Non-Normal Generalized Composition Operators of Higher Order

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Abstract

In the present paper, the non-normal properties of generalized composition operators of higher order are discussed along with their description.

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1 Introduction

Composition operators form an intriguing class of concrete linear operators in functional analysis. Their theory provides a fertile ground for the interaction of operator theory and the classical function theory where the operator theoretic properties of a composition operator are related to the function theoretic properties of its inducing function. Composition operators on the spaces of analytic functions have been discussed by various mathematicians over the years (see [5, 10, 11] and the references therein). Recently, the theory has been extensively extended over various function spaces, like Lebesgue spaces, Orlicz spaces, Lorentz spaces, Karamata spaces and Sobolev spaces, to name a few (see [1, 2, 7, 8, 12] and the references therein).

A new dimension is added to the theory of composition operators with the appearance of weighted sequence spaces, introduced by Kelley [9] around the year 1966. The attention is brought towards the study of the multiplication operators and the shift operators on weighted Hardy spaces by Shields [16]. Cowen and Kriete [4] have discussed the subnormality of the composition operator and the Denjoy-Wolff point of its induced map, while composition operators on weighted Hardy spaces are discussed by Zorboska in [17]. Normality and its weaker conditions have been discussed for composition operators on weighted Hardy spaces in [15]. The hyponormality of the composition operators is discussed by Zorboska in [18]. With the importance and wide appearance of differential operators, an experiment is done by connecting the theory of composition operators with the derivative of the functions and the study is lifted to generalized composition operators in [6].

The present paper is an attempt to discuss the structure of generalized composition and anti-differential operators of higher order. Normality and isometric properties of these operators are discussed along with the study of their Fredholm behavior. We begin with setting up necessary terminology to be used in the paper and the description of underlying weighted Hardy spaces.

Throughout the paper, the symbols $\mathbb{N}_0$ and $\mathbb{C}$ are used to denote the set of all integers greater than or equal to zero and the set of complex numbers respectively. By an operator, we mean a bounded linear mapping. The symbol $\mathfrak{B}(H)$ denotes the set of all operators on the Hilbert space H, while $Ker(T)$ and $R(T)$ denote respectively the kernel and the range of the operator T. The symbol $M^{\perp}$ denotes the orthogonal complement of the subspace M of H. For the weight sequence $\beta = \{\beta_n\}_{n\geq 0}$, a sequence of positive real numbers with $\beta_0 = 1$, the weighted Hardy space $H^2(\beta)$ is the collection of all formal power series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n, a_n \in \mathbb{C}$, for which $\|f\|_{\beta}^2 = \sum\limits_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty$. The space $H^2(\beta)$ is a Hilbert space with norm induced by the inner product defined as

$$\langle f, g\rangle = \sum_{n=0}^{\infty} a_n \overline{b}_n \beta_n^2,$$

where $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$ are elements of $H^2(\beta)$. The set $\{e_n\}_{n\geq 0}$, where $e_n(z) = \frac{z^n}{\beta_n}$, forms an orthonormal basis for the space $H^2(\beta)$. The symbol $H^{\infty}(\beta)$ denotes the set of formal power series ϕ such that $\phi(H^2(\beta)) \subseteq H^2(\beta)$. We refer to [16] for detailed descriptions of these spaces. It is worth mentioning here that for particular instances of the weight sequence β, the weighted Hardy spaces coincide with various well known spaces of analytic functions, for instance with the classical Hardy space, Bergman space and Dirichlet space (see [16]). This tendency of weighted Hardy spaces makes our study more productive.

2 Structural Behavior

Let $k \geq 1$ be an integer. The notion of k^{th}-order generalized composition operators defined on weighted Hardy spaces is discussed in [6]. We recall that a k^{th}-order generalized composition operator $C_{\phi,k}$ on $H^2(\beta)$ induced by the symbol $\phi \in H^2(\beta)$ is nothing but a continuous linear mapping defined as $f \mapsto f^{(k)} \circ \phi$ for each $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in H^2(\beta)$, where $f^{(k)}$ denotes the k^{th}-order derivative of f given by

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \alpha_n a_n z^{n-k},$$

with

$$\alpha_n = \begin{cases} 0 & \text{if } 0 \leq n \leq k-1 \\ \frac{n!}{(n-k)!} & \text{if } n \geq k. \end{cases}$$

For $k = 1$, this operator is the same as the generalized composition operator studied in [14].

It is known that the normality of a composition operator C_{ϕ}, induced by a non-singular measurable transformation ϕ, on Lebesgue spaces is characterized in terms of the Radon-Nikodym derivative of the inducing function [13]. Clearly, for a bounded linear operator $C_{\phi,k}$ on $H^2(\beta)$ induced by $\phi \in H^2(\beta)$, we have that $\phi^n \in H^2(\beta)$ for each $n \geq 0$. We begin with discussing the normality of generalized composition operators of higher order and via the next theorem, obtain that there does not exist any normal k^{th}-order generalized composition operator.

Theorem 2.1. *A k^{th}-order generalized composition operator $C_{\phi,k}$ induced by $\phi \in H^2(\beta)$ can not be hyponormal.*

Proof. For $C_{\phi,k}$ to be hyponormal, $\|C^*_{\phi,k}f\|^2_\beta \leq \|C_{\phi,k}f\|^2_\beta$ for each $f \in H^2(\beta)$. In particular for $f = e_0$, this provides that

$$\beta_0^4\Big(\frac{k!^2}{\beta_k^2} + \sum_{n=k+1}^{\infty} \frac{\alpha_n^2}{\beta_n^2} |< e_0, \phi^{n-k} > |^2 \Big) = \|C^*_{\phi,k}e_0\|^2_\beta \leq \|C_{\phi,k}e_0\|^2_\beta = 0,$$

which is not feasible. Hence the result. □

Using Theorem 2.1 together with the fact that the adjoint of a k^{th}-order generalized composition operator can not be a k^{th}-order generalized composition operator [6], we conclude that no ϕ in $H^2(\beta)$ can induce either a normal or a Hermitian generalized composition operator $C_{\phi,k}$.

Before we proceed ahead, we compute the adjoint of $C_{\phi,k}$ induced by the symbol $\phi(z) = az^p \in H^2(\beta)$, where $a \in \mathbb{C}$ and $p \geq 1$ is an integer. In this pursuance, for each $m \geq 0$ and for each $f(z) = \sum\limits_{n=0}^{\infty} f_n z^n \in H^2(\beta)$, the structure of $C_{\phi,k}$ helps to provide that

$$\begin{aligned}
\langle C^*_{\phi,k} z^m, f\rangle &= \langle z^m, \sum_{n=0}^{\infty} \alpha_{n+k} f_{n+k} a^n z^{np}\rangle \\
&= \begin{cases} \beta_m^2 \alpha_{t+k} \overline{f}_{t+k} \overline{a}^t & \text{if } m = tp \text{ for some } t \geq 0 \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \langle \frac{\beta_m^2}{\beta_{t+k}^2} \alpha_{t+k} \overline{a}^t z^{t+k}, f\rangle & \text{if } m = tp \text{ for some } t \geq 0 \\ 0 = \langle 0, f\rangle & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore, we find that for $\phi(z) = az^p$, the adjoint of $C_{\phi,k}$ is given as

$$C^*_{\phi,k} z^n = \begin{cases} \frac{\beta_n^2}{\beta_{t+k}^2} \alpha_{t+k} \overline{a}^t \; z^{t+k} & \text{if } n = tp \text{ for some } t \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

As an alternate approach, define the function k^β_ω, a point evaluation function, as

$$k^\beta_\omega(z) = \sum_{n=0}^{\infty} \frac{\overline{\omega^n}}{\beta_n^2} z^n$$

for a point w in the open unit disk in the complex plane. If the sequence $\beta = \{\beta_n\}_{n\geq 0}$ is such that $\beta_n \geq 1$ for each $n \geq 0$, then $k^\beta_w \in H^2(\beta)$ with $\|k^\beta_\omega\|^2_\beta = \sum\limits_{n=0}^{\infty} \frac{|w|^{2n}}{\beta_n^2} < \infty$ and $\langle f, k^\beta_\omega\rangle = f(w)$. This helps to provide that for $\phi(z) = az^p$,

$$(C^*_{\phi,k} z^n)(w) = \begin{cases} \frac{\beta_n^2}{\beta_{t+k}^2} \alpha_{t+k} \overline{a}^t w^{t+k} & \text{if } n = tp \text{ for some } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus equation (2.1) can be derived following this procedure as well.

We shall now discuss a weaker condition, namely quasi-normality, for $C_{\phi,k}$. We recall that an operator A on a Hilbert space is said to be quasi-normal if it commutes with A^*A. Condition for the composition operator on Lebesgue spaces to be quasi-normal is obtained in [13]. We proceed to obtain a characterization for the adjoint of generalized composition operator of higher order to be quasi-normal. We first draw the following remark.

Remark 2.1. *Let $k \geq 1$ be a fixed integer and $\{p_n\}_{n\geq 0}$ be a sequence of positive real numbers. Then the condition $p_n = p_{n+k}$ for each $n \geq 0$ is equivalent to the condition that for each $0 \leq i \leq k-1$, the sequence $\{p_{nk+i}\}_{n\geq 0}$ is a constant sequence.*

Theorem 2.2. *Let $C_{\phi,k} \in \mathfrak{B}(H^2(\beta))$, where $\phi(z) = az$, with $a \in \mathbb{C}$. Then $C^*_{\phi,k}$ is quasi-normal if and only if for each $0 \leq i \leq k-1$, $\{\frac{\beta_{nk+i}}{\beta_{(n+1)k+i}} \alpha_{(n+1)k+i} \, |a|^{nk+i}\}_{n\geq 0}$ is a constant sequence.*

Proof. It is a matter of routine computation to verify that

$$C_{\phi,k} C^*_{\phi,k} C^*_{\phi,k} z^n = \frac{\beta_n^2}{\beta_{n+2k}^2} \alpha_{n+k} \alpha_{n+2k}^2 \overline{a}^n |a|^{2n+2k} z^{n+k}$$

and

$$C^*_{\phi,k} C_{\phi,k} C^*_{\phi,k} z^n = \frac{\beta_n^4}{\beta_{n+k}^4} \alpha_{n+k}^3 \overline{a}^n |a|^{2n} z^{n+k}.$$

The result now follows directly using Remark 2.2. □

Remark 2.2. *Utilizing equation (2.1) and following parallel approach as in Theorem 2.3, we can easily verify the following for $\phi(z) = az^p$ with $p \geq 2$:*

1. *When k is not divisible by p then $C_{\phi,k} C^*_{\phi,k} C^*_{\phi,k} e_{p^2 m} \neq 0 = C^*_{\phi,k} C_{\phi,k} C^*_{\phi,k} e_{p^2 m}$, for each natural number m.*

2. *When k is divisible by p then $C_{\phi,k} C^*_{\phi,k} C^*_{\phi,k} e_{pm} \neq C^*_{\phi,k} C_{\phi,k} C^*_{\phi,k} e_{pm} = 0$, for each natural number m not divisible by p.*

These help to procure that the k^{th}-order generalized composition operator $C_{\phi,k}$ induced by the symbol $\phi(z) = az^p$ with $p \geq 2$ can not be quasi-normal.

We advance ahead now to discuss the isometric behavior of $C_{\phi,k}$. Observe that the structure of $C_{\phi,k}$ provides that $C_{\phi,k} e_0 = 0$ for each $\phi \in H^2(\beta)$. This helps to conclude the following.

Theorem 2.3. *No ϕ in $H^2(\beta)$ can induce an isometric k^{th}-order generalized composition operator $C_{\phi,k}$.*

There does not exist any isometric k^{th}-order generalized composition operator, but the existence of co-isometric generalized composition operators of higher order can be visualized by the next result.

Theorem 2.4. *A necessary and sufficient condition for $C_{\phi,k}$, where ϕ is given as $\phi(z) = az$, $a \in \mathbb{C}$, on $H^2(\beta)$ to be co-isometric is that $\alpha_{n+k}\frac{\beta_n}{\beta_{n+k}}|a|^n = 1$ for each $n \in \mathbb{N}_0$.*

Proof. It is straight forward to obtain that for each $n \geq 0$, $C_{\phi,k}C^*_{\phi,k}z^n = (\alpha_{n+k}\frac{\beta_n}{\beta_{n+k}}|a|^n)^2 z^n$. Consequently, $C_{\phi,k}$ is a co-isometry if and only if $\alpha_{n+k}\frac{\beta_n}{\beta_{n+k}}|a|^n = 1$ for each $n \geq 0$. Hence the result. □

As an example, on the weighted sequence space $H^2(\beta)$, with β defined as $\beta_n = n!$ for each $n \geq 0$, each symbol $\phi(z) = az \in H^2(\beta)$ for a unimodular complex number a, induces a co-isometric $C_{\phi,k}$.

Remark 2.3. *Since for $\phi(z) = az^p$, where $a \in \mathbb{C}$ and $p \geq 2$ is an integer, $C^*_{\phi,k}z^n = 0$ if n is not a multiple of p. Therefore, we conclude that $\phi(z) = az^p$ for $p \geq 2$ can never induce a co-isometric generalized composition operator.*

Motivated by the work of Zhao [19], who has obtained a characterization for a weighted composition operators on a class of weighted Hardy spaces to be Fredholm, we describe here the conditions so that a generalized composition operator of higher order becomes Fredholm. We first prove the following.

Lemma 2.1. *Let $\phi \in H^2(\beta)$ be such that $\{\phi^n : n \in \mathbb{N}_0\}$ is an orthogonal family. Then, $Ker(C_{\phi,k})$ is the linear span of the finite set $\{e_0, e_1, e_2, ..., e_{k-1}\}$.*

Proof. Since $C_{\phi,k}e_i = 0$ for each $0 \leq i \leq k-1$ and $C_{\phi,k}$ is linear, it immediately follows that the set spanned by $\{e_0, e_1, e_2, ..., e_{k-1}\}$ is a subset of $Ker(C_{\phi,k})$. Conversely, for any $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in Ker(C_{\phi,k})$, the orthogonality of the family $\{\phi^n : n \in \mathbb{N}_0\}$ provides that

$$0 = \|C_{\phi,k}\ f\|^2 = \sum_{n=k}^{\infty} |a_n|^2|\alpha_n|^2\|\phi^{n-k}\|^2,$$

which yields that $a_n = 0$ for each $n \geq k$. Hence the result. □

Lemma 2.2. *Let $\phi \in H^2(\beta)$ be such that $\{\phi^n : n \in \mathbb{N}_0\}$ is an orthogonal family spanning the space $H^2(\beta)$. Then, $Ker(C^*_{\phi,k}) = \{0\}$.*

Proof. Clearly, the family $\{\frac{\phi^n}{\|\phi^n\|} : n \in \mathbb{N}_0\}$ forms an orthonormal basis of $H^2(\beta)$. The proof is now straight forward, for any $g \in Ker(C^*_{\phi,k})$ can be expressed uniquely as $\sum\limits_{m=0}^{\infty} g_m\phi^m$, where $g_m \in \mathbb{C}$ and g satisfies that for each $n \geq 0$, $\langle (C^*_{\phi,k})g, e_n\rangle = 0$. This provides that $g_{n-k} = \langle g, \phi^{n-k}\rangle = 0$ for each $n \geq k$ so that $g = 0$. This completes the proof. □

Combining Lemma 2.7, Lemma 2.8 and the fact that $R(T)$ is closed if and only if T is bounded away from zero on $Ker(T)^{\perp}$, we prove the following theorem.

Theorem 2.5. *Let $\phi \in H^2(\beta)$ be such that $\{\phi^n : n \in \mathbb{N}_0\}$ is an orthogonal family which spans $H^2(\beta)$. Then the following statements are equivalent:*

1. *$C_{\phi,k}$ is a Fredholm operator.*
2. *$C_{\phi,k}$ has closed range.*
3. *There exists an $\epsilon > 0$ such that $\alpha_n \|\phi^{n-k}\| \geq \epsilon \beta_n$ for each $n \geq k$.*

Proof. The equivalence of (1) and (2) follows directly using the definition of a Fredholm operator and the fact that both $Ker(C_{\phi,k})$ and $Ker(C^*_{\phi,k})$ are finite dimensional.

To establish the equivalence of (2) and (3), first assume that $C_{\phi,k}$ has closed range. Then $C_{\phi,k}$ is bounded away from zero on $Ker(C_{\phi,k})^{\perp}$ and therefore, there exists an $\epsilon > 0$ such that $\|C_{\phi,k}e_n\| \geq \epsilon\|e_n\|$ for each $n \geq k$. Using the structure of $C_{\phi,k}$, we arrive at (3).

Conversely, let (3) holds true and $f(z) = \sum_{n=k}^{\infty} a_n z^n \in Ker(C_{\phi,k})^{\perp}$. Then, $\|C_{\phi,k} f\|^2 = \sum_{n=k}^{\infty} |a_n|^2 \alpha_n^2 \|\phi^{n-k}\|^2 \geq \epsilon^2\|f\|^2$. Thus we obtain that $C_{\phi,k}$ has closed range. This completes the proof. □

Towards the end, we focus our attention towards the study of anti-differential operators of higher order defined on $H^2(\beta)$ (see [6]). We would like to point out here that the k^{th}-order anti-differntial operator $D_{a,k}$ is the right inverse of the generalized composition operator $C_{\phi,k}$ for $\phi(z) = z$.

We first recall that $D_{a,k}$, the anti-differential operator of k^{th}-order on $H^2(\beta)$, is a bounded linear mapping defined as

$$D_{a,k}(f) = \sum_{n=0}^{\infty} \frac{a_n}{\gamma_n} z^{n+k},$$

for each $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H^2(\beta)$, where $\gamma_n = (n+1)(n+2)...(n+k)$. The adjoint of $D_{a,k}$ is given by $D^*_{a,k} e_n = \frac{\beta_n}{\gamma_{n-k}\beta n-k} e_{n-k}$ for each $n \geq k$ and $D^*_{a,k} e_n = 0$ for $0 \leq n \leq k-1$. We discuss the normality and isometric behaviour of $D_{a,k}$ and its adjoint. We begin with obtaining the necessary and sufficient condition for $D^*_{a,k}$ to become Fredholm.

Theorem 2.6. *The operator $D^*_{a,k}$ is Fredholm if and only if there exists an $\epsilon > 0$ such that $\frac{\beta_n}{\gamma_{n-k}\beta_{n-k}} \geq \epsilon$ for each $n \geq k$.*

Proof. Straight forward computations yield that both $Ker(D^*_{a,k})$ and $Ker(D_{a,k})$ are finite dimensional. In fact, $Ker(D^*_{a,k})$ is the linear span of the finite set $\{e_n : 0 \leq n \leq k-1\}$ and $Ker(D_{a,k}) = \{0\}$. It only remains to be be proved that $D^*_{a,k}$ has closed range if and only if the sequence β satisfies the given condition.

Following parallel steps as in Theorem 2.10, we arrive at the desired result. □

Since $H^2(\beta)$ coincides with the classical Hardy space H^2 when $\beta_n = 1$ for each $n \geq 0$, the above theorem also provides the following.

Corollary 2.1. *Let $D_{a,k} \in \mathfrak{B}(H^2)$. Then $D^*_{a,k}$ can never be a Fredholm operator.*

As an illustration to Theorem 2.11, consider the following example.

Consider the weighted sequence space $H^2(\beta)$ where β is defined as $\beta_n = 1$ for $0 \leq n \leq k-1$ and $\frac{\beta_{k+i}}{\beta_i} = (i+1)(i+2)\cdots(k+i)$, for each $i \geq 0$. Then, Theorem 2.11 provides that each $D^*_{a,k}$ on this sequence space is a Fredholm operator.

The following observations can easily be drawn for the anti-differential operator $D_{a,k}$.

1. $\|D^*_{a,k}e_0\|^2_\beta = 0$ and $\|D_{a,k}e_0\|^2_\beta = \frac{\beta_k^2}{k!^2}$.
2. $D^*_{a,k}D_{a,k}z^n = \frac{\beta^2_{n+k}}{\gamma_n^2\beta_n^2}z^n$, for each $n \geq 0$.
3. $D_{a,k}D^*_{a,k}z^n = \frac{\beta_n^2}{\gamma^2_{n-k}\beta^2_{n-k}}z^n$ for each $n \geq k$.
4. For each $n \geq 0$, $D^*_{a,k}D_{a,k}D_{a,k}z^n = \frac{\beta^2_{n+2k}}{\gamma_n\gamma^2_{n+k}\beta^2_{n+k}}z^{n+k}$ and $D_{a,k}D^*_{a,k}\ D_{a,k}z^n = \frac{\beta^2_{n+k}}{\gamma_n^3\beta_n^2}\ z^{n+k}$.

Observation (1) provides that $D^*_{a,k}$ can not be an isometry. Using observations (2) and (3), the following can be immediately obtained.

Theorem 2.7. *The following are equivalent for the operator $D_{a,k}$ defined on $H^2(\beta)$:*

1. *$D_{a,k}$ is an isometry.*
2. *$D^*_{a,k}$ is a partial isometry.*
3. *$\frac{\beta_{n+k}}{\gamma_n\beta_n} = 1$, for each $n \geq 0$.*

The observations (1) and (4) along with Remark 2.2 lead us to the following.

Theorem 2.8. *For the operator $D_{a,k}$ on $H^2(\beta)$, we have the following:*

1. *$D^*_{a,k}$ can not be a hyponormal operator.*
2. *$D^*_{a,k}$ can never be normal.*
3. *$D_{a,k}$ is quasi-normal if and only if for each $0 \leq i \leq k-1$, $\{\frac{\beta_{(n+1)k+i}}{\gamma_{nk+i}\ \beta_{nk+i}}\}_{n\geq 0}$, is a constant sequence.*

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