# Spectrum and fine spectrum of the generelized difference operators acting on some Banach sequence spaces - A Review

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## 1 Introduction

Spectral theory is one of the main branch of modern functional analysis and its applications. Roughly speaking, it is concerned with certain inverse operators, their general properties and their relations to the original operators. Such inverse operators arise quite naturally in connection with the problem of solving equations e.g. system of linear algebraic equations, differential equations, integral equations etc.

Spectral theory of operators in finite dimensional vector space is essentially a matrix eigenvalue theory and is much simpler than that of operators in infinite dimensional space. In fact, the spectrum of a bounded linear operator on a finite dimensional space is a pure point spectrum i.e. every spectral value is an eigenvalue while in case of infinite dimensional space, a bounded linear operator can have spectral values which are not eigenvalues e.g. the operator $T : l_2 \to l_2$ defined by

$$T(x_1, x_2, \cdots, x_n, \cdots) = (0, x_1, x_2, \cdots)$$

has zero as spectral value but it is not an eigenvalue. So one can say that the spectrum of a bounded linear operator is a generalisation of the notion of eigenvalue for matrices. The spectrum of an operator over a Banach space is partitioned into three parts i.e. the point spectrum, the continuous spectrum and the residual spectrum. Calculation of these parts of the spectrum of an operator is called determination of the fine spectrum.

In the present review, we will have a short survey regarding the spectrum and the fine spectrum of a bounded linear operators defined in terms of lower and upper triangle matrices over certain sequence spaces.

## 2 Basic definitions and known results

Let $X$ and $Y$ be two Banach spaces and $T : X \to Y$ be a bounded linear operator. We denote the range of $T$ as $R(T)$, that is, $R(T) = \{y \in Y : y = Tx, x \in X\}$. The set of all bounded linear operators on $X$ into itself is denoted by $B(X)$. Let $T \in B(X)$, then the adjoint $T^*$ of $T$ is a bounded linear operator on the dual space $X^*$ of $X$ defined by

$$(T^*\phi)(x) = \phi(Tx) \text{ for all } \phi \in X^* \text{ and } x \in X.$$

Let $X \neq \{0\}$ be a complex normed linear space and $T : D(T) \to X$ be a bounded linear operator with domain $D(T) \subseteq X$. With $T$, we associate the operator $T_\alpha = (T - \alpha I)$, where $\alpha$ is a complex number and $I$ is the identity operator on $D(T)$. The inverse of $T_\alpha$ (if exists) is denoted by $T_\alpha^{-1}$, where $T_\alpha^{-1} = (T - \alpha I)^{-1}$ and is known as the resolvent operator of $T$. Many properties of $T_\alpha$ and $T_\alpha^{-1}$ depend on $\alpha$ and spectral theory is concerned with those properties.

**Definition 2.1.** *Let $X \neq \{0\}$ be a complex normed linear space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A regular value of $T$ is a complex number $\alpha$ such that the following conditions R1-R3 hold* :
*(R1) $T_\alpha^{-1}$ exists*
*(R2) $T_\alpha^{-1}$ is bounded*
*(R3) $T_\alpha^{-1}$ is defined on a set which is dense in $X$.*

Resolvent set $\rho(T, X)$ of $T$ is the set of all regular values $\alpha$ of $T$. Its complement in the complex plane $\mathbb{C}$, i.e., $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ is called the spectrum of $T$. The spectrum $\sigma(T, X)$ is further partitioned into three disjoint sets, namely, point spectrum, continuous spectrum and residual spectrum which are defined as follows:

The point spectrum $\sigma_p(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that $T_\alpha^{-1}$ does not exist, i.e., the condition (R1) fails. The elements of $\sigma_p(T, X)$ are called eigenvalues of $T$.

The continuous spectrum $\sigma_c(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that the conditions (R1) and (R3) hold but the condition (R2) does not hold, i.e., $T_\alpha^{-1}$ exists, domain of $T_\alpha^{-1}$ is dense in $X$ but $T_\alpha^{-1}$ is unbounded.

The residual spectrum $\sigma_r(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that $T_\alpha^{-1}$ exists but does not satisfy the condition (R3), i.e., domain of $T_\alpha^{-1}$ is not dense in $X$. In this case, the condition (R2) may or may not hold good.

Let $T \in B(X)$, where $X$ is a Banach space, then there are three possibilities for $T_\alpha^{-1} = (T - \alpha I)^{-1}$ and three possibilities for $R(T - \alpha I)$. These possibilities are combined by Goldberg [10] and hence he has given nine subdivision of the spectrum of a bounded linear operator as shown in the following table.

Appell et al. [8] have defined three more subdivisions of the spectrum namely the approximate point spectrum $\sigma_{ap}(T, X)$, the defect spectrum $\sigma_\delta(T, X)$ and the compression spectrum $\sigma_{co}(T, X)$. Several authors have studied the spectrum and fine spectrum of linear operators defined by some particular matrices over some known sequence spaces. R. B. Wenger [28] has examined the fine spectrum of the integral power of the Cesaro operator over $c$ while B. E. Rhodes [24] has generelized this result to the weighted mean methods. J. B. Reade [23] has examined the spectrum of the Cesaro operator over the sequence space $c_0$. M. Gonzales [16] has studied the fine spectrum of the Cesaro operator of order 1 over the sequence space $l_p$. Weighted mean matrices of the operator on $l_p$ has been investigated by Cartlidge. J. T. Okutoyi [22], M. Yildirim [29] have examined the fine spectrum of the Rhally operators over the sequence space $c_0$ and $c$. The fine spectrum of the difference operator $\Delta$ over the sequence space $c_0$ and $c$ has been studied by Altay and Başar [6].

| | | 1<br>$T_\lambda^{-1}$ exists and bounded | 2<br>$T_\lambda^{-1}$ exists and unbounded | 3<br>$T_\lambda^{-1}$ does not exist |
|---|---|---|---|---|
| A | $R(T_\lambda) = X$ | $\lambda \in \rho(T, X)$ | — | $\lambda \in \sigma_p(T, X)$<br>$\lambda \in \sigma_{ap}(T, X)$ |
| B | $\overline{R(T_\lambda)} = X$ | $\lambda \in \rho(T, X)$ | $\lambda \in \sigma_c(T, X)$<br>$\lambda \in \sigma_\delta(T, X)$<br>$\lambda \in \sigma_{ap}(T, X)$ | $\lambda \in \sigma_p(T, X)$<br>$\lambda \in \sigma_\delta(T, X)$<br>$\lambda \in \sigma_{ap}(T, X)$ |
| C | $\overline{R(T_\lambda)} \neq X$ | $\lambda \in \sigma_r(T, X)$<br>$\lambda \in \sigma_\delta(T, X)$<br>$\lambda \in \sigma_{co}(T, X)$ | $\lambda \in \sigma_r(T, X)$<br>$\lambda \in \sigma_\delta(T, X)$<br>$\lambda \in \sigma_{co}(T, X)$<br>$\lambda \in \sigma_{ap}(T, X)$ | $\lambda \in \sigma_p(T, X)$<br>$\lambda \in \sigma_\delta(T, X)$<br>$\lambda \in \sigma_{co}(T, X)$<br>$\lambda \in \sigma_{ap}(T, X)$ |

Tab. 1: Subdivisions of spectrum of a linear operator

The fine spectra of the difference operator $\Delta$ over the sequence spaces $l_p$ and $bv_p$ is studied by Akhmedov and Başar [4], where $bv_p$ is the $p$-bounded variation sequences introduced by Başar and Atlay [9] with $1 \leq p < \infty$. Also the fine spectrum of $B(r, s, t)$ over the sequence spaces $c_o$ and $c$ has been studied by Furkan et al. [13]. In 2010 Srivastava and Kumar [26] have determined the spectra and fine spectra of the generalized difference operator $\Delta_v$ on $l_1$ where $\Delta_v$ is defined by $\left(\Delta_v\right)_{nn} = v_n$ and $\left(\Delta_v\right)_{n+1,n} = -v_n$ for all $n \in \mathbb{N}$ under suitable conditions on the sequence $v = (v_n)$. These author have further generalized the results by using difference operator $\Delta_{uv}$ defined as $\Delta_{uv}x = \left(u_n x_n + v_{n-1}x_{n-1}\right)_{n\in\mathbb{N}}$ for all $n \in \mathbb{N}$ [25]. Karakaya and Altun [21] have determined the fine spectrum of the upper triangular double band matrices over the sequence spaces $c_0$ and $c$. Akhmedov and El-Shabraway [4] have obtained the fine spectrum of the generalized difference operator $\Delta_{a,b}$ defined as a double matrix with the convergent sequences $\tilde{a} = (a_k)$ and $\tilde{b} = (b_k)$ having certain properties over the sequence spaces $c$.

Recently Karaisa [17, 18] and Karaisa & Başar [19] have determined the fine spectra of the matrix operators in the form of upper and lower triangular matrices $A(\tilde{r}, \tilde{s})$ and $B(\tilde{r}, \tilde{s})$ with the convergent sequences $\tilde{r} = (r_k)$ and $\tilde{s} = (s_k)$ in their band with certain properties, over the sequences space $l_p$ for $1 \leq p < \infty$ and $1 < p < \infty$ respectively. Later Karaisa and Başar [19] have determined the fine spectrum of the upper triangular triple band matrix $A(r, s, t)$ over some sequence spaces. Ali Karaisa et al. [20] have determined the fine spectra of the operator for which the corresponding upper and lower triangular matrices $A(r, s, t)$ and $B(r, s, t)$ are defined on the sequence spaces $c$ and $l_p$ where $0 < p < 1$ respectively.

## 3 Some Important Results on the fine spectrum

Akhmedov and Başar [3] have exaimned the fine spectrum of the Cesaro operator in the sequence spaces $c_0$ and $c$ where the Cesaro operator is denoted by $C_1$ and is represented by the matrix

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \frac{1}{n+1} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \end{pmatrix}$$

**Theorem 3.1.** *[3] For the Cesaro operator $C_1 : c_0 \to c_0$, the following statements hold:*

*(i)* $\sigma(C_1, c_0) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$

*(ii)* $\sigma_p(C_1, c_0) = \emptyset$

*(iii)* $\sigma_c(C_1, c_0) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| = \frac{1}{2},\ \alpha \neq 1\right\}$

*(iv)* $\sigma_r(C_1, c_0) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right\} \cup \{1\}$

**Theorem 3.2.** *[3] For the Cesaro operator $C_1 : c \to c$, the following statements hold:*

*(i)* $\sigma(C_1, c) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$

*(ii)* $\sigma_p(C_1, c) = \{1\}$

*(iii)* $\sigma_c(C_1, c) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| = \frac{1}{2}\right\}$

*(iv)* $\sigma_r(C_1, c) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right\}$

Akhmedov and Başar [2, 1] have derived the fine spectrum of the difference operator $\Delta$ on the sequence space $l_p$ and $bv_p$ respectively where $1 \leq p < \infty$. The difference operator $\Delta$ is represented by the matrix

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

**Theorem 3.3.** *[2] The spectrum and fine spectrum of $\Delta$ on $l_p$ are given by*

*(i)* $\sigma(\Delta, l_p) = \{\alpha \in \mathbb{C} : |\alpha - 1| \leq 1\}$

*(ii)* $\sigma_p(\Delta, l_p) = \emptyset$

*(iii)* $\sigma_c(\Delta, l_p) = \begin{cases} \{\alpha \in \mathbb{C} : |\alpha - 1| = 1\}, & 1 < p < \infty \\ \emptyset, & p = 1 \end{cases}$

*(iv)* $\sigma_r(\Delta, l_p) = \begin{cases} \{\alpha \in \mathbb{C} : |\alpha - 1| < 1\}, & 1 < p < \infty \\ \{\alpha \in \mathbb{C} : |\alpha - 1| \leq 1\}, & p = 1 \end{cases}$

**Theorem 3.4.** *[1] The spectrum and fine spectrum of* $\Delta$ *on* $bv_p$ *are given by*

*(i)* $\sigma(\Delta, bv_p) = \{\alpha \in \mathbb{C} : |\alpha - 1| \leq 1\}$

*(ii)* $\sigma_p(\Delta, bv_p) = \emptyset$

*(iii)* $\sigma_c(\Delta, bv_p) = \{\alpha \in \mathbb{C} : |\alpha - 1| = 1\}$

*(iv)* $\sigma_r(\Delta, bv_p) = \{\alpha \in \mathbb{C} : |\alpha - 1| < 1\}$

The spectrum and fine spectrum of the generalized differnece operator $B(r, s)$ have been obtained by Altay & Başar [7], Bilgiç & Furkan [12] and Furkan et al. [15] over the sequence spaces $c_0$, $c$ and $l_p$, $bv_p(1 < p < \infty)$ and $l_1$, $bv$ respectively. The operator $B(r, s)$ is represented by the following lower triangular double band matrix

$$B(r,s) = \begin{pmatrix} r & 0 & 0 & 0 & \cdots \\ s & r & 0 & 0 & \cdots \\ 0 & s & r & 0 & \cdots \\ 0 & 0 & s & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

**Theorem 3.5.** *[7, 12, 15] The spectrum and fine spectrum of* $B(r, s)$ *over the sequence spaces* $c_0$*,* $c$*,* $l_p(1 < p < \infty)$*,* $bv_p(1 < p < \infty)$*,* $l_1$*,* $bv$ *are given by*

*(i)* $\sigma(B(r, s), X) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}, \ X = c_0, c, l_1, l_p, bv, bv_p.$

*(ii)* $\sigma_p(B(r, s), X) = \emptyset, \ X = c_0, c, l_1, l_p, bv, bv_p.$

*(iii)* $\sigma_c(B(r, s), X) = \begin{cases} \emptyset, & X = l_1, bv \\ \{\alpha \in \mathbb{C} : |\alpha - r| = |s|\}, & X = c_0, l_p, bv_p \\ \{\alpha \in \mathbb{C} : |\alpha - r| = |s|\} \setminus \{r + s\}, & X = c. \end{cases}$

*(iv)* $\sigma_r(B(r, s), X) = \begin{cases} \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}, & X = l_1, bv \\ \{\alpha \in \mathbb{C} : |\alpha - r| < |s|\}, & X = c_0, l_p, bv_p \\ \{\alpha \in \mathbb{C} : |\alpha - r| < |s|\} \cup \{r + s\}, & X = c. \end{cases}$

The spectrum and fine spectrum of the generalized differnece operator $B(r, s, t)$ have been obtained by Furkan et al. [13], Furkan et al. [14] and Bilgiç & Furkan [11] over the sequence spaces $c_0$, $c$ and $l_p$, $bv_p(1 < p < \infty)$ and $l_1$, $bv$ respectively. The operator $B(r, s, t)$ is represented by the following lower triangular triple band matrix

$$B(r,s,t) = \begin{pmatrix} r & 0 & 0 & 0 & \cdots \\ s & r & 0 & 0 & \cdots \\ t & s & r & 0 & \cdots \\ 0 & t & s & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

**Theorem 3.6.** *[11, 13, 14] Let $s$ be a complex number such that $\sqrt{s^2} = -s$. Then the spectrum and fine spectrum of $B(r,s,t)$ over the sequence spaces $c_0$, $c$, $l_p(1 < p < \infty)$, $bv_p(1 < p < \infty)$, $l_1$, $bv$ are given by*

*(i)* $\sigma(B(r,s,t),X) = \left\{\alpha \in \mathbb{C} : 2|\alpha - r| \leq \left|-s + \sqrt{s^2 - 4t(r-\alpha)}\right|\right\}, X = c_0, c, l_1, l_p, bv, bv_p.$

*(ii)* $\sigma_p(B(r,s,t),X) = \emptyset$, $X = c_0, c, l_1, l_p, bv, bv_p$.

*(iii)* $$\sigma_c(B(r,s,t),X) = \begin{cases} \emptyset, & X = l_1, bv \\ \{\alpha \in \mathbb{C} : 2|\alpha - r| = |-s + \sqrt{s^2 - 4t(r-\alpha)}|\}, & X = c_0, l_p, bv_p \\ \{\alpha \in \mathbb{C} : 2|\alpha - r| = |-s + \sqrt{s^2 - 4t(r-\alpha)}|\} \setminus \{r+s+t\}, & X = c. \end{cases}$$

*(iv)* $$\sigma_r(B(r,s,t),X) = \begin{cases} \{\alpha \in \mathbb{C} : 2|\alpha - r| \leq |-s + \sqrt{s^2 - 4t(r-\alpha)}|\}, & X = l_1, bv \\ \{\alpha \in \mathbb{C} : 2|\alpha - r| < |-s + \sqrt{s^2 - 4t(r-\alpha)}|\}, & X = c_0, l_p, bv_p \\ \{\alpha \in \mathbb{C} : 2|\alpha - r| < |-s + \sqrt{s^2 - 4t(r-\alpha)}|\} \cup \{r+s+t\}, & X = c. \end{cases}$$

In 2017, Birbonshi and Srivastava [27] have obtained various spectral properties of $n$ band triangular matrices of constant bands defined over some known sequence spaces. The operator $T_1$ is considered in the form of

$$T_1 = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ a_2 & a_1 & 0 & \cdots & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & 0 & \cdots \\ 0 & a_n & a_{n-1} & \cdots & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Similar as lower triangular matrix they have also considered the operator $\tilde{T}$ represented by upper triangular matrix which is of the form

$$\tilde{T_1} = {T_1}^t.$$

**Theorem 3.7.** *[27] Let $P(z) = a_1 + a_2 z + \cdots + a_n z^{n-1}$ be a polynomial over $\mathbb{C}$ and $D$ be the closed unit disk in the complex plane $\mathbb{C}$. Then the following statements hold:*

*(i) Let $\mu$ be any one of the sequence spaces $c_0$, $c$, $l_\infty$, $l_1$, $l_p(1 < p < \infty)$, $bv_p(1 < p < \infty)$ and $bv$. Then the spectrum of the operator $T_1$ over $\mu$ is $P(D)$,*

(ii) *Let $\mu$ be any one of the sequence spaces $c_0$, $c$, $l_\infty$, $l_1$, $l_p(1 < p < \infty)$, $bv_p(1 < p < \infty)$ and $bv$. Then the point spectrum of the operator $T_1$ over $\mu$ is $\emptyset$,*

(iii) *Let $\mu$ be any one of the sequence spaces $c_0$, $l_p(1 < p < \infty)$ and $bv_p(1 < p < \infty)$. Then the residual spectrum of the operator $T_1$ over $\mu$ is $[P(D)]^\circ \setminus A$,*

(iv) *Let $\mu$ be any one of the sequence spaces $l_1$, $bv$. Then the residual spectrum of $T$ over $\mu$ is $P(D)$. Again, if $\mu = c$, then the residual spectrum of $T_1$ over $\mu$ is $([P(D)]^\circ \setminus A) \cup \{P(1)\}$,*

(v) *Let $\mu$ be any one of the sequence spaces $c_0$, $l_p(1 < p < \infty)$ and $bv_p(1 < p < \infty)$. Then the continuous spectrum of the operator $T_1$ over $\mu$ is $\partial P(D) \cup A$,*

(vi) *Let $\mu$ be any one of the sequence spaces $l_1$, $bv$. Then the continuous spectrum of $T_1$ over $\mu$ is $\emptyset$. Again, if $\mu = c$, then the continuous spectrum of $T_1$ over $\mu$ is $(\partial P(D) \cup A) \setminus \{P(1)\}$*

*where $[P(D)]^\circ$ denotes the interior part of $[P(D)]$ and $A$ is the set of all interior points of $P(D)$, which are the images of the boundary points of $D$ only (not the images of its interior points) and the cardinality of $A$ is finite.*

**Remark 3.1.** *It is proved in [27] that, a necessary condition for an interior point of the set $P(D)$ belonging to the set $A$ is that it is either self intersecting point of the image of the unit circle by the polynomial $P(z)$ or it is the image of those points where $P'(z) = 0$.*

**Theorem 3.8.** *[27] The following statements hold for $\tilde{T}_1$*

(i) *Let $\mu$ be any one of the sequence spaces $c_0$, $c$, $l_\infty$, $l_1$, $l_p(1 < p < \infty)$, then the spectrum of the operator $\tilde{T}_1$ over $\mu$ is $P(D)$,*

(ii) *Let $\mu$ be any one of the sequence spaces $c_0$, $l_1$ and $l_p(1 < p < \infty)$, then the point spectrum of the operator $\tilde{T}_1$ over $\mu$ is $[P(D)]^\circ \setminus A$,*

(iii) *The point spectrum of the operator $\tilde{T}_1$ over $c$ is $[P(D)]^\circ \setminus A \cup \{P(1)\}$ and over $l_\infty$ is $P(D)$.*

(iv) *Let $\mu$ be any one of the sequence spaces $c_0$, $c$, $l_\infty$, $l_1$, $l_p(1 < p < \infty)$. Then the residual spectrum of the operator $\tilde{T}_1$ over $\mu$ is $\emptyset$,*

(v) *Let $\mu$ be any one of the sequence spaces $c_0$, $l_1$ and $l_p(1 < p < \infty)$. Then the continuous spectrum of the operator $\tilde{T}_1$ over $\mu$ is $\partial P(D) \cup A$,*

(vi) *The continuous spectrum of the operator $\tilde{T}_1$ over $c$ is $(\partial P(D) \cup A) \setminus \{P(1)\}$ and over $l_\infty$ is $\emptyset$.*

Let $P(z)$ be a complex polynomials of degree $n$ i.e $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_0, a_1, \ldots, a_n$ are complex numbers . If $S$ be a bounded linear operator on a Banach space $X$ then $P(S)$ is also a bounded linear operator on $X$. Suppose $S$ is a right shift operator say $R$, then we denote the $n^{th}-$ band lower triangular matrix by $P(R)$ and if $S$ be a left shift operator say $L$, then we denote the $n^{th}-$ band upper triangular matrix by $P(L)$.

By chossing $P(z) = r + sz$, $P(R)$ gives the operator $B(r, s)$ and $P(L)$ gives the operator $U(r, s)$ and if $P(z) = r + sz + tr^2$ then $P(R)$ denotes the operator $B(r, s, t)$ and $P(L)$ denotes the operator $U(r, s, t)$, where $r, s, t$ are complex numbers. So the results of [7, 11, 12, 13, 14, 15] follows immediately by the above theory [27].

Recently Patra et al. [5] have obtained several spectral properties of difference operators represented as $n$ band lower and upper triangular matrices defined on the sequence space $l_p(1 < p < \infty)$ i.e., in the form of

$$T_2 = \begin{pmatrix} a_1^{(1)} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ a_1^{(2)} & a_2^{(1)} & 0 & \cdots & 0 & 0 & \cdots \\ a_1^{(3)} & a_2^{(2)} & a_3^{(1)} & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_1^{(n)} & a_2^{(n-1)} & a_3^{(n-2)} & \cdots & a_n^{(1)} & 0 & \cdots \\ 0 & a_2^{(n)} & a_3^{(n-1)} & \cdots & a_n^{(2)} & a_{n+1}^{(1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with

$$\lim_{m\to\infty} a_m^{(k)} = a^{(k)}, \quad \text{for } k = 1, 2, \cdots n,$$

$$a_m^{(k)} \neq 0,\ a_m^{(1)} \neq a^{(1)} \text{ for all } m \in \mathbb{N},\ k = 1, 2, \cdots, n \text{ and } a^{(n)} \neq 0$$

and $(a_m^{(1)})_{m\in\mathbb{N}}$ is a sequence of distinct complex numbers. Similarly the upper triangular operator $\tilde{T}$ is represented by $\tilde{T}_2 = {T_2}^t$ also considered.

**Theorem 3.9.** *[5] Let the sequences $(a_m^{(k)})_{m\in\mathbb{N}}$, $k = 1, 2, \cdots, n$ converges exponentially. Then the following holds.*

*(i) The spectrum of $T_2$ on $l_p$ is*

$$\sigma(T_2, l_p) = P(D) \cup B.$$

*(ii) The continuous spectrum of $T_2$ on $l_p$ is*

$$\sigma_c(T_2, l_p) = (A \cup \partial P(D)) \setminus B.$$

*(iii) The point spectrum of $T_2$ on $l_p$ is*

$$\begin{aligned} \sigma_p(T_2, l_p) \;=\; & \{a_i^{(1)} \in P(D) \cap B : (a_k^{(1)} - a_i^{(1)})x_k + a_{k-1}^{(2)} x_{k-1} + \cdots + a_{k-n+1}^{(n)} x_{k-n+1} = 0, \\ & k = i+1, i+2, \cdots \text{ has a solution in } l_p.\} \\ & \cup (B \setminus P(D)). \end{aligned}$$

*(iv) The residual spectrum of $T_2$ on $l_p$ is*

$$\sigma_r(T_2, l_p) = (([P(D)]^\circ \setminus A) \setminus B) \cup ((P(D) \cap B) \setminus \sigma_p(T_2, l_p)).$$

*Where $A$ is the set of all interior points of $P(D)$, which are the images of the boundary points of $D$ only (not the images of its interior points) and $B = \{a_i^{(1)} : i \in \mathbb{N}\}$.*

**Theorem 3.10.** *[5] The following statements hold for the operator $\tilde{T}_2$ :*

*(i) The spectrum of the operator* $\tilde{T}_2$ *on* $l_p$ *in* $P(D) \cup B$.

*(ii) The point spectrum of* $\tilde{T}_2$ *on* $l_p$ *is* $(P(D)^\circ \setminus A) \cup B$.

*(iii) The residual spectrum of* $\tilde{T}_2$ *on* $l_p$ *is* $\emptyset$.

*(iv) The continuous spectrum of* $\tilde{T}_2$ *on* $l_p$ *is* $(A \cup \partial P(D)) \setminus B$.

Let $B(1,3,3,1)$ be a bounded linear operator over the sequence space $\mu$, where $\mu = c_0, l_p, bv_p$ $(1 < p < \infty)$. Here $B(1,3,3,1)$ denote the 4-th band lower triangular matrix as below :

$$B(1,3,3,1) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 3 & 1 & 0 & 0 & \dots \\ 3 & 3 & 1 & 0 & \dots \\ 1 & 3 & 3 & 1 & \dots \\ 0 & 1 & 3 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly, $B(1,3,3,1)$ is represented by $P(R)$, where $P(z) = (z+1)^3$ is a polynomial of degree 3 and $R$ is the right shift operator. From Theorem 3.7, it is easy to show that the spectrum of the operator $P(R)$ over $\mu$ is $P(D)$, as shown in part (a) of Fig. 1. Now, by using Theorem 3.7, the point spectrum of $P(R)$ over $\mu$ is $\emptyset$, and the residual spectrum of $P(R)$ over $\mu$ is $[P(D)]^\circ \setminus A$, where $A$ is a finite set. We know that a necessary condition for an interior point of the set $P(D)$ belonging to the set $A$ is that it is either self intersecting point of the image of the unit circle by the polynomial $P(z)$ or it is the image of those points where $P'(z) = 0$. Now from part (a) of Fig. 1 it is clear that its interior does not contain any self intersecting point. So only possibility is that the image of $-1$ under the polynomial $P(z)$ may belongs to the set $A$, i.e, zero may belong to the set $A$. Since $P(z) = 0$ has only one solution $-1$, this gives the guaranty that zero belongs to $A$. Therefore $A = \{0\}$. Hence the residual spectrum of $P(R)$ over $\mu$ is $[P(D)]^\circ \setminus \{0\}$, as shown in part (b) of Fig. 1. Again by Theorem 3.7, the continuous spectrum of the operator $P(R)$ over $\mu$ is $\partial P(D) \cup A$, i.e, $\partial P(D) \cup \{0\}$, as shown in part (c) of Fig. 1.

Let us consider four band lower triangular matrix $T_3$ as

$$T_3 = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & \cdots \\ b_1 & a_2 & 0 & 0 & 0 & \cdots \\ c_1 & b_2 & a_3 & 0 & 0 & \cdots \\ d_1 & c_2 & b_3 & a_4 & 0 & \cdots \\ 0 & d_2 & c_3 & b_4 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a_n = 1 - \frac{1}{2^n}$, $b_n = 4 + \alpha^n$, $c_n = 1 + \beta^n$, $d_n = 1 + \gamma^n$ and $\alpha, \beta, \gamma$ are complex numbers such that $|\alpha|$ , $|\beta|$ and $|\gamma|$ lies in the interval $(0,1)$. Then the corresponding polynomial $P(z) = z^3 + z^2 + 4z + 1$.

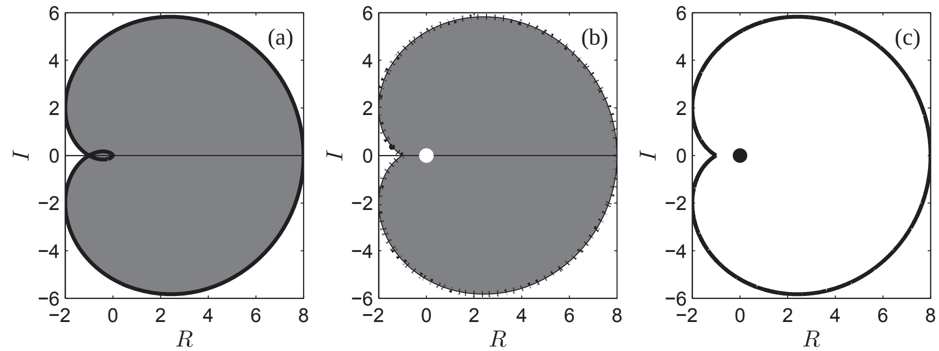


Fig. 1: (a) Spectrum; (b) Residual Spectrum; (c) Continuous Spectrum.

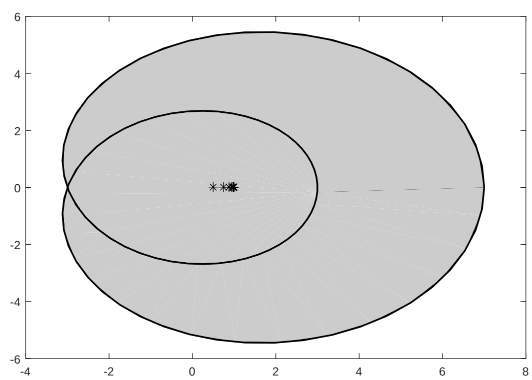

Fig. 2: Spectrum of $T$

From Figure 1 it is clear that the spectrum of $T_3$ is $P(D)$. The set $A$ contains those interior points of $P(D)$ which are images of boundary points of $D$ only. Since $P''(z)$ does not vanish on the unit circle, $A = \emptyset$. Hence by Theorem 3.9 the continuous spectrum

$$\sigma_c(T, l_p) = \partial P(D) \setminus \{a_n : n \in \mathbb{N}\} = \partial P(D).$$

Also

$$P(z) - a_n = z^3 + z^2 + 4z + \frac{1}{2^n}.$$

Then $P(z) - a_n = 0$ has one root located outside the unit circle. Hence the points denoted by $*$ are possible point spectrum. Therefore from Theorem 3.9 the point spectrum and the residual spectrum are given by

$$\begin{aligned}\sigma_p(T_3, l_p) &\subseteq \{a_n : n \in \mathbb{N}\},\\ \sigma_r(T_3, l_p) &= [P(D)]^\circ \setminus \sigma_p(T_3, l_p).\end{aligned}$$

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