Transitivity and Hypercyclicity of Operators

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Abstract

We prove that a continuous map T on a second countable, second category T_1- topological space X without isolated points is hypercyclic if and only if T is topologically transitive. This result generalizes the celebrated Birkhoff transitivity theorem([2]).

1 Introduction

Let T be a operator on a topological vector space X. The T orbit of a vector $x \in X$ is the set $O(x,T) = \{T^n(x); n \in \mathbb{N}\}$. The operator T is called hypercyclic if there exists a vector x in X such that T-orbit of x is dense in X. Such a vector x is said to be hypercyclic for T, and set of all hypercyclic vectors of T is denoted by HC(T). If T is a linear operator on a finite dimensional normed linear space and $O(x,T)$ is a orbit of x, then there are only three possibilities: (i) $limT^n x = 0$, (ii) $lim\|T^n x\| = \infty$, (iii) $\overline{O(x,T)}$ is compact and $0 \notin \overline{O(x,T)}$ ([3]). The forgoing result implies that hypercyclicity of a linear operator is an infinite dimensional phenomena (Proposition1.1 [5]). A complete topological vector space whose topology is generated by a countable family of seminorms is called a Frechet space.

Definition 1.1. *Let T be a map on a topological space X. Then T is said to be topologically transitive if for each non-empty open sets U and V of X there exists an $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \phi$.*

In 1929, Birkhoff proved that for a continuous linear operator on a separable Frechet space “topological transitivity”and “hypercyclicity”are equivalent notions. We state this result.
Birkhoff's transitivity theorem([2]) Let X be a separable F- space and T be a continuous linear operator on X. The following are equivalent:
(i) T is hypercyclic;
(ii) T is topologically transitive.
In that case, $HC(T)$ is a dense G_δ subset of X.
It is clear that the topological transitivity and hypercyclicity make sense in the more general setting of topological space.To that end we give the following definitions.

Definition 1.2. *A topological space X is called a T_1- space if, given any pair of distinct points x and y in X, there exist open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.*

We wish to emphasize that the concepts of hypercyclicity and topological transitivity make sense for any self map on an arbitrary topological space. We extend these twin ideas in this generalized setting. Further, we show the equivalence of topological transitivity and hypercyclicity of a continuous map on a more general setting of a topological space satisfying some additional conditions.

2 Main Results

As a starting point to the generalisation of Birkhoff's transitivity theorem, we prove the following lemma.

Lemma 2.1. *Let X be a T_1 space and X has no isolated points. Then $D \setminus \{x\}$ is also dense in X, whenever $x \in X$ and D is a dense subset of X.*

Proof. To prove that $D \setminus \{x\}$ is dense it suffices to show that x is a limit point of $D \setminus \{x\}$. Let U be an open set containing x. Since X has no isolated points, hence there exist a point $y (\neq x) \in U$. As D is dense in X, so either $y \in D$ or y is a limit point of D. If $y \in D$, then $y \in D \setminus \{x\}$ and so $y \in U \cap D \setminus \{x\}$. Further, if y is a limit point of D, then $U \cap D$ contains infinitely many points, as X is T_1. Therefore $U \cap D \setminus \{x\} \neq \phi$. Hence x is a limit point of $D \setminus \{x\}$. □

Theorem 2.1. *Let X be a separable T_1- topological space without isolated points and T be a map on X. If T is hypercyclic, then T is topologically transitive.*

Proof. Let U and V be non-empty open sets of X. Since T is hypercyclic, $O(x, T)$ is dense for some $x \in X$. Hence $U \cap O(x, T) \neq \phi$. Therefore $T^k(x) \in U$ for some non-negative integer k. Now it follows by Leema 2.1 that $O(x, T) \setminus \{x, Tx, T^2x, ...T^kx\}$ is also dense in X. Thus there exists a integer $m > k$, such that $T^m(x) \in V$. Again, $T^k(x) \in U$ implies $x \in T^{-k}(U)$. Hence $T^m(x) \in T^{m-k}(U)$. Therefore $T^m(x) \in T^{m-k}(U) \cap V$. Hence T is topologically transitive. □

Remark: We note that in the above theorem T is not even assumed to be continuous.
We now give a simple example which satisfies the above theorem.

Example 2.1. *1: Let $X = \mathbb{Z}^+ = \{0, 1, 2, 3, ...\}$ with cofinite topology. Define the map T on X as $T(n) = n + 1$, for $n \geq 1$ and $T(0) = 0$. Indeed X is T_1 and has no isolated points. We see that $O(1, T)$ is dense in X and hence T is hypercyclic. Now it follows from the theorem 2.2 that T is topologically transitive.*

The following example shows that the condition, X is a T_1 space, is necessary for the validity of theorem 2.2.

Example 2.2. *2: Let $X = \{1, \frac{1}{2}, \frac{1}{3}, ...\} \cup \{0\}$ with topology $\tau = \{\phi, X, \{1, \frac{1}{2}\}, \{\frac{1}{3}, \frac{1}{4}\}, \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}\}$. Define the map T on X as $T(\frac{1}{n}) = \frac{1}{n+1}$, for $n \geq 1$ and $T(0) = 0$. . We see that X is not a T_1 space, has no isolated points and $O(1, T)$ is dense in X. But T is not topologically transitive as, for $U = \{\frac{1}{3}, \frac{1}{4}\}$ and $V = \{1, \frac{1}{2}\}$, such that $T^nU \cap V = \phi$ for each $n \in \mathbb{N}$.*

Similarly, if a topological space X has isolated points, then again, the theorem 2.2 fails. This can be seen in the following example.

Example 2.3. *3: Let* $X = \{1, \frac{1}{2}, \frac{1}{3}, ...\} \cup \{0\}$ *with usual metric and define the map* T *on* X, *as* $T(\frac{1}{n}) = \frac{1}{n+1}$, *for* $n \geq 1$ *and* $T(0) = 0$.. *Then* X *is* T_1 *space with isolated points and* $O(1, T)$ *is dense in* X *but* T *is not topologically transitive.*

On the reverse direction, we have the following theorem.

Theorem 2.2. *Let* T *be a continuous map on a second countable and second category topological space* X. *If* T *is topologically transitive, then* T *is hypercyclic. In this case,* $HC(T)$ *is a dense* G_δ *subset of* X.

Proof. On contrary, suppose T is not hypercyclic. It means $\overline{O(x, T)} \neq X$ for all $x \in X$. Thus $X \setminus \overline{O(x, T)}$ is non-empty proper open set of X for each $x \in X$. Let $\mathbb{B} = \{V_1, V_2, V_3, ...\}$ be a countable base for X. Then, for each $x \in X$, there exists a positive integer $n(x)$ such that $V_{n(x)} \subset X \setminus \overline{O(x, T)}$. Indeed, $T^k(x) \notin V_{n(x)}$, or $x \notin T^{-k}V_{n(x)}$, for all $k \geq 0$. Since $V_{n(x)}$ is open, and T is continuous so $V^{(x)} = \bigcup_{k=0}^{\infty} T^{-k}(V_{n(x)})$ is also open. Let $W^{(x)}$ be the complement of $V^{(x)}$. We note that, for each $x \in X$, $x \notin V^{(x)}$. Hence $x \in W^{(x)}$ for each $x \in X$. Therefore $X = \bigcup_{x \in X} W^{(x)}$. Now, as T is topologically transitive, hence for each non-empty open set U, there exists a non-negative integer n such that $T^n(U) \cap V_{n(x)} \neq \phi$. Therefore $V^{(x)}$ intersects each non-empty open set U. This implies $W^{(x)}$ has empty interior for each $x \in X$. But $X = \bigcup_{x \in X} W^{(x)}$. This contradicts the fact that X is of second category. Further, we claim that $HC(T) = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq 0} (T^n)^{-1}(V_m)$. Let $y \in \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq 0} (T^n)^{-1}(V_m)$. Then $y \in \bigcup_{n \geq 0} (T^n)^{-1}(V_m)$ for each $m \in \mathbb{N}$. Hence for each $m \in \mathbb{N}$, there exists a non-negative integer n_m such that $y \in (T^{n_m})^{-1}(V_m)$. Therefore y is a hypercyclic vector for T. Conversely, let x is a hypercyclic vector for T, then for each basic open set V_j, there exists a n_j such that $T^{n_j}(x) \in V_j$. Hence $x \in (T^{n_j})^{-1}(V_j)$ for each j and therefore $x \in \bigcup_{n \geq 0} (T^n)^{-1}(V_j)$ for each j. Thus $x \in \bigcap_{j \in \mathbb{N}} \bigcup_{n \geq 0} (T^n)^{-1}(V_j)$. Hence $HC(T) = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq 0} (T^n)^{-1}(V_m)$. Therfore $HC(T)$ is a G_δ set. Also $HC(T)$ is a dense subset of X, as X is second category. □

The following example shows that if a topological space is not of second category, then the above result need not hold.

Example 2.4. *5: Let* $X = \{z \in \mathbb{C} : |z| = 1 \quad and \quad z^n = 1, for \quad some \quad n \in \mathbb{N}\}$. *Define a map* T *on* X, *as* $T(z) = z^2$. *Then* X *is not of second category and* T *is topologically transitive map whereas* T *does not have a dense orbit.*

Combining theorem 2.2 and 2.3 we get the following theorem, which generalizes the Birkhoff's transitivity theorem.

Theorem 2.3. *Let X be second countable, second category and T_1- topological space without isolated points and T be a continuous map on X. Then following statements are equivalent.*
(i) T is topologically transitive.
(ii) T is hypercyclic.
In this case $HC(T)$ is a dense G_δ subset of X.

Proof. It is immediate fallout of Theorem 2.2 and Theorem 2.3. □

References

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