(p,q)-Generalization of Szász-Schurer Operators

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Abstract

In this paper we introduce a generalization of Szász-Schurer operators based on (p,q)-integers. We give a recurrence relation for moments of these new operators and give the convergence theorem for these operators. We also give rate of convergence in terms of modulus of continuity in bounded intervals. We further study the convergence of these operators in weighted space of functions on a positive semi-axis.

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1 Introduction

The rapid development of q-calculus in the last two decades has led to the discovery of new generalizations in approximation theory. Lupas [8] was the first researcher who introduced a q-analogue of Bernstein polynomials. Since then an intensive research has been conducted on linear positive operators based on q-integers.

Recently Mursaleen et al. applied (p,q)-calculus in approximation theory. They first introduced (p,q)-analogue of Bernstein operators [9, 10]. Later on they introduced (p,q)-analogues of various operators such as Bernstein-Stancu operators [11], Bernstein-Schurer operators [13], Kantorovich type Bernstein operators [12], etc.

Acar [1] introduced (p,q)-analogue of Szász-Mirakjan operators,for $0 < q < p \leq 1, n \in N, f : [0, \infty) \to R$

$$S_{n,p,q}(f;x) = \frac{1}{E([n]_{p,q}x)} \sum_{k=0}^{\infty} f\left(\frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}\right) q^{\frac{k(k-1)}{2}} \frac{[n]_{p,q}^k x^k}{[k]_{p,q}!} \tag{1.1}$$

In the present paper we introduce a (p,q)-variant of Szász- Schurer operators and discuss their approximation properties. The paper is organized as follows. In section 2 we introduce the new operators and calculate their moments. In section 3 we give the convergence theorem using Korovkin's result. In section 4 we evaluate rate of convergence of these operators in terms of modulus of continuity in bounded intervals. In section 5 we study approximation properties of the new operators in weighted space.

Firstly we recall some basic notions based on (p,q)-integers.
Let $0 < q < p \leq 1$. For each nonnegative integer $k, n, n \geq k \geq 0$, the (p, q)-integer $[k]_{p,q}$, (p, q)-factorial $[k]_{p,q}!$ and (p, q)-binomial coefficient are defined by

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}$$

with, $[k]_{p,q} = 1$ for $k = 0$.

$$[k]_{p,q}! = \begin{cases} [k]_{p,q}[k-1]_{p,q} \ldots 1 & , k \geq 1, \\ 1 & , k = 0 \end{cases}$$

$$\left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$$

The (p,q)-Binomial expansion is

$$(ax + by)_{p,q}^n = \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} a^{n-k} b^k x^{n-k} y^k$$

and

$$(x + y)_{p,q}^n = (x + y)(px + qy)(p^2 x + q^2 y) \ldots (p^{n-1} x + q^{n-1} y)$$

For p = 1 the above notations reduce to their q-analogues. More details on (p, q)-calculus can be found in [2], [3], [6], [7], [15],[16].
There are two (p,q)-analogues of exponential functions,

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!}$$

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!} \tag{1.2}$$

These functions satisfy the equality $e_{p,q}(x)E_{p,q}(-x) = 1$. For $p = 1$, $e_{p,q}(x)$ and $E_{p,q}(x)$ reduce to q-exponential functions.

2 Construction of operators and Auxiliary Results

Let $0 < q < p \leq 1, n \in N$. For $f : [0, \infty) \to R$ we introduce (p,q)-Szász Schurer operators which is defined for fixed $r \in N_0$ by

$$S^*_{n,p,q}(f;x;r) = \frac{1}{E_{p,q}([n+r]_{p,q}x)} \sum_{k=0}^{\infty} f\left(\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right) q^{\frac{k(k-1)}{2}} \frac{[n+r]^k_{p,q}x^k}{[k]_{p,q}!} \tag{2.1}$$

where $E_{p,q}(x)$ is given by (1.2). These are linear positive operators and in the case $r = 0$, these operators reduce to the slightly modified form of the Acar's (p,q)-Szász-Mirakjan operators. On taking $p = 1$ in (2.1), one can get q-Szász-Schurer operators introduced in [14]. Also it is clear from (1.2) that

$$S^*_{n,p,q}(1;x;r) = 1 \tag{2.2}$$

for each fixed $r \in N_0$.

Now we give a recurrence relation for the moments of the operators $S^*_{n,p,q}$.

Lemma 2.1. *Let $0 < q < p \leq 1$, $r \in N_0$ and $n \in N$*

$$S^*_{n,p,q}(t^{m+1};x;r) = \sum_{j=0}^{m} \binom{m}{j} \frac{x[n+r]_{p,q}p^j}{[n]^{m+1-j}_{p,q}q^j} S^*_{n,p,q}(t^j;x;r) \tag{2.3}$$

Proof. Using the identity

$$[k]_{p,q} = q^{k-1} + p[k-1]_{p,q}$$

we get

$$\begin{aligned}
&S^*_{n,p,q}(t^{m+1};x;r)\\
&=\frac{1}{E_{p,q}([n+r]_{p,q}x)}\sum_{k=0}^{\infty}\frac{[k]_{p,q}^{m+1}}{q^{(m+1)(k-1)}[n]_{p,q}^{m+1}}q^{\frac{k(k-1)}{2}}\frac{[n+r]_{p,q}^k x^k}{[k]_{p,q}!}\\
&=\frac{1}{E_{p,q}([n+r]_{p,q}x)}\sum_{k=1}^{\infty}\frac{[k]_{p,q}^{m}}{q^{m(k-1)}[n]_{p,q}^{m+1}}q^{\frac{(k-1)(k-2)}{2}}\frac{[n+r]_{p,q}^k x^k}{[k-1]_{p,q}!}\\
&=\frac{1}{E_{p,q}([n+r]_{p,q}x)}\sum_{k=1}^{\infty}\frac{\left(q^{k-1}+p[k-1]_{p,q}\right)^m}{q^{m(k-1)}[n]_{p,q}^{m+1}}q^{\frac{(k-1)(k-2)}{2}}\frac{[n+r]_{p,q}^k x^k}{[k-1]_{p,q}!}\\
&=\frac{1}{E_{p,q}([n+r]_{p,q}x)}\sum_{k=1}^{\infty}\sum_{j=0}^{m}\binom{m}{j}q^{(k-1)(m-j)}p^j[k-1]_{p,q}^j\frac{q^{\frac{(k-1)(k-2)}{2}}[n+r]_{p,q}^k x^k}{q^{m(k-1)}[n]_{p,q}^{m+1}[k-1]_{p,q}!}\\
&=\frac{1}{E_{p,q}([n+r]_{p,q}x)}\sum_{j=0}^{m}\binom{m}{j}\frac{p^j}{[n]_{p,q}^{m+1-j}}\sum_{k=1}^{\infty}\frac{q^{\frac{(k-1)(k-2)}{2}}[n+r]_{p,q}^k x^k[k-1]_{p,q}^j}{q^{(k-1)j}[n]_{p,q}^j[k-1]_{p,q}!}\\
&=\frac{1}{E_{p,q}([n+r]_{p,q}x)}\sum_{j=0}^{m}\binom{m}{j}\frac{x[n+r]_{p,q}p^j}{[n]_{p,q}^{m+1-j}q^j}\sum_{k=0}^{\infty}\frac{q^{\frac{k(k-1)}{2}}[n+r]_{p,q}^k x^k[k]_{p,q}^j}{q^{(k-1)j}[n]_{p,q}^j[k]_{p,q}!}\\
&=\sum_{j=0}^{m}\binom{m}{j}\frac{x[n+r]_{p,q}p^j}{[n]_{p,q}^{m+1-j}q^j}S^*_{n,p,q}(t^j;x;r)
\end{aligned}$$

□

Using this recurrence formula we derive the following lemma.

Lemma 2.2. *Let $0<q<p\leq 1$, $r\in N_0$ and $n\in N$. Then we have*

$$S^*_{n,p,q}(1;x;r)=1 \tag{2.4}$$

$$S^*_{n,p,q}(t;x;r)=x\frac{[n+r]_{p,q}}{[n]_{p,q}} \tag{2.5}$$

$$S^*_{n,p,q}(t^2;x;r)=x\frac{[n+r]_{p,q}}{[n]_{p,q}^2}+x^2\frac{p[n+r]_{p,q}^2}{q[n]_{p,q}^2} \tag{2.6}$$

$$S^*_{n,p,q}(t^3;x;r)=x\frac{[n+r]_{p,q}}{[n]_{p,q}^3}+x^2\frac{(2pq+p^2)}{q^2}\frac{[n+r]_{p,q}^2}{[n]_{p,q}^3}+x^3\frac{p^3[n+r]_{p,q}^3}{q^3[n]_{p,q}^3} \tag{2.7}$$

$$\begin{aligned}
S^*_{n,p,q}(t^4;x;r)=&x\frac{[n+r]_{p,q}}{[n]_{p,q}^4}+x^2\frac{p(3q^2+3pq+p^2)}{q^3}\frac{[n+r]_{p,q}^2}{[n]_{p,q}^4}\\
&+x^3\frac{(3q^2+2pq+p^2)}{q^5}\frac{p^3[n+r]_{p,q}^3}{[n]_{p,q}^4}+x^4\frac{p^6[n+r]_{p,q}^4}{q^6[n]_{p,q}^4}
\end{aligned} \tag{2.8}$$

Proof. First equality follows from (2.2). From the recurrence relation given in Lemma 2.1 we get

$$S^*_{n,p,q}(t;x;r) = x\frac{[n+r]_{p,q}}{[n]_{p,q}}S^*_{n,p,q}(1;x;r)$$
$$= x\frac{[n+r]_{p,q}}{[n]_{p,q}}$$

$$S^*_{n,p,q}(t^2;x;r) = x\binom{1}{0}\frac{[n+r]_{p,q}}{[n]^2_{p,q}}S^*_{n,p,q}(1;x;r) + \binom{1}{1}\frac{xp[n+r]_{p,q}}{q[n]_{p,q}}S^*_{n,p,q}(t;x;r)$$
$$= x\frac{[n+r]_{p,q}}{[n]^2_{p,q}} + \frac{x^2p[n+r]^2_{p,q}}{q[n]^2_{p,q}}$$

$$S^*_{n,p,q}(t^3;x;r) = x\binom{2}{0}\frac{[n+r]_{p,q}}{[n]^3_{p,q}}S^*_{n,p,q}(1;x;r) + \binom{2}{1}\frac{xp[n+r]_{p,q}}{[n]^2_{p,q}q}S^*_{n,p,q}(t;x;r)$$
$$+ \binom{2}{2}\frac{xp^2[n+r]_{p,q}}{[n]_{p,q}q^2}S^*_{n,p,q}(t^2;x;r)$$
$$= x\frac{[n+r]_{p,q}}{[n]^3_{p,q}} + \frac{2x^2p[n+r]^2_{p,q}}{q[n]^3_{p,q}} + \frac{xp^2[n+r]_{p,q}}{[n]_{p,q}q^2}\left\{x\frac{[n+r]_{p,q}}{[n]^2_{p,q}}\right.$$
$$\left. + x^2\frac{p[n+r]^2_{p,q}}{q[n]^2_{p,q}}\right\}$$
$$= x\frac{[n+r]_{p,q}}{[n]^3_{p,q}} + x^2\frac{(2pq+p^2)}{q^2}\frac{[n+r]^2_{p,q}}{[n]^3_{p,q}} + x^3\frac{p^3[n+r]^3_{p,q}}{q^3[n]^3_{p,q}}$$

and

$$
\begin{aligned}
&S^*_{n,p,q}(t^4;x;r)\\
&=x\binom{3}{0}\frac{[n+r]_{p,q}}{[n]^4_{p,q}}S^*_{n,p,q}(1;x;r)+\binom{3}{1}\frac{xp[n+r]_{p,q}}{q[n]^3_{p,q}}S^*_{n,p,q}(t;x;r)\\
&\quad+\binom{3}{2}\frac{xp^2[n+r]_{p,q}}{q^2[n]^2_{p,q}}S^*_{n,p,q}(t^2;x;r)+\binom{3}{3}\frac{xp^3[n+r]_{p,q}}{q^3[n]_{p,q}}S^*_{n,p,q}(t^3;x;r)\\
&=x\frac{[n+r]_{p,q}}{[n]^4_{p,q}}+\frac{3x^2p[n+r]^2_{p,q}}{q[n]^4_{p,q}}+\frac{3xp^2[n+r]_{p,q}}{q^2[n]^2_{p,q}}\left\{x\frac{[n+r]_{p,q}}{[n]^2_{p,q}}\right.\\
&\quad\left.+x^2\frac{p[n+r]^2_{p,q}}{q[n]^2_{p,q}}\right\}+\frac{xp^3[n+r]_{p,q}}{q^3[n]_{p,q}}\left\{x\frac{[n+r]_{p,q}}{[n]^3_{p,q}}\right.\\
&\quad\left.+x^2\frac{(2pq+p^2)}{q^2}\frac{[n+r]^2_{p,q}}{[n]^3_{p,q}}+x^3\frac{p^3[n+r]^3_{p,q}}{q^3[n]^3_{p,q}}\right\}\\
&=x\frac{[n+r]_{p,q}}{[n]^4_{p,q}}+x^2\frac{p(3q^2+3pq+p^2)}{q^3}\frac{[n+r]^2_{p,q}}{[n]^4_{p,q}}\\
&\quad+x^3\frac{(3q^2+2pq+p^2)}{q^5}\frac{p^3[n+r]^3_{p,q}}{[n]^4_{p,q}}+x^4\frac{p^6[n+r]^4_{p,q}}{q^6[n]^4_{p,q}}
\end{aligned}
$$

□

Using Lemma 2.2 and linearity of operators $S^*_{n,p,q}$ we can easily obtain the following formulas for the central moments.

Corollary 2.1. *Let $0<q<p\le 1$, $r\in N_0$ and $n\in N$. Then*

$$
\begin{aligned}
S^*_{n,p,q}(t-x;x;r)&=x\left(\frac{[n+r]_{p,q}}{[n]_{p,q}}-1\right)\\
S^*_{n,p,q}((t-x)^2;x;r)&=x\frac{[n+r]_{p,q}}{[n]^2_{p,q}}+x^2\left\{\frac{[n+r]_{p,q}}{[n]_{p,q}}\left(\frac{p[n+r]_{p,q}}{q[n]_{p,q}}-2\right)+1\right\}
\end{aligned}
$$

Remark 2.1. *For $q\in(0,1)$ and $p\in(q,1]$ we see that*

$$\lim_{n\to\infty}[n]_{p,q}=\frac{1}{p-q}$$

Hence the operators $S^*_{n,p,q}(f)$ given in (2.1) are not approximation process. In order to ensure the convergence properties of the operators we assume that $q=(q_n)$ and $p=(p_n)$ such that $0<q_n<p_n\le 1$ and $q_n\to 1, p_n\to 1, q_n^n\to\alpha, p_n^n\to\beta$ as $n\to\infty$. Such sequences (q_n) and (p_n) exist. For example if we take $c,d\in R^+$ such that $c>d$ and choose

$$q_n=\frac{n}{n+c}\ \text{and}\ p_n=\frac{n}{n+d}$$

then we see that

$$q_n \to 1, p_n \to 1, q_n^n \to e^{-c}, p_n^n \to e^{-d}, [n]_{p,q} \to \infty \text{ as } n \to \infty.$$

3 Convergence Theorem

Let $C_B[0,\infty)$ denote the space of real-valued continuous and bounded functions defined on $[0,\infty)$. The norm $||\cdot||$ on the space $C_B[0,\infty)$ is given by

$$||f|| = \sup_{x\in[0,\infty)} |f(x)|$$

Theorem 3.1. *Let us fix $r \in N_0$, $q = (q_n)$ and $p = (p_n)$ satisfying $0 < q_n < p_n \le 1$ such that $q_n \to 1, p_n \to 1$ as $n \to \infty$. Then for each $f \in C_B[0,\infty)$*

$$\lim_{n\to\infty} S^*_{n,p_n,q_n}(f;x;r) = f \tag{3.1}$$

uniformly on $C_B[0,\infty)$.

Proof. We prove the theorem with the help of the well known Korovkin theorem. So it suffices to show that

$$\lim_{n\to\infty} S^*_{n,p_n,q_n}(e_i;x;r) = x^i, i = 0,1,2, r \in N_0.$$

uniformly on $[0,\infty)$,where $e_i(t) = t^i$.

Clearly from (2.2) we have

$$\lim_{n\to\infty} S^*_{n,p_n,q_n}(e_0;x;r) = 1.$$

By simple calculations we get

$$\lim_{n\to\infty} \frac{[n+r]_{p_n,q_n}}{[n]_{p_n,q_n}} = 1, r \in N_0 \ (as\ 0 < q_n < p_n \le 1) \tag{3.2}$$

Also we get

$$\lim_{n\to\infty} \frac{[n+r]_{p_n,q_n}}{[n]^2_{p_n,q_n}} = 0, r \in N_0 \ (as\ 0 < q_n < p_n \le 1) \tag{3.3}$$

Thus using (3.2)-(3.3) in (2.5) and (2.6) of Lemma 2.2 we have

$$\lim_{n\to\infty} S^*_{n,p_n,q_n}(e_1;x;r) = x.$$

and

$$\lim_{n\to\infty} S^*_{n,p_n,q_n}(e_2;x;r) = x^2.$$

This completes the proof. □

4 Rate of Convergence

We recall the definitions of weighted spaces and corresponding modulus of continuity. Let $C[0,\infty)$ be the set of all continuous functions defined on $[0,\infty)$ and $B_2[0,\infty)$ the set of all functions defined on $[0,\infty)$ satisfying the condition

$$|f(x)| \leq M((1+x^2))$$

with some positive constant M which may depend only on f. $C_2[0,\infty)$ denotes the subspace of all continuous functions in $B_2[0,\infty)$. In this section we consider the space

$$C_2^*[0,\infty) = \left\{ f \in C_2[0,\infty) : \exists \lim_{x\to\infty} \frac{|f(x)|}{(1+x^2)} < \infty \right\}$$

endowed with the norm

$$||f||_2 = \sup_{x\in[0,\infty)} \frac{|f(x)|}{((1+x^2))}$$

The modulus of continuity of f on the closed interval $[0,a], a > 0$ is defined by

$$\omega_a(f,\delta) = \sup_{|s-x|<\delta} |f(s) - f(x)|, s, x \in [0,a]$$

. It is well known that for a function $f \in C_2[0,\infty)$ we have

$$\lim_{\delta\to 0} \omega_a(f,\delta) = 0.$$

Now we obtain the rate of convergence of the operators $S^*_{n,p,q}(f;x;r)$ to $f(x)$ on $C_2^*[0,\infty)$.

Theorem 4.1. *Let $f \in C_2^*[0,\infty), r \in N_0, q = q_n \in (0,1), p = p_n \in (q,1]$, such that $q_n \to 1, p_n \to 1$ as $n \to \infty$ and $\omega_{a+1}(f,\delta)(a > 0)$ be its modulus of continuity on the finite interval $[0, a+1] \subset [0,\infty)$. Then*

$$\begin{aligned} &||S^*_{n,p,q}(f;x;r) - f(x)||_{C[0,a]} \\ &\quad \leq N_f(1+a^2)\left[a\frac{[n+r]_{p,q}}{[n]^2_{p,q}} + a^2\left\{ \frac{[n+r]_{p,q}}{[n]_{p,q}}\left(\frac{p[n+r]_{p,q}}{q[n]_{p,q}} - 2\right) + 1\right\}\right] \\ &\qquad + 2\omega_{a+1}(f,\delta) \end{aligned}$$

where $\delta = \left[a\frac{[n+r]_{p,q}}{[n]^2_{p,q}} + a^2\left\{ \frac{[n+r]_{p,q}}{[n]_{p,q}}\left(\frac{p[n+r]_{p,q}}{q[n]_{p,q}} - 2\right) + 1\right\}\right]^{1/2}$ and N_f is a positive constant depending on f.

Proof. For $x \in [0,a]$ and $t > a+1$, since $t - x > 1$, we have

$$\begin{aligned} |f(t) - f(x)| &\leq M_f(2 + x^2 + t^2) \\ &\leq M_f(2 + 3x^2 + 2(t-x)^2) \\ &\leq N_f(1+a^2)(t-x)^2 \qquad (4.1) \end{aligned}$$

where$N_f = 6M_f$. For $x \in [0, a]$ and $t \leq a+1$, we have

$$|f(t) - f(x)| \leq \omega_{a+1}(f, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right) \omega_{a+1}(f, \delta), \delta > 0 \tag{4.2}$$

Combining (4.1) and (4.2), we get for all $x \in [0, a]$ and $t \geq 0$

$$|f(t) - f(x)| \leq N_f(1 + a^2)(t - x)^2 + \left(1 + \frac{|t-x|}{\delta}\right) \omega_{a+1}(f, \delta)$$

On applying the operators to both sides of above inequality, we get

$$\begin{aligned}|S^*_{n,p,q}(f; x; r) - f(x)| \leq & S^*_{n,p,q}(|f(t) - f(x)|; x; r) \\ \leq & N_f(1 + a^2) S^*_{n,p,q}((t - x)^2; x; r) \\ & + \left(1 + \frac{S^*_{n,p,q}(|t - x|; x; r)}{\delta}\right) \omega_{a+1}(f, \delta)\end{aligned}$$

By Cauchy-Schwarz inequality and Corollary 2.1 we obtain

$$\begin{aligned}&|S^*_{n,p,q}(f; x; r) - f(x)| \\ &\leq S^*_{n,p,q}(|f(t) - f(x)|; x; r) \\ &\leq N_f(1 + a^2) S^*_{n,p,q}((t - x)^2; x; r) + \left(1 + \frac{\{S^*_{n,p,q}((t - x)^2; x; r)\}^{1/2}}{\delta}\right) \omega_{a+1}(f, \delta) \\ &\leq N_f(1 + a^2)\left[a\frac{[n+r]_{p,q}}{[n]^2_{p,q}} + a^2\left\{\frac{[n+r]_{p,q}}{[n]_{p,q}}\left(\frac{p[n+r]_{p,q}}{q[n]_{p,q}} - 2\right) + 1\right\}\right] \\ &\quad + \left(1 + \frac{\left[a\frac{[n+r]_{p,q}}{[n]^2_{p,q}} + a^2\left\{\frac{[n+r]_{p,q}}{[n]_{p,q}}\left(\frac{p[n+r]_{p,q}}{q[n]_{p,q}} - 2\right) + 1\right\}\right]^{1/2}}{\delta}\right) \omega_{a+1}(f, \delta)\end{aligned}$$

Choosing $\delta = \left[a\frac{[n+r]_{p,q}}{[n]^2_{p,q}} + a^2\left\{\frac{[n+r]_{p,q}}{[n]_{p,q}}\left(\frac{p[n+r]_{p,q}}{q[n]_{p,q}} - 2\right) + 1\right\}\right]^{1/2}$, we get the desired result. □

5 Weighted Approximation By $S^*_{n,p,q}$

Firstly we show that $S^*_{n,p,q}$ are linear positive operators acting from $C_2[0, \infty)$ to $B_2[0, \infty)$. We make use of the results given in [4, 5].

Lemma 5.1 ([4, 5]). *The sequence of positive linear operators $(L_n)_{n\geq1}$ which act from $C_2[0, \infty)$ to $B_2[0, \infty)$ if and only if there exists a positive constant k such that*

$$L_n(\rho; x) \leq k\rho(x), \text{ i.e.}$$

$$||L_n(\rho; x)||_\rho \leq k$$

where $\rho(x) = (1 + x^2)$ is the weight function.

Lemma 5.2. *If* $f \in C_2^*[0,\infty), r \in N_0, q = q_n \in (0,1), p = p_n \in (q,1]$, *such that* $q_n \to 1, p_n \to 1$ *as* $n \to \infty$ *then*

$$||S^*_{n,p_n,q_n}(1+t^2;x;r)||_2 \leq 1 + M$$

where M is a positive constant.

Proof. By linearity of operators $S^*_{n,p,q}$ and Lemma 2.2 we get

$$\begin{aligned} S^*_{n,p_n,q_n}(1+t^2;x;r) =& 1 + x\frac{[n+r]_{p_n,q_n}}{[n]^2_{p_n,q_n}} + \frac{x^2 p_n [n+r]^2_{p_n,q_n}}{q_n [n]^2_{p_n,q_n}} \\ ||S^*_{n,p_n,q_n}(1+t^2;x;r)||_2 =& \sup_{x\in[0,\infty)} \left\{ \frac{1}{(1+x^2)} + \frac{x}{(1+x^2)} \frac{[n+r]_{p_n,q_n}}{[n]^2_{p_n,q_n}} \right. \\ & \left. + \frac{x^2}{(1+x^2)} \frac{p_n [n+r]^2_{p_n,q_n}}{q_n [n]^2_{p_n,q_n}} \right\} \\ \leq & 1 + \frac{[n+r]_{p_n,q_n}}{[n]^2_{p_n,q_n}} + \frac{p_n [n+r]^2_{p_n,q_n}}{q_n [n]^2_{p_n,q_n}} \end{aligned}$$

from (3.2), (3.3) and by the fact that $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} p_n = 1$ there exists a positive constant M such that

$$||S^*_{n,p_n,q_n}(1+t^2;x;r)||_2 \leq 1 + M$$

□

By using Lemma 5.1 we can easily see that S^*_{n,p_n,q_n} are linear positive operators acting from $C_2[0,\infty)$ to $B_2[0,\infty)$.

Theorem 5.1 ([4, 5])**.** *Let* $(L_n)_{n\geq 1}$ *be the sequence of positive linear operators which act from* $C_2[0,\infty)$ *to* $B_2[0,\infty)$ *satisfying the conditions*

$$\lim_{n\to\infty} ||L_n e_i - e_i||_2 = 0, i = 0,1,2$$

then for any function $f \in C_2^*[0,\infty)$

$$\lim_{n\to\infty} ||L_n f - f||_2 = 0.$$

Next we prove a weighted korovkin type approximation theorem for operators $S^*_{n,p,q}$ with the help of Theorem 5.1.

Theorem 5.2. *Let* $r \in N_0$ *be fixed ,* $q = q_n \in (0,1), p = p_n \in (q,1]$, *such that* $q_n \to 1, p_n \to 1$ *as* $n \to \infty$. *Then for each function* $f \in C_2^*[0,\infty)$

$$\lim_{n\to\infty} ||S^*_{n,p_n,q_n}(f;x;r) - f||_2 = 0$$

Proof. According to the weighted Korovkin type theorem given by Theorem 5.1, it is sufficient to prove the following three conditions

$$\lim_{n\to\infty} ||S^*_{n,p_n,q_n}(e_i;x;r) - e_i||_2 = 0, i = 0,1,2 \tag{5.1}$$

By (2.4) we see that (5.1) holds for $i = 0$.
From (2.5) we get

$$\begin{aligned} ||S^*_{n,p_n,q_n}(e_1;x;r) - e_1||_2 &= \sup_{x\in[0,\infty)} \frac{x}{(1+x^2)} \left\{ \frac{[n+r]_{p_n,q_n}}{[n]_{p_n,q_n}} - 1 \right\} \\ &\leq \left\{ \frac{[n+r]_{p_n,q_n}}{[n]_{p_n,q_n}} - 1 \right\} \end{aligned}$$

and by using (3.2) we obtain

$$\lim_{n\to\infty} ||S^*_{n,p_n,q_n}(e_1;x;r) - e_1||_2 = 0$$

Again from (2.6)

$$\begin{aligned} &||S^*_{n,p_n,q_n}(e_2;x;r) - e_2||_2 \\ &\quad = \sup_{x\in[0,\infty)} \left\{ \frac{x}{(1+x^2)} \frac{[n+r]_{p_n,q_n}}{[n]^2_{p_n,q_n}} + \frac{x^2}{(1+x^2)} \left(\frac{p_n[n+r]^2_{p_n,q_n}}{q_n[n]^2_{p_n,q_n}} - 1 \right) \right\} \\ &\quad \leq \frac{[n+r]_{p_n,q_n}}{[n]^2_{p_n,q_n}} + \frac{p_n[n+r]^2_{p_n,q_n}}{q_n[n]^2_{p_n,q_n}} - 1 \end{aligned}$$

and thus by (3.2) and (3.3)

$$\lim_{n\to\infty} ||S^*_{n,p_n,q_n}(e_2;x;r) - e_2||_2 = 0$$

Thus (5.1) holds for $i = 0, 1, 2$. Hence the proof is completed. □

References

[1] T. Acar, (p,q)-Generalization of Szász-Mirakjan operators, Mathematical Methods in the Applied Sciences, DOI: 10.1002/mma.3721

[2] I.M. Burban, A.U. Klimyk, p, q-Differentiation, (p,q)-integration and (p,q)-hypergeometric functions related to quantum groups, Integral Transforms Spec. Funct., 2 (1994), 15–36.

[3] R. Chakrabarti and R. Jagannathan, A (p,q)-oscillator realization of two parameter quantum algebras, J. Phys. A: Math. Gen., 24 (1991), 711–718.

[4] A.D. Gadjiev, On P. P. Korovkin type theorems, Mat. Zametki, 20 (1976), 781–786; Transl. in Math. Notes, (56) (1978), 995-998.

[5] A.D. Gadjiev, The convergence problem for a sequence of positive linear operators on bounded sets and theorems analogous to that of P.P. Korovkin, Dokl. Akad. Nauk SSSR 218 (5), (1974); Transl. in Soviet Math. Dokl. 15 (5) (1974), 1433-1436.

[6] M. N. Hounkonnou, J. Desire, B. Kyemba, R(p,q)-calculus: differentiation and integration, SUT Journal of Mathematics, Vol. 49, No. 2 (2013), 145–167.

[7] R. Jagannathan, K. S. Rao, Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, Proceedings of the International Conference on Number Theory and Mathematical Physics, 20-21 December 2005.

[8] A. Lupaş, A q-analogue of the Bernstein operator, Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, 9 (1987), 85–92.

[9] M. Mursaleen, Khursheed J. Ansari, Asif Khan, On (p,q)-analogue of Bernstein operators, Appl. Math. Comput., 266 (2015), 874–882.

[10] M. Mursaleen, Khursheed J. Ansari, Asif Khan, On (p,q)-analogue of Bernstein operators(Revised), arXiv:1503.07404v2 [math.CA] 20 Nov 2015.

[11] M. Mursaleen, Khursheed J. Ansari and Asif Khan, Some approximation results by (p, q)-analogue of Bernstein-Stancu operators, Appl. Math. Comput., 264 (2015), 392–402.

[12] M. Mursaleen, Khursheed J. Ansari, Asif Khan, Some approximation results for Bernstein-Kantorovich operators based on (p,q)- integers, arXiv:1504.05887 [math.CA], 5 JUNE 2015 (Pre-print).

[13] M. Mursaleem, Md. Nasiruzzaman and Ashirbayev Nurgali, Some approximation results on Bernstein-Schurer operators defined by (p,q)- integers, J. of Ineq. and Appl., Vol. 2015, Issue 1, 249 (2015), DOI 10.1186/s13660-015-0767-4.

[14] M. A. Özarslan , q-Szász Schurer Operators, Miskolc Mathematical Notes, Vol. 12 (2011), No. 2, pp. 225–235

[15] P. N. Sadjang, On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas, arXiv:1309.3934 [math.QA].

[16] V. Sahai, S. Yadav, Representations of two parameter quantum algebras and (p,q)-special functions, J. Math. Anal. Appl., 335 (2007), 268–279.