsion of ϕ, given by

$$[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

A normal contact metric manifold is called a Sasakian manifold. It can be proved that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X\phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.9}$$

for any X, Y on M.

On the other hand, given an almost contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, we say that M is a generalized Sasakian-space-form if there exist three functions f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z \;=\;& f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &-g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{2.10}$$

for any vector fields X, Y, Z on M [13]. Such a manifold is denoted by $M^{2n+1}(f_1, f_2, f_3)$. This kind of manifold appears as a generalization of the well known Sasakian-space-form, which can be obtained as a particular case of generalized Sasakian-space-form by taking $f_1 = \frac{C+3}{4}$, $f_2 = f_3 = \frac{C-1}{4}$.

In a $(2n+1)-$dimensional generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$, we have the following relations [20];

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.11}$$

$$R(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X)], \tag{2.12}$$

$$R(\xi,X)\xi = (f_1 - f_3)[\eta(X)\xi - X)], \tag{2.13}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.14}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.15}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.16}$$

$$(2.17) \qquad S(X,\xi) = 2n(f_1 - f_3)\eta(X),$$

$$(2.18) \qquad S(\phi X, \phi Y) = S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y),$$

$$(2.19) \qquad S(\xi,\xi) = 2n(f_1 - f_3),$$

$$(2.20) \qquad Q\xi = 2n(f_1 - f_3)\xi,$$

$$(2.21) \qquad r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3,$$

where R, S and r denote the curvature tensor, Ricci tensor of type $(0,2)$ and scalar curvature of the space-form, respectively, and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$. We know that [13] the $\phi-$sectional curvature of a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$, is $f_1 + 3f_2$.

Again a Sasakian manifold is said to be a $\phi-$recurrent manifold if there exists a non zero $1-$form A such that

$$\phi^2((\nabla_W R)(X,Y)Z) = \alpha(X)R(Y,Z)W,$$

for all vector fields X, Y, Z, W orthogonal to ξ. A Riemannian manifold (M^{2n+1}, g) is called generalized recurrent [19], if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = \alpha(X)R(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z],$$

where, α and β are two $1-$forms, β is non zero and these are defined by

$$\alpha(X) = g(X,\rho_1) \text{ and } \beta(X) = g(X,\rho_2), \forall X \in TM,$$

ρ_1 and ρ_2 being the vector fields associated to the $1-$form α and β.

3 Generalized $\phi-$recurrent generalized Sasakian-space-forms

Definition 3.1. *A generalized Sasakian-space-form $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be locally $\phi-$symmetric if the relation*

$$(3.1) \qquad \phi^2((\nabla_W R)(X,Y)Z) = 0,$$

holds for any arbitrary vector field X, Y, Z and W.

Definition 3.2. *A generalized Sasakian-space-form $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be $\phi-$recurrent if there exist a non zero $1-$form α such that*

$$\phi^2((\nabla_W R)(X,Y)Z) = \alpha(W)R(X,Y)Z, \tag{3.2}$$

for any arbitrary vector field X, Y, Z and W.

Definition 3.3. *A generalized Sasakian-space-form $M^{2n+1}(\phi,\xi,\eta,g)$ is called generalized $\phi-$recurrent if its curvature tensor R satisfies the condition*

$$\phi^2((\nabla_W R)(X,Y)Z) = \alpha(W)R(X,Y)Z + \beta(W)[g(Y,Z)X - g(X,Z)Y], \tag{3.3}$$

where α and β are two $1-$forms, β is non zero and these are defined by

$$\alpha(W) = g(W,\rho_1) \quad and \qquad \beta(W) = g(W,\rho_2), \forall W \in TM,$$

ρ_1 and ρ_2 being the vector fields associated to the $1-$form α and β.

Definition 3.4. *A generalized Sasakian-space-form $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be $\eta-$Einstein manifold if its Ricci tensor S is of the form*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

for any vector fields X and Y , where a and b are smooth functions on $M^{2n+1}(\phi,\xi,\eta,g)$. If $b = 0$, then it becomes Einstein manifold.

Let us consider a generalized Sasakian-space-forms $M^{2n+1}(\phi,\xi,\eta,g), (n > 1)$, which is generalized $\phi-$recurrent. Then by virtue of (2.2), (3.3) yields

$$\begin{aligned} & -(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z) \\ = \ & \alpha(W)R(X,Y)Z + \beta(W)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.4}$$

From which it follows that

$$\begin{aligned} & -g((\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)(U) \\ = \ & \alpha(W)g(R(X,Y)Z,U) + \beta(W)[g(Y,Z)g(X,U) \\ & -g(X,Z)g(Y,U)]. \end{aligned} \tag{3.5}$$

Let $\{e_i\}$, $i = 1, 2, ..., 2n+1$ be an orthonormal basis of the tangent space at any point of the space form. Then replacing $X = U = e_i$ in (3.5) and taking summation over i, $1 \le i \le 2n+1$, we obtain

$$\begin{aligned} & -(\nabla_W S)(Y,Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)(e_i) \\ = \ & \alpha(W)S(Y,Z) + 2n\beta(W)g(Y,Z). \end{aligned} \tag{3.6}$$

In the second term of (3.6), replacing $Z = \xi$. The equation (3.6) takes the form $g((\nabla_W R)(e_i, Y)\xi, \xi)g(e_i, \xi)$. Consider

$$(3.7)\qquad \begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) &= g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(\nabla_W e_i, Y)\xi, \xi) \\ &\quad - g(R(e_i, \nabla_W Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi), \end{aligned}$$

at $p \in M$.Since $\{e_i\}$ is an orthonormal basis, so $\nabla_X e_i = 0$ at p. Using (2.2), (2.11) and (2.9), we have

$$(3.8)\qquad g(R(e_i, \nabla_W Y), \xi) = (f_1 - f_3)\{\eta(\nabla_W Y)\eta(e_i) - \eta(e_i)\eta(\nabla_W Y)\} = 0.$$

Now from (3.7) and (3.8), we have

$$(3.9)\qquad g((\nabla_W R)(e_i, Y)\xi, \xi) = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi).$$

Since $(\nabla_W g) = 0$, we have

$$g(\nabla_W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_W \xi) = 0,$$

which implies that

$$(3.10)\qquad g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla_W \xi) - g(R(e_i, Y)\nabla_W \xi, \xi).$$

Now from (2.8) and (3.10), we have

$$(3.11)\qquad g((\nabla_W R)(e_i, Y)\xi, \xi) = g(R(e_i, Y)\xi, \phi W) - g(R(e_i, Y)\xi, \phi W) = 0.$$

Putting $Z = \xi$ in (3.6) and using (2.5) and (2.17), we have

$$(3.12)\qquad (\nabla_W S)(Y, \xi) = -[2n(f_1 - f_3)\alpha(W) + 2n\beta(W)]\eta(Y).$$

Also, we know that

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).$$

Using (2.4), (2.8) and (2.17) in the above equation, we get

$$(3.13)\qquad (\nabla_W S)(Y, \xi) = -2n(f_1 - f_3)g(Y, \phi W) + S(Y, \phi W).$$

In view of (3.12) and (3.13), we obtain

$$-[2n(f_1 - f_3)A(W) + 2nB(W)]\eta(Y) = -2n(f_1 - f_3)g(Y, \phi W) + S(Y, \phi W).$$

Putting $Y = \xi$ in the above relation and using (2.2) and (2.6), we have

$$(3.14)\qquad (f_1 - f_3)\alpha(W) + \beta(W) = 0.$$

Again replacing Y by ϕY in (3.12) and then using (2.3), (2.5) and (2.17), we obtain

$$(3.15)\qquad S(Y, W) = 2n(f_1 - f_3)g(Y, W),$$

and

$$S(\phi Y, W) = 2n(f_1 - f_3)g(\phi Y, W).$$

Thus, we state the following:

Theorem 3.1. *A generalized $\phi-$recurrent generalized Sasakian-space-forms $M^{2n+1}(\phi, \xi, \eta, g)$ satisfying is an Einstein manifold and more over, the $1-$forms α and β are satisfying* $(f_1 - f_3)\alpha(W) + \beta(W) = 0$.

Now from (2.2) and (3.3), we get

$$\begin{aligned}(\nabla_W R)(X, Y)Z &= \eta((\nabla_W R)(X, Y)Z)\xi - \alpha(W)R(X, Y)Z \\ &\quad -\beta(W)[g(Y, Z)X - g(X, Z)Y].\end{aligned}$$

Then using second Bianchi's identity in above equation and again using (3.14), we get

$$\begin{aligned}&\alpha(W)R(X, Y)Z + (f_1 - f_3)\alpha(W)[g(Y, Z)X - g(X, Z)Y] \\ &+\alpha(X)R(Y, W)Z + (f_1 - f_3)\alpha(X)[g(W, Z)Y - g(Y, Z)W] \\ &+\alpha(Y)R(W, X)Z + (f_1 - f_3)\alpha(Y)[g(X, Z)W - g(W, Z)X] \\ = \;& 0.\end{aligned} \tag{3.16}$$

Replacing $Y = Z = \{e_i\}$, where $\{e_i\}$ be an orthonormal basis of the tangent space at any point of the space form, in (3.16) and taking summation over $i, 1 \leq i \leq 2n + 1$, we obtain

$$\begin{aligned}&\alpha(W)[S(X, U) + (2n - 1)(f_1 - f_3)g(X, U)] \\ &-\alpha(X)[S(W, U) + (2n - 1)(f_1 - f_3)g(W, U)] \\ &-g(R(W, X)U, \rho_1) \\ = \;& 0.\end{aligned} \tag{3.17}$$

Contracting (3.17) with respect to X, U; we find

$$r\alpha(W) + 2n(2n - 1)(f_1 - f_3)\alpha(W) = 2S(W, \rho_1).$$

Using (2.21) in above equation, we obtain

$$(4n^2 f_1 + 3nf_2 - n(2n + 1)f_3)\alpha(W) = S(W, \rho_1). \tag{3.18}$$

Thus, we state the following:

Theorem 3.2. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a generalized $\phi-$recurrent generalized Sasakian-space-forms. Then*

$$(4n^2 f_1 + 3nf_2 - n(2n + 1)f_3)\alpha(W) = S(W, \rho_1),$$

holds.

Now,

$$\begin{aligned}(\nabla_W R)(X, Y)\xi &= \nabla_W R(X, Y)\xi - R(\nabla_W X, Y)\xi \\ &\quad -R(X, \nabla_W Y)\xi - R(X, Y)\nabla_W \xi.\end{aligned} \tag{3.19}$$

By virtue of $(2.8), (2.11)$ and (3.19), we can easily get

$$(3.20)\qquad (\nabla_W R)(X,Y)\xi = (f_1 - f_3)[(\nabla_W \eta)(Y)X - (\nabla_W \eta)(X)Y] - R(X,Y)\phi W.$$

If we consider X, Y orthogonal to ξ , then in view of (2.14), we get

$$\eta((\nabla_W R)(X,Y)\xi) = 0.$$

Hence

$$(3.21)\qquad \eta((\nabla_{\phi W} R)(X,Y)\xi) = 0.$$

From (3.20), we get

$$(3.22)\qquad (\nabla_{\phi W} R)(X,Y)\xi = (f_1 - f_3)[(\nabla_{\phi W} \eta)(Y)X - (\nabla_{\phi W} \eta)(X)Y] - R(X,Y)\phi^2 W.$$

Using (2.2) in above equation, we obtain

$$(3.23)\qquad (\nabla_{\phi W} R)(X,Y)\xi = (f_1 - f_3)[g(W,Y)X - g(W,X)Y] - R(X,Y)W.$$

Suppose the space form is generalized $\phi-$recurrent. Then in view of (3.4) and (3.23), we get

$$(3.24)\qquad \eta((\nabla_{\phi W} R)(X,Y)\xi)\xi - \alpha(\phi W)(f_1 - f_3)[\eta(Y)X - \eta(X)Y] - \beta(\phi W)[\eta(Y)X - \eta(X)Y] = (f_1 - f_3)[g(W,Y)X - g(W,X)Y] - R(X,Y)W.$$

Using (3.21) and (3.14) in the above equation, we obtain

$$(3.25)\qquad R(X,Y)W = (f_1 - f_3)[g(Y,W)X - g(X,W)Y],$$

for all X, Y, W.

Thus, we state the following:

Theorem 3.3. *A generalized $\phi-$recurrent generalized Sasakian-space-forms $M^{2n+1}(\phi, \xi, \eta, g)$ is of constant curvature .*

By the definition, we have

$$(3.26)\qquad g((\nabla_W R)(X,Y)Z,U) = g(\nabla_W R(X,Y)Z,U) + R(\nabla_W X,Y,Z,U) + R(X,\nabla_W Y,Z,U) + R(X,Y,U,\nabla_W Z),$$

where $g(R(X,Y)Z,U) = R(X,Y,Z,U)$ and the property of curvature tensor have been used. Since ∇ is a metric connection, it follows that

$$(3.27)\qquad g(\nabla_W R(X,Y)Z,U) = g(R(X,Y)\nabla_W U,Z) - \nabla_W g(R(X,Y)U,Z),$$

and

$$\nabla_W g(R(X,Y)U,Z) = g(\nabla_W R(X,Y)U,Z) + g(R(X,Y)U,\nabla_W Z). \tag{3.28}$$

From (3.27) and (3.28), we obtain

$$\begin{aligned} g(\nabla_W R(X,Y)Z,U) &= g(R(X,Y)\nabla_W U,Z) - g(\nabla_W R(X,Y)U,Z) \\ &\quad -g(R(X,Y)U,\nabla_W Z). \end{aligned} \tag{3.29}$$

Hence from (3.29), (3.26) reduces to

$$g((\nabla_W R)(X,Y)Z,U) = -g((\nabla_W R)(X,Y)U,Z). \tag{3.30}$$

Using (2.2), (2.5) and (3.30) in (3.3), we obtain

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= -g((\nabla_W R)(X,Y)\xi,Z)\xi - \alpha(W)R(X,Y)Z \\ &\quad -\beta(W)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.31}$$

Again using (2.1), (2.4), (2.6) and (2.11) in the above equation, we can easily get

$$\begin{aligned} (\nabla_W R)(X,Y)\xi &= (f_1 - f_3)[g(\phi Y,W)X - g(\phi X,W)Y] \\ &\quad -R(X,Y)\phi W. \end{aligned} \tag{3.32}$$

By virtue of (3.32) and (3.31), we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= \{(f_1 - f_3)[g(\phi X,W)g(Y,Z) - g(\phi Y,W)g(X,Z)] \\ &\quad +g(R(X,Y)\phi W,Z)\}\xi - \alpha(W)R(X,Y)Z \\ &\quad -\beta(W)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.33}$$

Conversely, if in a generalized Sasakian-space-forms $M^{2n+1}(\phi,\xi,\eta,g)$ the relation (3.33) holds, then applying ϕ on both sides of (3.33) and keeping mind that X,Y,Z and W are orthogonal to ξ, we obtain (3.3).

Thus, we state the following:

Theorem 3.4. *A generalized Sasakian-space-form $M^{2n+1}(\phi,\xi,\eta,g)$ is generalized $\phi-$recurrent if and only if the relation*

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= \{(f_1 - f_3)[g(\phi X,W)g(Y,Z) - g(\phi Y,W)g(X,Z)] \\ &\quad +g(R(X,Y)\phi W,Z)\}\xi - \alpha(W)R(X,Y)Z \\ &\quad -\beta(W)[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

holds for all vector fields X,Y,Z,W on M.

4 On generalized concircular $\phi-$recurrent generalized Sasakian-space-forms

Definition 4.1. *A generalized Sasakian-space-forms $M^{2n+1}(\phi, \xi, \eta, g)$ is called generalized concircular $\phi-$recurrent if its concircular curvature tensor (Yano, K; Kon, M, 1984)*

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y],$$

satisfies the condition [17]

$$\phi^2((\nabla_W C)(X,Y)Z) = \alpha(W)C(X,Y)Z + \beta(W)[g(Y,Z)X - g(X,Z)Y], \tag{4.1}$$

where α and β are two $1-$forms, β is non zero and these are defined by

$$\alpha(W) = g(W, \rho_1) \quad and \qquad \beta(W) = g(W, \rho_2), \forall W \in TM,$$

ρ_1 and ρ_2 being the vector fields associated to the $1-$form α and β.

In this section we consider a generalized concircular $\phi-$recurrent generalized Sasakian-space-forms $M^{2n+1}(\phi, \xi, \eta, g)$. Then from (2.2) and (4.1), we have

$$\begin{aligned} & -(\nabla_W C)(X,Y)Z + \eta((\nabla_W C)(X,Y)Z)\xi \\ = \; & \alpha(W)C(X,Y)Z + \beta(W)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{4.2}$$

from the above equation it follows that

$$\begin{aligned} & -g((\nabla_W C)(X,Y)Z, U) + \eta((\nabla_W C)(X,Y)Z)\eta(U) \\ = \; & \alpha(W)g(C(X,Y)Z, U) + \beta(W)[g(Y,Z)g(X,U) \\ & -g(X,Z)g(Y,U)]. \end{aligned} \tag{4.3}$$

Let $\{e_i\}, i = 1, 2, ..., 2n+1$, be an orthonormal basis of the tangent space at any point of the space form. Then putting $Y = Z = \{e_i\}$ in the above equation and taking summation over $i, 1 \leq i \leq 2n+1$,we find

$$\begin{aligned} & -(\nabla_W S)(X,U) + \frac{\nabla_W r}{2n+1} g(X,U) \\ & +(\nabla_W S)(X,\xi)\eta(U) - \frac{\nabla_W r}{2n+1}\eta(X)\eta(U) \\ = \; & \alpha(W)[S(X,U) - \frac{r}{2n+1} g(X,U) \\ & +2n\beta(W)g(X,U)]. \end{aligned} \tag{4.4}$$

Now, taking $U = \xi$ in (4.4) and then using (2.5) and (2.17), we get

$$\alpha(W)[2n(f_1 - f_3) - \frac{r}{2n+1}]\eta(X) + 2n\beta(W))\eta(X) = 0,$$

$$\eta(X) \neq 0, \qquad \alpha(W)[2n(f_1 - f_3) - \frac{r}{2n+1}] + 2n\beta(W) = 0,$$

i.e.

$$2n\beta(W) = \alpha(W)[\frac{r}{2n+1} - 2n(f_1 - f_3)]. \tag{4.5}$$

Now using (2.21) in (4.5), we get

$$\beta(W) = [\frac{(3f_2 + (2n-1)f_3)}{(2n+1)}]\alpha(W).$$

Thus, we state the following:

Theorem 4.1. *In a generalized concircular $\phi-$recurrent generalized Sasakian-space-forms $M^{2n+1}(\phi, \xi, \eta, g)$, the $1-$forms α and β are satisfying*

$$\beta(W) = [\frac{(3f_2 + (2n-1)f_3)}{(2n+1)}]\alpha(W).$$

5 Generalized Sasakian-space-forms satisfying $S(X, \xi).R = 0$

We consider a generalized Sasakian-space-forms $M^{2n+1}(\phi, \xi, \eta, g), (n > 1)$ satisfying the condition

$$(S(X, \xi).R)(U, V)Z = 0. \tag{5.1}$$

By the definition, we obtain

$$\begin{aligned} (S(X, \xi).R)(U, V)Z &= ((X \wedge_S \xi).R)(U, V)Z \\ &= (X \wedge_S \xi)R(U, V)Z + R((X \wedge_S \xi)U, V)Z \\ &\quad + R(U, (X \wedge_S \xi)V)Z + R(U, V)(X \wedge_S \xi)Z, \end{aligned} \tag{5.2}$$

where the endomorphism $X \wedge_S Y$ is defined by

$$X \wedge_S Y = S(Y, Z)X - S(X, Z)Y. \tag{5.3}$$

Using the above definition in (5.2), by virtue of (2.17), we get

$$\begin{aligned} (S(X, \xi).R)(U, V)Z &= 2n(f_1 - f_3)[\eta(R(U, V)Z)X + \eta(U)R(X, V)Z \\ &\quad + \eta(V)R(U, X)Z + \eta(Z)R(U, V)X] \\ &\quad - S(X, R(U, V)Z)\xi - S(X, U)R(\xi, V)Z \\ &\quad - S(X, V)R(U, \xi)Z - S(X, Z)R(U, V)\xi. \end{aligned} \tag{5.4}$$

From (5.1) and (5.4), we get

$$\begin{aligned} &2n(f_1 - f_3)[\eta(R(U, V)Z)X + \eta(U)R(X, V)Z \\ &+ \eta(V)R(U, X)Z + \eta(Z)R(U, V)X] \\ &- S(X, R(U, V)Z)\xi - S(X, U)R(\xi, V)Z \\ &- S(X, V)R(U, \xi)Z - S(X, Z)R(U, V)\xi = 0. \end{aligned} \tag{5.5}$$

Taking the inner product with ξ on both sides of (5.5), we get

$$\begin{aligned}(5.6)\qquad & 2n(f_1-f_3)[\eta(R(U,V)Z)\eta(X)+\eta(U)\eta(R(X,V)Z)\\ & +\eta(V)\eta(R(U,X)Z)+\eta(Z)\eta(R(U,V)X)]\\ & -S(X,R(U,V)Z)\xi-S(X,U)\eta(R(\xi,V)Z)\\ & -S(X,V)\eta(R(U,\xi)Z)-S(X,Z)\eta(R(U,V)\xi)=0.\end{aligned}$$

Substituting $U=Z=\xi$ in (5.6) and using (2.11), (2.14), (2.17), (2.18) and (2.20), we obtain

$$\begin{aligned}(5.7)\qquad & (f_1-f_3)S(X,V)+2n(f_1-f_3)^2g(X,V)\\ & -(2n+1)(f_1-f_3)^2\eta(X)\eta(V)\\ = & \ 0.\end{aligned}$$

Since $(f_1-f_3)\neq 0$, therefore

$$(5.8)\qquad S(X,V)=-2n(f_1-f_3)g(X,V)+(2n+1)(f_1-f_3)\eta(X)\eta(V),$$

which is of the form

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y),$$

where $a=-2n(f_1-f_3)$ and $b=(2n+1)(f_1-f_3)$, which shows that a generalized Sasakian-space-forms is an $\eta-$Einstein manifold.

Thus, we state the following:

Theorem 5.1. *A generalized Sasakian-space-forms $M^{2n+1}(\phi,\xi,\eta,g)$, $(n>1)$ satisfying the condition $S(X,\xi).R=0$ with $(f_1-f_3)\neq 0$, is an $\eta-$Einstein manifold.*

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