ON GENERALIZED RECURRENT SASAKIAN MANIFOLD WITH SPECIAL CURVATURE TENSOR $J(X, Y, Z)$

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Abstract

In this paper, we have studied the nature of the associated vector field ξ , in case of a special curvature tensor J satisfies the condition of a generalized recurrent Sasakian manifold. In sequence we have also studied the nature of the 1-forms α and β , and the contact form η in a generalized recurrent Sasakian manifold satisfying the condition $J(X, Y, Z) = 0$.

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1 Introduction

Let (M^n, g) be a contact Riemannian manifold with a contact form η, the associated vector field ξ, (1-1) tensor field Φ and the associated Riemannian metric g. If ξ is a killing vector field, then M^n is called a K - contact Riemannian manifold. A K - contact Riemannian manifold is called a Sasakian manifold if

$$(D_X\Phi)(Y) = g(X, Y)\xi - \eta(Y)X \tag{1.1}$$

holds, where D denotes the operator of covariant differentiation with respect to the metric g. This paper deals with a type of Sasakian manifold in which

$$J(X, Y, Z) = 0, \tag{1.2}$$

where J is the special curvature tensor [7] defined by

$$J(Y, Z, W) = K(Y, Z, W) + K(Y, W, Z), \tag{1.3}$$

where K is the Riemannian curvature tensor which is skew-symmetric in first two slots, that is, $K(X,Y,Z) = -K(Y,X,Z)$ for all vector fields X,Y,Z. It is known that in a Sasakian manifold M^n, besides the relation (1.1), the following relations also hold

$$\Phi(\xi) = 0, \tag{1.4}$$

$$\eta(\xi) = 1, \tag{1.5}$$

$$g(\xi, X) = \eta(X), \tag{1.6}$$

$$Ric(\xi, X) = (n-1)\eta(X), \tag{1.7}$$

$$K(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \tag{1.8}$$

$$K(\xi, X)\xi = -X + \eta(X)\xi, \tag{1.9}$$

$$g(K(\xi, X)Y, \xi) = g(X,Y) - \eta(X)\eta(Y), \tag{1.10}$$

$$D_X\xi = -\Phi X, \tag{1.11}$$

$$\eta(\Phi X) = 0, \tag{1.12}$$

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1.13}$$

and

$$(D_X\Phi)(Y) = K(\xi, X)Y, \tag{1.14}$$

for any vector fields X, Y. The above results will be used in the next section. Our work is also motivated by the following references [1, 2, 3, 4, 8, 9, 10, 11, 12, 13]

2 GENERALIZED RECURRENT SASAKIAN MANIFOLD WITH SPECIAL CURVATURE TENSOR $J(X,Y,Z)$

A Riemannian manifold is called a generalized recurrent Riemannian manifold (see [5], [6]) if the curvature tensor K satisfies the condition

$$(D_XK)(Y,Z,W) = \alpha(X)K(Y,Z,W) + \beta(X)[g(Z,W)Y - g(Y,W)Z], \tag{2.1}$$

where α and β are two 1-forms; β is non - zero and these are defined by

$$\alpha(X) = g(X, A); \beta(X) = g(X, B), \tag{2.2}$$

where A and B are the vector fields associated with 1 - forms α and β respectively. Contracting (2.1) with respect to Y, we get

$$(D_X Ric)(Z, W) = \alpha(X)Ric(Z, W) + (n-1)\beta(X)g(Z, W). \tag{2.3}$$

In this case, the Riemannian manifold is called a generalized Ricci recurrent manifold, where α and β are as stated earlier. A non - flat n - dimensional Riemannian manifold in which the special curvature tensor J satisfies the condition :

$$(D_X J)(Y, Z, W) = \alpha(X)J(Y, Z, W) + \beta(X)[g(Z, W)Y - g(Y, W)Z], \tag{2.4}$$

where α and β are two 1-forms; β is non-zero and the special curvature tensor J is defined by (1.3). Such an n-dimensional Riemannian manifold shall be called a generalized recurrent Riemannian manifold with special curvature tensor J. Taking covariant derivative of (1.3) with respect to X, we get
$(D_X J)(Y, Z, W) = (D_X K)(Y, Z, W) + (D_X K)(Y, W, Z)$,
which in view of (2.4) and (1.3) gives

$$\begin{aligned} &\alpha(X)[K(Y, Z, W) + K(Y, W, Z)] + \beta(X)[g(Z, W)Y - g(Y, W)Z] \\ &\qquad = (D_X K)(Y, Z, W) + (D_X K)(Y, W, Z). \end{aligned} \tag{2.5}$$

Taking $Y = W = \xi$ in (2.5), we have

$$\begin{aligned} &\alpha(X)[K(\xi, Z, \xi) + K(\xi, \xi, Z)] + \beta(X)[g(Z, \xi)\xi - g(\xi, \xi)Z] \\ &\qquad = (D_X K)(\xi, Z, \xi) + (D_X K)(\xi, \xi, Z). \end{aligned} \tag{2.6}$$

The right hand side of (2.6), clearly can be written in the form
$(D_X K)(\xi, Z, \xi) + (D_X K)(\xi, \xi, Z)$
$= XK(\xi, Z, \xi) - K(D_X\xi, Z, \xi) - K(\xi, D_X Z, \xi) - K(\xi, Z, D_X\xi)$
$+ XK(\xi, \xi, Z) - K(D_X\xi, \xi, Z) - K(\xi, D_X\xi, Z) - K(\xi, \xi, D_X Z)$,
which in view of (1.9), (1.11) and skew-symmetric property of curvature tensor $K(X, Y, Z)$ reduces
$(D_X K)(\xi, Z, \xi) + (D_X K)(\xi, \xi, Z)$
$= X[-Z + \eta(Z)\xi] + K(\Phi X, Z, \xi) - [-D_X Z + \eta(D_X Z)\xi] + K(\xi, Z, \Phi X)$
$= Xg(Z, \xi)\xi - K(Z, \Phi X, \xi) - g(D_X Z, \xi)\xi + g(Z, \Phi X)\xi$
$- \eta(\Phi X)Z$, using (1.6) and (1.8)
$= g(D_X Z, \xi)\xi + g(Z, D_X\xi)\xi - K(Z, \Phi X, \xi) - g(D_X Z, \xi)\xi$
$+ g(Z, \Phi X)\xi, using(1.12)$
$= -g(Z, \Phi X)\xi - K(Z, \Phi X, \xi) + g(Z, \Phi X)\xi, using(1.12)$
$= -K(Z, \Phi X, \xi)$,
while the left hand side of (2.6) equals
$\alpha(X)[K(\xi, Z, \xi) + K(\xi, \xi, Z)] + \beta(X)[g(Z, \xi)\xi - g(\xi, \xi)Z]$

$= \alpha(X)[-Z+\eta(Z)\xi+0]+\beta(X)[\eta(Z)\xi-Z], using (1.6) and (1.9)$
$= [\alpha(X)+\beta(X)][-Z+\eta(Z)\xi].$
Hence $[\alpha(X)+\beta(X)][-Z+\eta(Z)\xi] = -K(Z,\Phi X,\xi).$
Taking $Z=\xi$ in the above relation and then using (1.5) and (1.9), we get
$[\alpha(X)+\beta(X)][-\xi+\xi] = [-\Phi X+\eta(\Phi X)\xi],$
which in view of (1.12) gives
$\Phi X = 0,$
which in view of (1.11) gives
$D_X\xi = 0.$
which shows that the associated vector field ξ is constant. Thus we get the following theorem:
Theorem 2.1 If a special curvature tensor J satisfies the condition of a generalized recurrent Sasakian manifold, then the associated vector field ξ is a constant.

3 GENERALIZED RECURRENT SASAKIAN MANIFOLD SATISFYING THE CONDITION $J(X,Y,Z)=0$

Let us suppose that in a Sasakian manifold

(3.1) $$J(Y,Z,W) = 0.$$

Then it follows from (1.3) that

(3.2) $$K(Y,Z,W)+K(Y,W,Z) = 0.$$

Taking covariant derivative of (3.2) with respect to X , we get

$$(D_XK)(Y,Z,W)+(D_XK)(Y,W,Z) = 0.$$

Using (2.1) in the above relation, we have
(3.3)
$$\alpha(X)[K(Y,Z,W)+K(Y,W,Z)]+\beta(X)[g(Z,W)Y-g(Y,W)Z+g(W,Z)Y-g(Y,Z)W] = 0.$$

Taking $Y=Z=\xi$ in the relation (3.3), we get

$$\alpha(X)[K(\xi,\xi,W)+K(\xi,W,\xi)]+\beta(X)[g(\xi,W)\xi-g(\xi,W)\xi+g(W,\xi)\xi-g(\xi,\xi)W] = 0,$$

which in view of (1.6) and (1.9), gives

$$\alpha(X)[-W+\eta(W)\xi]+\alpha(X)K(\xi,\xi,W)+\beta(X)[-W+\eta(W)\xi] = 0$$
$$or,\quad [\alpha(X)+\beta(X)][-W+\eta(W)\xi]+\alpha(X)K(\xi,\xi,W) = 0,$$

which in view of (1.9), gives

(3.4) $$[\alpha(X)+\beta(X)]K(\xi,W,\xi)+\alpha(X)K(\xi,\xi,W) = 0.$$

Using the skew-symmetric property of $K(X,Y,Z)$, that is $K(X,Y,Z) = -K(Y,X,Z)$ in the relation (3.4), we have

$$[\alpha(X) + \beta(X)]K(\xi, W, \xi) - \alpha(X)K(\xi, \xi, W) = 0. \tag{3.5}$$

Adding the relations (3.4) and (3.5), we have

$$2[\alpha(X) + \beta(X)]K(\xi, W, \xi) = 0.$$

But $2 \neq 0$ and also $K(\xi, W, \xi) \neq 0$. Therefore

$$\alpha(X) + \beta(X) = 0.$$

which shows that the sum of the 1-forms α and β is zero. Thus we get the following theorem:
Theorem 3.1 If a generalized recurrent Sasakian manifold satisfies the condition $J(X,Y,Z) = 0$, then the sum of the 1-forms α and β is zero.
Contracting (3.2)) with respect to Y, we have

$$Ric(Z, W) + Ric(W, Z) = 0. \tag{3.6}$$

Since Ricci tensor of type $(0, 2)$ is symmetric, therefore from (3.6), we have

$$2Ric(Z, W) = 0 \;\; or \;\; Ric(Z, W) = 0 \; as \; 2 \neq 0. \tag{3.7}$$

Taking $Z = \xi$ in (3.7), we have

$$(n - 1)\eta(W) = 0.$$

But for $n > 1, (n - 1) \neq 0$.Therefore

$$\eta(W) = 0.$$

which shows that the contact form η is zero. Thus we can state the following theorem:
Theorem 3.2 If a special curvature tensor J in a Sasakian manifold is flat, that is $J(X,Y,Z) = 0$, then the contact form η must be vanish.
Note: The theorem 3.1 can also be stated as follows:
Theorem 3.3 If a generalized recurrent Sasakian manifold satisfies the condition $J(X,Y,Z) = 0$, then the 1-forms are same in the magnitude but opposite in direction.

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