Triangles and Circles in Infinite Dimension

M.A. Sofi [1]

Department of Mathematics
University of Kashmir, Srinagar-190006, India
aminsofi@gmail.com

Abstract

In this short survey surrounding certain elementary facts involving the existence of equilateral triangles and circles with certain special properties in the 2-dimensional plane, attention is drawn to higher dimensional analogues of these properties including the case of infinite dimension and to the progress that has been achieved on these questions in the latter setting. Wherever possible, an attempt has been made to point out the difficulties in extending these ideas to higher dimensional spaces. It is expected that the problems raised in the survey shall motivate young researchers to take up these problems for further study.

1 Introduction

We begin by recalling the following well-known but important geometrical facts from plane Euclidean geometry.
(A) Euclidean case:

1. (Euclid, 300 BC): In the plane, there exists an equilateral triangle. (Equivalently, there exist three equidistant points in the plane).
2. (Jung [8], 1901): Given a subset of the plane having diameter (maximum distance between two points of the set) equal to 1, the minimum radius r of a circle inside which it can be enclosed is $1/\sqrt{3}$.
3. (H. Steinhauss [16], 1965): For every $n \geq 1$, there exists a circle containing exactly n lattice points in its interior.

(B) Noneuclidean case:

1. (C. M. Petty [15], 1971): With the plane being equipped with the 'max-norm' $\|\square\|_\infty$(denoted ℓ^2_∞), there are exactly four equidistant points. Further, that is the only (other) possibility (in the plane).
2. (Jung [9], 1901): For ℓ^2_∞, the value of r in $A(2)$ above is equal to 1.

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3. (Exercise): For each $n \geq 1$, there exists a square containing exactly n lattice points in its interior. (In other words, the stated property holds in ℓ_2^∞).

(C) **Notation**

(a) For $1 \leq p \leq \infty$, we denote the $(\mathbb{R}^n, \|\square\|_p)$ by ℓ_p^n where

$$\|(x_i)\|_p = \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}, \qquad (x_i) \in \mathbb{R}^n.$$

(b) For a Banach space X, $e(X)$ = largest cardinality of a set in X of equidistant points.

(c) *Jung's constant* $J(X)$ of a Banach space X is defined to be the smallest positive number r such that each set of diameter 1 in X is contained in a ball of radius $r/2$.
Equivalently,

$$J(X) = \sup\left\{\inf_{x\in X}\ \sup_{y\in A} \|x-y\|\,;\, A \subset X, d(A) \leq 1\right\}.$$

$J(X)$ may also be defined as the infimum of all $c > 0$ such that any family of mutually intersecting balls of unit radius in X intersect if the radius of each ball is (dilated by) c.

(d) The property (S) shall denote the statement of 1.A.3 above.

With this notation, the above results can be restated as:

a. $e(\ell_2^2) = 3$, $e(\ell_\infty^2) = 4$.

b. $J(\ell_2^2) = \frac{2}{\sqrt{3}}$, $J(\ell_\infty^2) = 1$. (In other words, ℓ_∞^2 has the binary intersection property).

c. The spaces ℓ_2^2 and ℓ_∞^2, both have property (S).

2 Higher Dimensional Analogue

Equilateral Sets: Perhaps the most important open problem involving the size of an equilateral set in an n-dimensional normed is the following:
Conjecture: For each n-dimensional Banach space X, $e(X) \geq n+1$.

We summarize below some elementary facts regarding the above problem.

Theorem 2.1. *(i) (Petty [15],1971): $e(X) \leq 2^n$ where X is an n-dimensional normed space.*

(ii) $e(\ell_\infty^n) \geq 2^n$. In particular, $e(\ell_\infty^n) = 2^n$.

(iii) $e(\ell_1^n) \geq 2n$.

(iv) $e(\ell_p^n) \geq n+1$, $1 \leq p < \infty$.

(v) $e(\ell_2^n) = n + 1$.

Proof. (i) Let $m = e(X)$. Let $e_1, e_2, \dots, e_m$ be an equilateral set in X and let $P = \mathrm{conv}\{e_1, e_2, \dots, e_m\}$ be the convex hull of the points $e_1, e_2, \dots, e_m$. Assuming that P is full dimensional, we set $P_i = \frac{1}{2}(P + e_i)$ for $i = 1, 2, ..., m$. We note the following:

(a) $P_i \subsetneq P$, $i = 1, 2, \dots, m$.

(b) $P_i \subset B(e_i, 1/2)$, $i = 1, 2, \dots, m$.

(c) $P_i \cap P_j \neq \phi, i \neq j, i = 1, 2, \dots, m$.

Using the inclusion $\bigcup_{i=1}^m P_i \subset P$ and the translation invariance of the Lebesgue measure λ, we get

$$\sum_{i=1}^m \lambda(P_i) = \lambda\left(\bigcup_{i=1}^m P_i\right) \leq \lambda(P).$$

Finally since $\lambda(P_i) = 1/2^n\ \lambda(P)$, we obtain $m \leq 2^n$. The case of P not being full dimensional follows by using induction on n.
(ii) Consider the set $\{(\epsilon_i); \epsilon_i = \pm 1,\ i = 1, 2, \dots, n\}$.
(iii) Here the set $\{\pm e_i; i = 1, 2, \dots, n\}$ is 2-equilateral.
(iv) We choose $c \in \mathbb{R}$ in such a way that the set

$$\left\{ e_1, e_2, \dots, e_n, c\sum_{i=1}^n e_i \right\}$$

is $2^{1/p}$–equilateral. Indeed, for $j = 1, 2, \dots, n$, we note that the equation $\left\| c\sum_{i=1}^n e_i - e_j \right\| = 2^{1/p}$ yields that

$$(n-1)c^p + (c-1)^p - 2 = 0$$

which is satisfied for some real value of c.
(v) In view of (iv), it suffices to show $e(\ell_2^n) \leq n + 1$. Let S be an equilateral set in ℓ_2^n. For each $s \in S$, define

$$P_s(x) = 1 - \|x - s\|^2,\ \ x = (x_1, x_2, \dots, x_n) \in \ell_2^n,$$

which is a polynomial in n variables. It is easily checked that the set $\{1\} \cup \{P_s; s \in S\}$ is linearly independent in the (real) vector space $R[x_1, x_2, \dots, x_n]$. Further, simplifying the expression for P_s, it follows that

$$P_s \in \mathrm{span}\left[1, \sum_{i=1}^n x_i^2, x_1, x_2, \dots, x_n\right],\ \ s \in S.$$

This gives: $1 + |S| \leq n + 2$ and so $|S| \leq n + 1$. This completes the proof.
Remarks:

(a) By (ii), the upper bound of 2^n in (i) is attained for $X = \ell_\infty^n$. It turns out that the equality $e(\ell_\infty^n) = 2^n$ actually characterizes ℓ_∞^n isometrically among n-dimensional normed spaces.

(b) It is still unknown if the inequality in (iii) could be strengthened to an equality: $\ell_1^n = 2n$ for $n > 4$. Whereas the equality clearly holds for $n = 2$, the cases $n = 3$ and $n = 4$ were settled by Bandelt et al [1] and Koolen et al [12] in 1998 and 2000, respectively. However, in view of the little progress having been made on this problem, it is desirable to reformulate the question and ask if the modified question may yield an answer.

We shall use the symbol $h(\ell_1^n)$ to denote the maximum size of a set S in ℓ_1^n which is equilateral such that the sum $\sum_{i=1}^n x_i$ remains the same for each $x = (x_1, x_2, \dots, x_n) \in S$. In other words, we are looking for equilateral sets in ℓ_1^n which are located in a hyperplane. We shall express $h(\ell_1^n)$ in terms of a quantity that arises in combinatorial geometry. Clearly, $h(\ell_1^n) \le e(\ell_1^n)$. In the opposite direction, we have:

Theorem 2.2. *For each $n \ge 1$, we have $e(\ell_1^n) \le h(\ell_1^{2n-1}) + 1$.*

To get an exact formula for $h(\ell_1^n)$, we recall the following theorem of R.A. Fisher [5].

Theorem 2.3. *Let $F_1, F_2, \dots, F_m$ be m subsets of $\{1, 2 \dots, n\}$ such that there exist integers $0 \le p \le q$ with $|F_i| = q$ for all i and $|F_i \cap F_j| = p$ for all $i \ne j$.. Then $m \le n$.*

The appropriate quantity we are looking for comes from a generalization of Fisher's theorem that we now describe. Here we deal with what are called multi-sets where elements may be repeated. More precisely, a multi-set on $\{1, 2, \dots, n\}$ is a function $f : \{1, 2, \dots, n\} \to \mathbb{N} \cup \{0\}$ where $f(n)$ is the number of times n is repeated in the multi-set. Thus a multi-set F may be represented as $F = \left\{1^{f(1)}, 2^{f(2)}, \dots, n^{f(n)}\right\}$ with i being repeated $f(i)$ times. Obviously, the number of elements of F is given by $|F| = \sum_i^n f(i)$ and the intersection of F and $G = \left\{1^{g(1)}, 2^{g(2)}, \dots, n^{g(n)}\right\}$ is defined by: $F \cap G = \left\{1^{h(1)}, 2^{h(2)}, \dots, n^{h(n)}\right\}$ where $h(i) = \min\{f(i), g(i)\}$ We pose the following question.

Question 2.1. *Let$F_1, F_2, \dots, F_m$ be m multi-sets on $\{1, 2, \dots, n\}$ such that there exist integers $0 \le p \le q$ with $|F_i| = q$ for all i and $|F_i \cap F_j| = p$ for all $i \ne j$. Denote $\varphi(n) = \max\ m$ where m is the integer with the above properties. Estimate $\varphi(n)$ in terms of appropriate parameters.*

The case of the functions representing the multi-sets being $\{0, 1\}$-valued reduces to Fisher's theorem in which case we have $\varphi(n) \le n$. However, $\varphi(n)$ may be much larger than n, in general.

The following theorem provides the promised relationship:

Theorem 2.4. *([13]): For all $n \ge 1$, we have: $\varphi(n) = h(\ell_1^n)$.*

The assertions in (iv) and (v) provide special, but classical situations witnessing the validity of the conjecture. These include the Hilbert space case where the conjecture holds as an equality. In the general case when the given space is 'very close' to being a Hilbert space in the sense of the Banach Mazur distance, the equality still holds as the following theorems show:

Theorem 2.5. *(P. Brass [3], 1999): $e(X) \le n + 1$, for X with $\dim X = n$ and $d(X, \ell_2^n) \le 1 + \frac{1}{n+2}$.*

Theorem 2.6. *(P. Brass [3], 1999): $e(X) \geq n+1$, for X with* $\dim X = n$ *and* $d(X, \ell_2^n) \leq 1 + \frac{1}{n+1}$.

Here the Banach-Mazur distance between Banach spaces X and Y is meant in the following sense:

$$d(X,Y) = \inf\left\{ \|T\|\|T^{-1}\|; T : X \to Y \text{is an isomorphism} \right\}.$$

In particular, $e(X) = n+1$, for X with $\dim X = n$ and $d(X, \ell_2^n) \leq 1 + \frac{1}{n+2}$.

We also note that the conjecture has been settled for $n \leq 3$ (Petty [15]). Also, Swanepoel ([21], 2004) verified the conjecture for ℓ_4^n, whereas Swanepoel and Villa [21] showed the validity of the conjecture for all n-dimensional normed spaces X such that $d(X, \ell_2^n) \leq 3/2$. They also show that there exists an absolute constant $c > 0$ such that $e(X) \geq e^{c\sqrt{\log n}}$ for each $n \geq 1$.

Recently, T. Kobos ([11], 2013) has shown that the conjecture holds for

(i) Permutation invariant norms on $\mathbb{R}^n$.

(ii) Orlicz-Musielak spaces: $\mathbb{R}^n$ equipped with the Luxemberg norm (obtained by the Minkowski functional on the set K given by:

$$\left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n; \sum_{i=1}^{n} |f_i(x_i)| \leq 1 \right\}.$$

2.7. *Jung's constant*

(i) $1 \leq J(X) \leq 2$, *for each Banach space X (Exercise !).*

(ii) (Bohnenblust [2], 1938): $J(X) \leq \frac{2n}{n+1}$, *if* $\dim X = n$.

(iii) (Jung [9], 1901): $J(\ell_\infty^n) = 1$, $n \geq 1$ *(and conversely).*

(iv) (jung [9], 1901): $J(\ell_2^n) = \left(\frac{2n}{n+1}\right)^{1/2}$.

Let us see how (iv) may be derived in the special case of $n = 2$. We begin by observing that in the case of the set S of diameter 1 consisting of an equilateral triangle having each side equal to 1, the circle having the smallest radius and containing S is the circumcircle of the triangle (having circumradius equal to $1/\sqrt{3}$). In the general case of S being an arbitrary set of diameter ≤ 1, we proceed as follows. Draw a circle of radius $1/\sqrt{3}$ about each point of S. For the moment, let us claim that all these circles have a point in common, say O. Then it is easy to see that the circle C centred at O and having radius equal to $1/\sqrt{3}$ contains all the points of S. For otherwise, there exists a point P in S which lies outside C, in which case the length of the segment OP exceeds $1/\sqrt{3}$. In other words, O lies outside the circle centred at P and having radius equal to $1/\sqrt{3}$, contradicting that O is common to all the circles of the collection having centres at points of S.

To justify the above claim, we use the 2-dimensional version of Helly's intersection theorem:

(*) Given a family of compact convex subsets of the plane having the property that each subfamily of three sets from the given family have a non-empty intersection, then the given family has a non-empty intersection.

The desired family consists of all circles of radius $1/\sqrt{3}$ drawn around each point of S as its centre. In view of Helly's theorem (*), it suffices to show that any family of three circles in the family has a point of intersection. Let A, B, C in S be the centres of these three circles. In the triangle ABC, let angle $C \geq \pi/3$. It is easily checked that the circumradius of the triangle ABC is given by: $R = AB/2\sin C$. Since S has diameter ≤ 1, we have $AB \leq 1$ and this gives:

$$R \leq \frac{1}{2\sin C}.$$

Case 1: $C \leq \pi/2$. Thus $\sin C \geq \sin \pi/3 = \sqrt{3}/2$, and it follows that

$$R \leq \frac{1}{2}.\frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

In other words, the circumcircle O lies at a distance of at most $1/\sqrt{3}$ from each of the points A, B and C and so is common to each of these circles.
Case 2: $C > \pi/2$. In this case, the circle with AB as its diameter contains C in its interior. Letting O denote the midpoint of AB, we see that $OC < OA = OB \leq 1/2 < 1/\sqrt{3}$. This means that O is common to all the circles around A, B and C.

2.8. *Steinhauss Property*

(i) For each n ≥ 1, *the (3-dimensional) sphere contains exactly n lattice points.*

(ii) More generally, for each n ≥ 1, *the space* ℓ_2^n *has property (S).*

(iii) (Analogous property): For each n ≥ 1, *the circle in the plane as well as the 3-dimensional sphere each contains exactly n lattic points on its boundary.*

We provide an easy argument due to H. Steinhauss involving a proof of (ii) in the special case corresponding to $n = 2$. The simple but clever idea is the observation that no two lattice points are the same distance from the distinguished point $P(\sqrt{2}, 1/3)$. Once this is verified, the lattice points can be arranged in a sequence $\{p_1, p_2, \ldots, p_n, \ldots\}$ according to their distance from P. Finally, for each m, the circle with centre at P and radius p_{m+1} contains exactly m lattice points in its interior.
Remarks: The same method works for the sphere in the 3-dimensional space ℓ_2^3 where the corresponding point P may be chosen to be $(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Also the same result remains valid in ℓ_∞^2 (the construction of a square having exactly m lattice points inside it), but the proof is different and slightly more involved.

The analogous problem involving the existence of a circle having exactly m lattice points on its boundary also has an affirmative answer. The same holds for a square in place of a circle and also for the sphere in dimension 3.

3 Infinite dimensional case

The following theorems provide infinite dimensional analogues of some of the results discussed earlier in the finite dimensional setting.

a. Equilateral Sets

(i) Each infinite dimensional Banach space contains equilateral sets of size n for each $n \geq 1$ (This follows by combining the comment following Theorem 2.6 with Dvoretzky's spherical sections theorem which says that every infinite dimensional Banach space X contains for each $\epsilon > 0$ and each $n \geq 1$, an n dimensional subspace Y such that $d(X, \ell_2^n) < 1 + \epsilon$).

(ii) (Mercoulakis and Vassiliadis [14], 2016): Each infinite dimensional Banach space containing an isomorphic copy of c_0 admits an infinite equilateral set. (This is a consequence of the so-called 'non-distorsion property' of c_0 which asserts that under any equivalent norm $\|\square\|$ on c_0, there exists a subspace Y of c_0 isomorphic to c_0 such that $d((Y, \|\square\|), (c_0, \|\square\|_\infty)) < 1 + \epsilon$).

(iii) (Freeman, Odell, Sari and Schlumprecht [6], 2016): Each infinite dimensional uniformly smooth Banach space admits an infinite equilateral set.

(iv) (Swanepoel [21], 1997, Mercoulakis and Vassiliadis [14], 2014): Each infinite dimensional Banach space can be renormed to admit an infinite equilateral set. Further, the new norm can be chosen to be 'sufficiently/arbitrarily close' to the original norm.

(v) (Terenzi [22], 1989): There exists an infinite dimensional Banach spaces admitting no infinite equilateral set. (The Banach space with this property is actually isomorphic to ℓ_1).

(vi) (Glakousakis and Mercourakis [7], 2016): There exist infinite dimensional spaces with the property as in (v) above but not isomorphic to a subspace of ℓ_1.

b. Jung's Constant

(i) **Theorem 3.1**(Routledge [16], 1952): $J(\ell_2) = \sqrt{2}$. (A simpler proof of this equality was obtained by Semrl [17]).

(ii) **Theorem 3.2**(Davis [4], 1977): $J(X) = 1$ if and only if X is injective.

c. Steinhauss Property

Definition 3.3: A subset A of a metric space X is said to be *quasi finite* if A is countable and each ball in X intersects A in a finite set.

Theorem 3.4 (P. Zwolenski [23], 2013): Given a Hilbert space X and a quasi finite subset A of X, then the set of points y in X such that for each $n \geq 1$, there exists a ball centred at y containing exactly n points of A is (nonempty and) dense in X.

Theorem 3.5 (T. Kania and T. Kochanek [10], 2014): The above property holds in each strictly convex Banach space. Moreover, converse is not true.

4 Open Problems

: We list below some problems, including those which have remained open over a long period of time and those which are interesting in their own right and which arise naturally in this circle of ideas.

a. Finite dimensional setting

1. Conjecture: For each n-dimensional Banach space X, $e(X) \geq n+1$.
2. Conjecture: For each $n \geq 1$, $e(\ell_1^n) = 2n$.
3. Compute the (exact) value/asymptotic growth of $J(\ell_p^n)$, $p \neq 1, 2, \infty$
4. Banach spaces X verifying $J(X) = \lambda_1(X)$.
5. Is it true that ℓ_∞^n has property (S) for $n \geq 3$?
6. Does there exist a ball in ℓ_∞^n $(n > 2)$ which contains m lattice points on its boundary for each $m \geq 2$?

A word clarifying the statement in (4) above is in order. It turns that $J(X) \leq \lambda_1(X)$ where the latter quantity is defined to be the infimum of all $c > 0$ such that for any Banach space Z containing X as a 1-codimensional subspace, there exists a projection P of Z onto X with $\|P\| \leq c$. Further, the class of Banach spaces verifying (4) include all injective spaces.

b. Infinite dimensional setting:

1. Characterise Banach spaces admitting an infinite equilateral set.
2. Given $c \in [1, 2]$, characterise Banach spaces X such that $J(X) = c$.
3. (Conjecture [8]): $J(X) = H(X)$ for each Banach space X. ($J(X) \geq H(X)$, Semrl [17]. The conjecture has been verified in certain special cases including Hilbert spaces, injective Banach spaces and the space c_0 and certain finite dimensional spaces.
4. Characterise Banach spaces verifying property (S) (for quasi finite sets).

Some comments regarding the constant $H(X)$ appearing in (3) above are in order. An old result of Mazur-Ulam states that a surjective isometry $f : X \to Y$ between (real) normed spaces X and Y such that $f(0) = 0$ is always linear. A question that typically arises in the stability theory of functional equations deals with the possibility of approximating 'approximate' objects in a particular setting by 'proper' objects within that setting. In the case of stability issues involving isometries between Banach spaces, the appropriate question asks whether these exists a constant $c > 0$ such that for each Banach space X, each $\epsilon > 0$ and each subjective ϵ-isometry $f : X \to Y$, there exists a (proper) surjective isometry $g : X \to Y$ such that

$$\|f(x) - g(x)\| \leq c\epsilon, \qquad \text{for each} \quad x \in X.$$

The Hyesr-Ulam constant $H(X)$ appearing above is defined to be the infimum of all c's appearing in the above inequality. Here f being an ϵ-isometry is meant in the sense that

$$|\|f(x) - g(x)\| - \|x - y\|| \leq \epsilon, \qquad \text{for each} \quad x, y \in X$$

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