From Euclidean Geometry to Manifold Theory And Some Basic Properties of the Curvature Tensors in Riemannian Geometry

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Abstract

In the present talk we explain how the notion of manifolds come from Euclidean geometry. Next some basic properties of curvature tensors in Riemannian geometry have been discussed. In particular, 2-dimensional and 3-dimensional Riemannian space have been considered.

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1 History of Euclidean Geometry

About **300 B. C., Euclid** wrote a book **The Elements**. First he defined the terms-points, lines, planes and so on. Then he wrote down five postulates. He defined points, lines etc in the following way:
A point is that which has no part. A line is a breathless length. A straight line is a line which lies evenly with the points on itself. A plane angle is the inclination to one another of two lines which meet. When a straight line set upon a straight line makes adjacent angles equal to one another, each of the equal angles is a right angle. Euclid did not define length, distance, inclination or "set upon". This book was a compilation of knowledge that became the center of mathematical teaching for 2300 years. An Arabic version of it appears at the end of 8th century. More than two thousand edition of **The Elements** have been published since it was first printed in 1482. This is the first printed mathematics books published in Latin in Venice. In the 19th century, **B. L. VANDER WAERDEN** assesses importance of **The Elements** as follows:
"Next to the Bible, **The Element** may be the most translated, published and studied of all the books produced in the Western World".
The five Euclid's postulates are
1. *To draw a straight line from any point to any point.*
2. *To produce a finite straight line continuously in a straight line.*

3. *To describe a circle with any center and distance.*
4. *That, all right angles are equal to one another.*
5. *That, if a straight line falling on two straight lines make the interior angle on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.*

The famous fifth postulate is known as the **parallel postulate**, since, it deals with parallelity of lines. After more than two millennia of study, this postulate was found to be independent of others. Euclid did not feel it necessary to state the following postulate though he used it in his very first Theorem.
Two circles, the sum of whose radii is greater than the distance between their centers, and the difference of whose radii is less than the distance must have a point of intersection.
PROCLUS, PTOLEMY, PLAFAIR, WALLIS and many other Mathematicians tried to prove the fifth postulate from the other four, but did not succeed. What all of them did was that they assumed some obvious 'property'which turned out to be equivalent to the fifth postulate.
PROCLUUS did give the following postulate which is equivalent to the fifth postulate.
Plafair's Axioms: Given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line.
(**PLAFAIR** wrote a famous commentary on **EUCLID** in 1795 in which he proposed replacing the fifth postulate by this axiom.)
In 1663 **WALLIS** deduced the fifth postulate which is equivalent to
"To each triangle, there exists a similar triangle of arbitrary magnitude".
SACCHERI, LAMBERT and some others tried in a different way. They assumed the fifth postulate false and attempted to derive a contradiction, but everything went in vain.
In 1766 **LAMBERT** noticed that, in his new geometry, the angle sum of a triangle increased as the area of the triangle decreased.
A. M. LEGENDRE (1752 − 1833) did spent 40 years working on parallel postulate. In the most interesting attempts, **LEGENDRE**, like Saccheri approached the question on the sum of the angles of a triangle and proved the Euclid's fifth postulate is equivalent to:
"The sum of the angles of a triangle is equal to two right angles".
Since the major problems of the geometry was the parallel postulate, the French geometer **D' ALEMBERT** in 1767, called the problem *"the scandal of elementary geometry".*
However in 1817, the German Mathematician **GAUSS** came to a conclusion that the fifth postulate was independent of the other four, but for some reason he kept his result secret. **GAUSS** wrote a letter on December 17, 1799 on the theory of parallels to his friend, the Hungarian geometer **W. BOLYAI** who made several false proof of the parallel postulate. **W. BALYAI** advised his son **JANOS BOLYAI** not to waste one hour's time on that problem of the fifth postulate. But **JANOS BOLYAI** did work on the problem and he had an opinion that the new geometry was possible. **GAUSS**, after reading the 24 pages article **BOLYAI**, described **JANOS BOLYAI** in this words while writing to a friend: I regard this young geometer **BOLYAI** as a genius of the first order. But **GAUSS** told **BOLYAI** that he had discovered all these earlier but had not published.
In 1829 **LOBACHEVSKY** published a work on non-Euclidean geometry. Neither **BOLYAI** nor **GAUSS** knew of **LOBACHEVSKY's** work mainly because it was only published in Russian in the Kazan messenger a local university publication. **LOBACHEVSKY** replaced the fifth postulate of **EUCLID** by

"There exist two lines parallel to a given line through a given point not on the line".
The Italian Mathematician **E. BETTRAMI's** (1835 – 1900) was the first person to put the **BOLYAI-LOBACHEVSKY-GAUSS's** non-Euclidean geometry on the same footing as Euclidean geometry. In 1868 he produced a model for 2-dimensional non-Euclidean geometry within 3-dimensional Euclidean geometry. The model was obtain on the surface of revolution of a tractrix about its asymptote.
BETTRAMI's work a model of **BOLYAI-LOBACHEVSKY-GAUSS's** non-Euclidean geometry was completed by **F. KLEIN** in 1871.
GEORGE FREIDRICH BERNHAND RIEMANN (1826 – 1866), a research student of **GAUSS**, gave an inaugural lecture on 10th June 1854 in which he reformulated the whole, concept of geometry which he saw as a space with extra structure to be able to measure things like length. But this lecture was published in 1868, two years after **RIEMANNIAN's** death. **RIEMANNIAN** briefly discussed a 'spherical' geometry in which every line through a point P not on the line AB rneets the line AB. In this geometry no parallels is possible.
Conclusion. When we investigate the vast object in galaxy, we find Euclidean geometry model has short comings. Here non-Euclidean geometry (Riemannain geometry) works. Thus we shall agree that alternative to **Euclid's** parallel postulate are not simply abstract concepts but also have practical manifestations.

2 From Euclidean Geometry to Manifold Theory

TOPOLOGY literally means the study of surface (**topos** meaning 'surface' and **logos** meaning 'study'). Topology is considered to be the modern day analysis, but actually it has more relations with Geometry, which also deals with study of various surfaces. In geometry we consider collection of certain objects and an equivalence relation, and then instead of the objects we consider the equivalence classes, and finally study those properties of objects which remain invariant under this classification. Likewise in topology. Although, the equivalence relations being different in these two fields. In geometry we take the congruence and in topology the homeomorphism as the respective equivalence relations. This process is used in almost every field of mathematics, whether it is Set theory (the equivalence relation being equipottency), Group Theory (here isomorphism plays the role), or any other field, we carry on the same thing, study the classes rather than the objects themselves.

The journey from Geometry or more specifically from Euclidean geometry to topology can well be divided into two parts:

i. Geometry to Metric Space.

ii. Metric space to Topological Space.

It is very common in mathematics that whenever a branch of mathematics becomes rich in theory as well as in application purpose, we usually think of dropping some basic postulates and thus a new branch of mathematics is born. In this process we always take care of the proper blance between depth and applicability.

When Euclidean geometry reaches that level some people thought to ignore all the postulates except the third one, i.e., the one working with **distance**. They generalize the concept of distance and thus come to the concept of METRIC.

In an abstract set X defining the metric as a function $d : X \times X \longrightarrow \mathbb{R}$ satisfying certain properties we obtain a Metric Space as an ordered pair (X, d).

The second transition starts from Metric Space. While studying Metric Space we found that most of the concepts in a Metric Space can well be explained independently in terms of a new tool, namely **OPEN SET**. We have also seen that the family τ of all open sets in a metric space (X, d) satisfy some properties such as :

1. $\phi \in \tau$ as well as $X \in \tau$.

2. τ is closed under arbitrary unions.

3. τ is closed under finite intersections.

4. Given two district points x, y in X, there exit U, $V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

After a certain level of advancement of metric space some people thought of working with just the first three properties of family of open sets. In this process of abstraction of metric space comes to the notion of TOPOLOGICAL SPACE.

In an abstract set X, we take a certain family τ of its subsets, or in another way the members of $P(X)$ satisfying the first three properties and then call the ordered pair (X, τ) as a topological space.

In topological spaces with the help of the open sets we can study limit, continuity very well. So if an arbitrary set is equipped with a topological structure, then studying limit, continuity is not a problem. But our aim is to study the whole analysis. So we extend our discussion towards differential calculus. As all of us know derivative means rate of change, so we need to study a fraction. Alas, we can not, since there is no concept of division in topology. This is the main reason we study yet a new structure, named **Differential Manifold**.
We are aware of the analysis of $\mathbb{R}^n$, n-dimensional Euclidean space. So, what we do is borrow the differential calculus of $\mathbb{R}^n$ in an arbitrary set.
Let M be an arbitrary set. We say M is a differentiable manifold if
(i) for every $x \in M$, there exists a set $O_x \subseteq M$ containing x, an injective mapping α_x such that $\alpha_x(O_x)$ is an open set in $\mathbb{R}^n$,
ii) $O_x \cap O_y \neq \phi$, then $\alpha_x \circ \beta_y^{-1} : \beta_y(O_x \cap O_y) \longrightarrow \alpha_x(O_x \cap O_y)$ and its inverse are differentiable.

What basically all these things mean is that a manifold is locally Euclidean, i.e., if we concentrate on a neighborhood of a point, we will find a copy of Euclidean space there. For differentiability related issue we first transfer the whole problem in $\mathbb{R}^n$ with the help of the chart maps, then solving the problem there we come back to the manifold with the help of inverse maps. In this regard we must know a very useful term, diffeomorphism. A mapping f is called diffeomorphism if f and f^{-1} both are differentiable. We should also mention that we can start from a topological structure on the set, and if it is second countable and Hausdorff, we can also define metric on the set. The fact is that if we start from a mere set we still draw a topological structure in it with the help of chart maps and chart domains.

3 Some curvature properies of Riemannian Geometry

A Riemannian space of dimension n is denoted by M^n. A Riemannian space has three main notions of curvature:

- The Riemannian curvature
- The Ricci Curvature
- The scalar curvature.

In M^n, the Riemannian curvature tensor of type $(1,3)$, Ricci tensor of type $(0,2)$ and scalar curvature are denoted by R^h_{ijk}, R_{ij} and R respectively. Also the Riemannian curvature tensor of type $(0,4)$ is defined by $R_{lijk} = g_{hl}R^h_{ijk}$.
First we note that for the dimension $n = 2$, the three curvatures are equivalent. First we prove this statement.

For M^2, we have

$$\frac{R_{11}}{g_{11}} = \frac{R_{22}}{g_{22}} = \frac{R_{12}}{g_{12}} = -\frac{R_{1212}}{g}.$$

In M^2, the Riemannian curvature tensor has only one non-vanishing independent component which is R_{1212}.
We have $R_{ij} = g^{hk}R_{ihkj}$.
For M^2,

$$R_{ij} = g^{11}R_{i11j} + g^{12}R_{i12j} + g^{21}R_{i21j} + g^{22}R_{i22j}.$$

Therefore,

$$\begin{aligned} R_{11} &= g^{11}R_{1111} + g^{12}R_{1121} + g^{21}R_{1211} + g^{22}R_{1221} \\ &= g^{22}R_{1221} = -g^{22}R_{1212} = -\frac{g_{11}}{g}R_{1212}. \end{aligned}$$

Hence $\frac{R_{11}}{g_{11}} = -\frac{R_{1212}}{g}$. Similarly, the others.

Now, we prove that $R_{ijkl} = -\frac{R}{2}(g_{ik}g_{jl} - g_{il}g_{jk})$.
Since in M^2, the only non-vanishing independent components of R_{ijkl} are R_{1212}, therefore we are to show that

$$R_{1212} = -\frac{R}{2}(g_{11}g_{22} - g_{12}g_{21}).$$

Now,

$$\begin{aligned} R &= g^{ij}R_{ij} = g^{11}R_{11} + g^{12}R_{12} + g^{21}R_{21} + g^{22}R_{22} \\ &= g^{11}. - \frac{R_{1212}}{g}g_{11} + g^{12}. - \frac{R_{1212}}{g}g_{21} \\ &\quad + g^{21}. - \frac{R_{1212}}{g}g_{12} + g^{22}. - \frac{R_{1212}}{g}g_{22} \\ &= -\frac{R_{1212}}{g}[g^{11}g_{11} + g^{12}g_{21} + g^{21}g_{12} + g^{22}g_{22}] \\ &= -\frac{R_{1212}}{g}.2 \end{aligned}$$

i.e.,

$$R_{1212} = -\frac{R}{2}g = -\frac{R}{2}(g_{11}g_{22} - g_{12}g_{21}).$$

Next we prove that for M^3:

$$\begin{aligned} R_{ijkl} &= g_{il}R_{jk} - g_{ik}R_{jl} + g_{jk}R_{il} - g_{jl}R_{ik} \\ &\quad -\tfrac{R}{2}(g_{jk}g_{il} - g_{jl}g_{ik}). \end{aligned} \tag{A}$$

We know that in M^n, the total number of independent components of the covariant curvature tensor is $\frac{n^2(n^2-1)}{12}$.

Now we prove equation (A). We observe that the equation (A) is a tensor equation. So, we can choose a convenient coordinate system to obtain it. In M^3, we can choose a coordinate system in which $g_{ij} = 0$ for $i \neq j$ and $g_{ij} = 1$ for $i = j$.
By this choice, using $g^{ij} = \frac{\text{cofactor of } g_{ij} \text{ in } g}{g}$, it can be shown that $g^{ij} = 0$ for $i \neq j$ and $g^{ij} = \frac{1}{g_{ij}}$, otherwise.

Again in M^3 the number of independent components of R_{ijkl} are 6. They are R_{1212}, R_{1313}, R_{2323}, R_{1213}, R_{2123} and R_{3231}.

Now, we know that

$$\begin{aligned} R_{ij} &= g^{11}R_{i11j} + g^{12}R_{i12j} + g^{13}R_{i13j} + g^{21}R_{i21j} \\ &\quad g^{22}R_{i22j} + g^{23}R_{i23j} + g^{31}R_{i31j} + g^{32}R_{i32j} + g^{33}R_{i33j}. \end{aligned} \tag{B}$$

Putting $i = 1$ and $j = 2$ we get

$$\begin{aligned} R_{12} &= g^{11}R_{1112} + g^{12}R_{1122} + ... \\ &= g^{33}R_{1332} \\ &= \frac{1}{g_{33}}R_{1332}, \end{aligned}$$

since $g^{ij} = 0$ for $i \neq j$.

Similarly, we get the expressions for R_{13}, R_{23}.
Again, in (B), putting $i = j = 1$, we get

$$\begin{aligned} R_{11} &= g^{22}R_{1221} + g^{33}R_{1331} \\ &= \frac{1}{g_{22}}R_{1221} + \frac{1}{g_{33}}R_{1331}. \end{aligned}$$

Similarly, we can find R_{22}, R_{33}.

Hence generalizing we get

$$R_{ii} = \tfrac{1}{g_{jj}}R_{ijji} + \tfrac{1}{g_{kk}}R_{ikki}, \qquad i \neq j \neq k. \tag{1}$$

Again the scalar curvature R is given by

$$\begin{aligned} R = g^{ij}R_{ij} &= g^{11}R_{11} + g^{12}R_{12} + g^{13}R_{13} + g^{21}R_{21} + ... \\ &= g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \\ &= \frac{1}{g_{11}}R_{11} + \frac{1}{g_{22}}R_{22} + \frac{1}{g_{33}}R_{33}. \end{aligned}$$

Using R_{11}, R_{22} and R_{33} we get

$$R = \textstyle\sum_{i,j} \frac{1}{g_{ii}}\frac{1}{g_{jj}}R_{ijji}. \tag{2}$$

In virtue of these relations, we get

$$R_{ijji} - g_{ii}R_{jj} - g_{jj}R_{ii} + \tfrac{1}{2}Rg_{ii}g_{jj} = 0. \tag{3}$$

We now define a tensor T_{ijkl} as follows:

$$\begin{aligned} T_{ijkl} &= R_{ijkl} - g_{il}R_{jk} + g_{ik}R_{jl} - g_{jk}R_{il} \\ &\quad + g_{jl}R_{ik} + \tfrac{R}{2}[g_{jk}g_{il} - g_{ik}g_{jl}]. \end{aligned} \tag{4}$$

Then it can be verified that

$$T_{ijji} = 0, \qquad i \neq j$$

and

$$T_{ijki} = 0, \qquad i \neq j \neq k.$$

Hence T_{ijkl} is a zero tensor. Therefore from (4) we get (A). Hence (A) is the special form of R_{ijkl} in M^3.

Space of constant curvature. A Riemannian space M^n is said to be a space of constant curvature if $R^h_{ijk} = \lambda[\delta^h_k g_{ij} - \delta^h_j g_{ik}]$, where λ is a constant.

As for example Euclidean space E^n with Cartesian coordinate system is a space of constant curvature zero.
As n-dimensional sphere S^n in R^{n+1} given by $(x^1)^2 + (x^2)^2 + ... + (x^n)^2 = c^2$ is a space of constant curvature $\frac{1}{c^2}$.

Einstein Space. Albert Einstein in 1913 proposed that the field equation for the interaction of gravitation and other fields take the form $R_{ij} - \frac{R}{2}g_{ij} = T_{ij}$, where T_{ij} is the energy-momentum tensor.
A Riemannian space $M^n, (n \geq 2)$ is said to be Einstein space if the Ricci tensor is of the form $R_{ij} = \frac{R}{2}g_{ij}$.
It can be easily proved that for $n > 2$, $R =$ constant. Therefore we can write $R_{ij} = \lambda g_{ij}$, where λ is constant.
From the definition it follows that a space of constant curvature implies Einstein space, but the converse is not necessarily true. However for $n = 3$, the converse is true.

Result 1. A 3-dimensional Einstein space is a space of constant curvature.

Proof. By hypothesis $R_{ij} = \frac{R}{3}g_{ij}$. But in a M^3,

$$\begin{aligned} R_{ijkl} &= g_{il}R_{jk} - g_{ik}R_{jl} + g_{jk}R_{il} - g_{jl}R_{ik} - \frac{R}{2}(g_{jk}g_{il} - g_{jl}g_{ik}) \\ &= g_{il}\frac{R}{3}g_{jk} - g_{ik}\frac{R}{3}g_{jl} + g_{jk}\frac{R}{3}g_{il} - g_{jl}\frac{R}{3}g_{ik} - \frac{R}{2}(g_{jk}g_{il} - g_{jl}g_{ik}) \\ &= \frac{R}{6}(g_{jk}g_{il} - g_{jl}g_{ik}), \end{aligned}$$

which implies that M^3 is a space of constant curvature.
Example 1. A space with schwarzschild metric is an Einstein space, but not a space of constant curvature.
Example 2. Einstein Universe is neither an Einstein space nor a space of constant curvature.
Example 3. De Sitter's Universe is an Einstein space.
Example 4. Consider a sphere of radius c in E^3 with equation of the form: $x^1 = c\sin u^1 \cos u^2$, $x^2 = c\sin u^1 \sin u^2$, $x^3 = c\cos u^1$, where c is a constant, in E^3, is an Einstein space.
Solution. For the sphere the line element is given by $ds^2 = c^2(du^1)^2 + c^2 sin^2u^1(du^2)^2$, Here the coefficients of the first fundamental form of the sphere are $a_{11} = c^2, a_{12} = a_{21} = 0$ and $a_{22} = c^2 sin^2 u^1$.
Now $a^{11} = \frac{1}{c^2}, a^{12} = a^{21} = 0$ and $a^{22} = \frac{1}{c^2 sin^2 u^1}$.
From this we can calculate $R_{1212} = c^2 sin^2 u^1$.
Therefore, $R_{11} = a^{22}R_{1221} = -a^{22}R_{1212} = \frac{1}{c^2 sin^2 u^1}(-c^2 sin^2 u^1) = -1$.
$R_{22} = a^{11}R_{2112} = -sin^2 u^1$.
Also the scalar curvature $R = a^{11}R_{11} + a^{22}R_{22} = -\frac{2}{c^2}$.
Now, $R_{11} = -1 = -\frac{2}{2c^2}c^2 = \frac{R}{2}a_{11}$ and $R_{22} = \frac{R}{2}a_{22}$.
So the sphere of constant radius c is an Einstein space.

Result 2. A necessary and sufficient condition for a Riemannian space $M^n, (n \geq 2)$ to be an Einstein space is $R_{ij}R^{ij} = \frac{R^2}{n}$.

Proof. Let us define a covariant tensor a_{ij} by $a_{ij} = R_{ij} - \frac{R}{n}g_{ij}$. Then $a^{ij} = R^{ij} - \frac{R}{n}g^{ij}$. Hence we get $a_{ij}a^{ij} = R_{ij}R^{ij} - \frac{R^2}{n}$.
Suppose the space is an Einstein space. Then $a_{ij} = 0$ which implies $R_{ij}R^{ij} = \frac{R^2}{n}$.
Conversely, if $R_{ij}R^{ij} = \frac{R^2}{n}$, then from $a_{ij}a^{ij} = R_{ij}R^{ij} - \frac{R^2}{n}$ it follows that $a_{ij}a^{ij} = 0$. Since the metric is positive definite, $a_{ij} = 0$, which implies that the space is an Einstein space.

4 Weyl conformal curvature tensor and Weyl projective curvature tensor

Let an n-dimensional Riemannian space M^n equipped with two metric tensor g and $\bar{g}$. If a transformation of M^n does not change the angle between two tangent vectors at a point with respect to g and $\bar{g}$, then such a transformation is said to be a conformal transformation of the metrics on the Riemannian manifold.
Let us consider a Riemannian manifold M^n with two metric tensors g and $\bar{g}$ such that they are related by

$$\bar{g}(X, Y) = e^{2\sigma} g(X, Y), \tag{4.1}$$

where σ is a real function. The angle between two tangent vectors at a point $p \in M$ does not change with respect to the change of metrics given by the above equation. Then (M^n, g) and $(M^n, \bar{g})$ are called conformally related spaces and the correspondence between (M^n, g) and $(M^n, \bar{g})$ is known as conformal transformation.
A tensor C^h_{ijk} of type $(1, 3)$ that remains invariant under conformal transformation for an n-dimensional Riemannian space M^n, is given by

$$\begin{aligned} C^h_{ijk} &= R^h_{ijk} + \frac{1}{n-2}\{R_{ik}\delta^h_j - R_{ij}\delta^h_k + g_{ik}R^h_j - g_{ij}R^h_k\} \\ &\quad + \frac{R}{(n-1)(n-2)}\{g_{ij}\delta^h_k - g_{ik}\delta^h_j\}. \end{aligned} \tag{4.2}$$

The curvature tensor defined by the equation (4.2) is known as **Weyl conformal curvature tensor** due to **H. Weyl**. If the conformal curvature tensor of a Riemannain space of dimension > 3 vanishes identically then the manifold is called **conformally flat**. However for a 3-dimensional Riemannian manifold $C^h_{ijk} = 0$. This can be proved by substituting the expression of R_{hijk} from (A) in (4.2).

The Weyl conformal curvature tensor satisfies the following properties:

- $C^h_{ijk} = -C^h_{ikj}$
- $C^h_{ijk} + C^h_{jki} + C^h_{kij} = 0.$

Result 1. If a Riemannian space M^n of dimension $n > 3$ is Einstein and conformally flat, then it is a manifold of constant curvature.
Result 2. If a Riemannian space M^n of dimension $n > 3$ is of constant curvature, then the manifold is conformally flat.

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. If we consider a transformation in a Riemannian space such that the geodesic goes to geodesic such a transformation is know as projective transformation. The curvature tensor corresponding to projective transformation is know as **Weyl projective curvature tensor**. The Projective curvature tensor is defined by

$$W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}[R_{ij}\delta^h_k - R_{ik}\delta^h_j].$$

The projective curvature tensor satisfies the following properties:

- $W^h_{ijk} = -W^h_{ikj}$
- $W^h_{ijk} + W^h_{jki} + W^h_{kij} = 0.$

Definition. A Riemannian space of dimension $n > 2$ is said to be projectively flat if the projective curvature tensor vanishes at each point of the space.
Now we prove the following important result:
Result 3. A Riemannian space M^n, $(n > 2)$ is projectively flat if and only if the space is of constant curvature.

Proof. Projective curvature tensor can be written in the following form

$$W_{hijk} = R_{hijk} - \frac{1}{n-1}[g_{hk}R_{ij} - g_{hj}R_{ik}]. \tag{4.3}$$

Suppose M^n is projectively flat. Then from (4.3) we get

$$R_{hijk} = \frac{1}{n-1}[g_{hk}R_{ij} - g_{hj}R_{ik}], \tag{4.4}$$

from which it follows that

$$R_{ihjk} = \frac{1}{n-1}[g_{ik}R_{hj} - g_{ij}R_{hk}]. \tag{4.5}$$

Since $R_{hijk} + R_{ihjk} = 0$, therefore from (4.4) and (4.5) it follows that

$$g_{hk}R_{ij} - g_{hj}R_{ik} + g_{ik}R_{hj} - g_{ij}R_{hk} = 0. \tag{4.6}$$

Applying g^{ij} in (4.6) yields

$$R_{hk} = \frac{R}{n}g_{hk}, \tag{4.7}$$

which implies that the space is an Einstein space.
From (4.4) and (4.7) we get

$$R_{hijk} = \frac{R}{n(n-1)}[g_{hk}g_{ij} - g_{hj}g_{ik}], \tag{4.8}$$

since R is constant, therefore the Riemannian space is a space of constant curvature.
Conversely, suppose that the space is a space of constant curvature λ. Then

$$R_{hijk} = \lambda[g_{hk}g_{ij} - g_{hj}g_{ik}], \tag{4.9}$$

which implies $R_{hk} = \lambda(n-1)g_{hk}$.
Substituting the value of R_{ij} in (4.3), we get

$$W_{hijk} = 0, \tag{4.10}$$

that is, the space $M^n(n > 2)$ is projectively flat.
Also it can be easily prove that in a Riemannian space $M^n(n > 2)$, div W=0 implies div C=0, that is, $W^h_{ijk,h} = 0$ implies $C^h_{ijk,h} = 0$.

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