Pseudo-differential operators and wavelet transform associated with fractional Fourier transform

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Abstract

Pseudo-differential operators are generalization of partial differential operators and have been defined on various function spaces and distributional spaces by using many integral transforms. In fact, these operators played an important role to study the elliptic partial differential equations. These operators are also useful to consider many problems such as existence and uniqueness of boundary value problems in linear partial differential equations, regularity of elliptic partial differential equations and also yield many significant results in non linear partial differential equations.

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1 Introduction:

The theory of pseudo-differential operators was developed by Kohn-Nirenberg, Hörmander, Wong, Zaidman, Taylor, Treves etc. [1, 2, 4, 6, 8, 11, 12, 13, 14, 15, 21, 30, 31, 32, 33]. They applied pseudo-differential operators associated with different types of symbols on the Schwartz space $S(\mathbb{R}^n)$ and got important resuts by exploiting the Fourier transformation. Our main aim of this review article is to study the different works of pseudo-differential operators and wavelet transform on Schwartz space and W-type spaces by using fractional Fourier transform technique. Pathak and Pandey [9] studied the continuous wavelet transform on the space of type-W and found many results. Upadhyay et al. [7] generalized the results of [9]. Novel fractional wavelet transform was given by Shi et al. [3]. Using $L^2(\mathbb{R})$- space and they got many important properties. Prasad et al. [4], Prasad and Mahato [11] studied the generalized continuous wavelet transform associated with fractional Fourier transform on the $L^2(\mathbb{R})$ and W-type spaces.

This review article consists of five sections. Section 1 is introductory which gives the adequate information of the definitions, properties of W-type spaces, Fourier transform, fractional Fourier transform, pseudo-differential operators, wavelet transform, Schwartz space, tempered distribution, relation between Fourier transform and fractional Fourier transform. In section 2, characterization of W-type spaces is given and various mapping properties, an integral representation of pseudo-differential operators obtained by using the fractional Fourier transform. In section 3, Relations between W and W^p type spaces are found by exploiting the fractional Fourier transform. The uniqueness class of Cauchy problem is also studied by using the aforesaid transform. In section 4, various properties of W type spaces are discussed by exploiting the n-dimensional continuous fractional wavelet transform (CFrWT). Various mapping properties of fractional wavelet transform on Gel'fand and Shilov spaces are discussed. In section 5, the concept of generalized Sobolev space $\mathcal{G}_\alpha^{s,p}(\mathbb{R})$, involving fractional Fourier transform is defined. Some bounded estimates of pseudo-differential operator associated with L^p-norm for $1 \le p < \infty$ are discussed and an asymptotic series of general symbol of pseudo-differential operator is also obtained by using the theory of fractional Fourier transform.

$$\text{Let } \mathbb{R}^n = \{(x_1, ..., x_n)\colon x_j\text{'s are real numbers}\}$$

be the usual Euclidean space. If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ are the elements of $\mathbb{R}^n$. Then the inner product of x and y is defined by

$$\langle x, y\rangle = x.y = \sum_{j=1}^{n} x_j.y_j \tag{1.1}$$

and the norm of x is defined by

$$|x| = \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}} = \left(x_1^2 + \cdots + x_n^2\right)^{\frac{1}{2}}. \tag{1.2}$$

Definition 1.1 The space $S(\mathbb{R}^n)$, called the Schwartz space in $\mathbb{R}^n$, is the set of all $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\gamma_{\mu,\nu}(\phi) = \sup_{x\in\mathbb{R}^n} |x^\mu D^\nu \phi(x)| < \infty, \tag{1.3}$$

for all multi-indices $\mu, \nu \in \mathbb{Z}_+^n$.

From [3] the n-dimensional fractional Fourier transform with parameter α of $f(x)$ on $x \in \mathbb{R}^n$ is denoted by $(F_\alpha f)(\xi)$ and defined as

$$\hat{f}_\alpha(x) = (F_\alpha f)(\xi) = \int_{\mathbb{R}^n} K_\alpha(x,\xi) f(x)dx, \quad \xi \in \mathbb{R}^n \tag{1.4}$$

where

$$K_\alpha(x,\xi) = \begin{cases} C_\alpha e^{\frac{i(|x|^2+|\xi|^2)\cot\alpha}{2} - i\langle x,\xi\rangle \csc\alpha} & \text{if } \alpha \ne n\pi \\ \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-i\langle x,\xi\rangle} & \text{if } \alpha = \frac{\pi}{2}, \end{cases} \qquad \forall\, n \in \mathbb{Z},$$

and

$$C_\alpha = (2\pi i \sin\alpha)^{\frac{-n}{2}} e^{\frac{in\alpha}{2}} = \frac{1}{[\pi(1-e^{-2i\alpha})]^{\frac{n}{2}}}.$$

The corresponding inversion formula is given by

$$f(x) = \int_{\mathbb{R}^n} \overline{K_\alpha(x,\xi)}\ \hat{f}_\alpha(\xi)\,d\xi, \quad x \in \mathbb{R}^n \tag{1.5}$$

where the kernel

$$\overline{K_\alpha(x,\xi)} = C'_\alpha e^{\frac{-i(|x|^2+|\xi|^2)\cot\alpha}{2} + i\langle x,\xi\rangle \csc\alpha},$$

and

$$C'_\alpha = (2\pi i \sin\alpha)^{-\frac{n}{2}} e^{\frac{in\alpha}{2}} = \left[\pi(1-e^{-2i\alpha})\right]^{-\frac{n}{2}}.$$

Now we recall the definitions of W-type spaces from [4, 5], which are given below.

Let μ_j and w_j, $j = 1, ..., n$, be continuous and increasing functions on $[0,\infty)$ with $\mu_j(0) = w_j(0) = 0$ and $\mu_j(\infty) = w_j(\infty) = \infty$.
We define

$$M_j(x_j) = \int_0^{x_j} \mu_j(\xi_j)d\xi_j, \qquad (x_j \geq 0) \tag{1.6}$$

$$\Omega_j(y_j) = \int_0^{y_j} w_j(\eta_j)d\eta_j, \qquad (y_j \geq 0) \tag{1.7}$$

where $j = 1, ..., n$. The functions $M_j(x_j)$ and $\Omega_j(y_j)$ are continuous, increasing and convex with $M_j(0) = \Omega_j(0) = 0$ and $M_j(\infty) = \Omega_j(\infty) = \infty$,
we have

$$M_j(-x_j) = M_j(x_j), \qquad M_j(x_j) + M_j(x'_j) \leq M_j(x_j + x'_j), \tag{1.8}$$

$$\Omega_j(-y_j) = \Omega_j(y_j), \qquad \Omega_j(y_j) + \Omega_j(y'_j) \leq \Omega_j(y_j + y'_j). \tag{1.9}$$

We set

$$\mu(\xi) = (\mu_1(\xi_1)), ..., (\mu_n(\xi_n)),$$
$$w(\eta) = (w_1(\eta_1)), ..., (w_n(\eta_n)).$$

The space $W_M(\mathbb{R}^n)$ consists of all C^∞- complex valued functions $\phi(x)$ on $\mathbb{R}^n$ which for some $a \in \mathbb{R}^n_+$ satisfy the inequality

$$\left| D_x^k \phi(x) \right| \leq C_k\, exp\,[-M(ax)], \tag{1.10}$$

where, $exp\,[-M(ax)] = exp\left[-M_1(a_1x_1) - \cdots - M_n(a_nx_n)\right]$,

$D_x^k = D_{x_1}^{k_1} \cdots D_{x_n}^{k_n}$ and $a_1, ..., a_n, C_k$ are positive constants depending on the function $\phi(x)$ and the space $W_{M,a}(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\phi(x)$ which satisfy the inequality

$$\left|D_x^k \phi(x)\right| \leq C_{k,\delta}\, exp\,[-M(a-\delta)x] \text{ for } x \in \mathbb{R}^n, \tag{1.11}$$

where

$$\begin{aligned} & exp\,[-M(a-\delta)x] \\ &= exp\left[-M_1(a_1-\delta_1)x_1 - \cdots - M_j(a_j-\delta_j)x_j - \cdots - M_n(a_n-\delta_n)x_n\right] \end{aligned}$$

and $a, \delta \in \mathbb{R}_+^n$, depend on the function $\phi(x)$. The space $W^{\Omega}(\mathbb{C}^n)$ consists of all entire analytic functions $\phi(z)$, where $z = x + iy$ and $x, y \in \mathbb{R}^n$, which for some $b \in R_+^n$, satisfy the inequality

$$\left|z^k \phi(z)\right| \leq C_k\, exp\,[\Omega(by)], \tag{1.12}$$

where

$$z^k = z_1^{k_1} \ldots z_n^{k_n},$$

$$exp\,[\Omega(by)] = exp\left[\Omega_1(b_1y_1) + \cdots + \Omega_j(b_jy_j) + \cdots + \Omega_n(b_ny_n)\right],$$

and $C_k, b_1, ..., b_n$ are positive constants depending on the function $\phi(x)$,
and the space $W^{\Omega,b}$ consists of all entire analytic functions $\phi(z)$ such that for $k \in \mathbb{Z}_+^n, \rho \in \mathbb{R}_+^n$, there exists a constants $C_{k,\rho} > 0$ such that

$$\left|z^k \phi(z)\right| \leq C_{k,\rho}\, exp\,[\Omega(b+\rho)y], \tag{1.13}$$

where

$$\begin{aligned} & exp\,[\Omega(b+\rho)y] \\ &= exp\left[\Omega_1(b_1+\rho_1)y_1 + \cdots + \Omega_j(b_j+\rho_j)y_j + \cdots + \Omega_n(b_n+\rho_n)y_n\right]. \end{aligned}$$

The space $W_M^{\Omega}(\mathbb{C}^n)$ consists of all entire analytic functions $\phi(z)$ such that there exist $a, b \in \mathbb{R}_+^n$ and $C > 0$ such that

$$|\phi(z)| \leq C\, exp\,[-M[(ax)] + \Omega[(by)]], \tag{1.14}$$

where $exp\,[-M(ax)]$ and $exp\,[\Omega(by)]$ have usual meaning like (1.10) and (1.12),
and the space $W_{M,a}^{\Omega,b}(\mathbb{C}^n)$ consists of all entire analytic functions $\phi(z)$ such that for $\rho, \delta \in \mathbb{R}_+^n$ and $C_{\rho,\delta} > 0$,

$$|\phi(z)| \leq C_{\rho,\delta}\; exp\,[-M[(a-\delta)x] + \Omega[(b+\rho)y]], \tag{1.15}$$

where $exp\,[-M[(a-\delta)x]]$ and $exp\,[\Omega[(b+\rho)y]]$ have usual meaning like (1.11) and (1.13), and the constants $C_{\rho,\delta}, a, b$ and ρ, δ, depend only on the function $\phi(z)$.

Let $M_j(x_j)$ and $\Omega_j(y_j)$ be the functions defined by (1.6) and (1.7), where the functions $\mu_j(\xi_j)$ and $w_j(\eta_j)$ which occur in these equations are mutually inverse, that is $\mu_j(w_j(\eta_j)) = \eta_j$ and $w_j(\mu_j(\xi_j)) = \xi_j$, then the corresponding functions $M_j(x_j)$ and $\Omega_j(y_j)$ are said to be the dual in sense of Young. In this case, the Young inequality

$$x_j y_j \leq M_j(x_j) + \Omega_j(y_j), \tag{1.16}$$

holds for any $x_j \geq 0$, $y_j \geq 0$.

2 Pseudo-differential operators and characterization of W-type spaces

The concept of pseudo-differential operators of infinite order has been studied by Cappiello [2], Zanghirati [14], Bouted de Monvel [1] and Upadhyay et al. [21] on Gevrey, Gelfand and Shilov type of spaces by using the Fourier transform. Pseudo-differential operator associated with fractional Fourier transform was given by Pathak et al. [9] and others see [23, 26, 34, 10].

In this section, the characterization of W-type spaces is investigated and various mapping properties of pseudo-differential operators studied by exploiting the fractional Fourier transform.

Definition 2.1 Let $m \in \mathbb{R}$. Then we define the symbol class U^m to be the space of all entire analytic functions $\theta(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{C}^n)$ in ξ such that for any two multi-indices μ and ϑ , there is a positive constant $C_{\mu,\vartheta}$ depending upon μ and ϑ such that

$$\left|(D_x^\mu D_\xi^\vartheta \theta)(x, \xi)\right| \leq C_{\mu,\vartheta}(1 + |\xi|)^{m-|\vartheta|}\, exp[\Omega(a_0 t)] \tag{2.1}$$

$\forall\ x \in \mathbb{R}^n$, $\xi \in \mathbb{C}^n$ and $\xi = u + it$.

If we put $t = 0$ in (2.1), then the symbol class U^m reduces to the symbol class introduced by Kohn and Nirenberg [8], see also Wong [13].

Definition 2.2 Let $\theta(x, \xi)$ be a symbol belonging to U^m, then the pseudo-differential operator $A_{\theta,\alpha}$ associated with $\theta(x, \xi)$ is defined as

$$(A_{\theta,\alpha}\phi)(x) = \int_{\mathbb{R}^n} \overline{K_\alpha(x, \xi)}\ \theta(x, \xi)\hat{\phi}_\alpha(\xi)\, d\xi, \quad \phi \in W_M(\mathbb{R}^n) \tag{2.2}$$

where $\hat{\phi}_\alpha(\xi)$ is the fractional Fourier transform of ϕ which is defined in (1.4).

Theorem 2.3 Let $M(x)$ and $\Omega(y)$ be a pair of functions which are dual in sense of Young . Then

$$F_\alpha\left[W_{M,a}\right] \subset W^{\Omega, \frac{1}{a}}. \tag{2.3}$$

Proof. The proof of above theorem is given in [23].

Theorem 2.4 Let $M(x)$ be the function which is dual in the sense of Young of $\Omega(y)$. Then

$$F_\alpha\left[W^{\Omega,b}\right] \subset W_{M,\frac{1}{b}}.$$

Proof. The proof of above theorem is given in [23].

Theorem 2.5 Let $M_0(x)$ and $\Omega_0(y)$ be dual in sense of Young to the functions $M(x)$ and $\Omega(y)$, respectively.Then

$$F_\alpha\left[W^{\Omega,b}_{M,a}\right] \subset W^{\Omega_0,\frac{1}{a}}_{M_0,\frac{1}{b}}.$$

Proof. The proof of above theorem is given in [23].

The mapping properties of pseudo-differential operators defined by (2.2) on $W_M(\mathbb{R}^n)$ are obtained.

Theorem 2.6 Let $\theta(x,\xi) \in U^m$, where $m \in \mathbb{R}$. Then $A_{\theta,\alpha}$ maps $W_M(\mathbb{R}^n)$ into itself.
Proof. The proof of above theorem is given in [23].

Theorem 2.7 Let $\theta(x,\xi) \in U^m$ be a symbol, where $m \in \mathbb{R}$. Then $A_{\theta,\alpha}$ maps $W_M(\mathbb{R}^n)$ continuously into itself.
Proof. The proof of above theorem is given in [23].

3 Characterization of W^p-type spaces

The spaces of W-type were studied by Gurevich [15], Gelfand and Shilov [5] and Friedman [4]. They found the behaviour of Fourier transform on W-type spaces and got many results. The importance of W -type spaces are in the sense because they suitably applied in the theory of partial differential equations. Pathak and Upadhyay [36] investigated the spaces $W^p_M, W^p_{M,a}$, $W^{\Omega,b,p}, W^{\Omega,p}, W^{\Omega,p}_M, W^{\Omega,b,p}_{M,a}$ in terms of L^p norms for $1 \leq p < \infty$. They have shown that Fourier transform F is to be a continuous linear mapping as follows: $F : W^p_{M,a} \to W^{\Omega,\frac{1}{a},r}$, $F : W^{\Omega,b,p} \to W^r_{M,\frac{1}{b}}$, $F : W^{\Omega,b,p}_{M,a} \to W^{\Omega,\frac{1}{a},r}_{M,\frac{1}{a}}$. Betancor and Mesa [38] gave a new characterization of the spaces We^p_μ-type and established the results, $We^p_{\mu,M,a} = We_{M,a}, We^{p,\Omega,b} = We^{\Omega,b}, We^{p,\Omega,b}_{M,a} = We^{\Omega,b}_{M,a}$ by using the theory of Hankel transformation. Also, Upadhyay [11] established the results of the following types: $W^p_{M,a} = W_{M,a}, W^{p,\Omega,b} = W^{\Omega,b}, W^{p,\Omega,b}_{M,a} = W^{\Omega,b}_{M,a}$ by exploiting the theory of Fourier transform. Motivated by the work of Pathak and Upadhyay [36], Upadhyay [11] and Upadhyay et. al [23, 24] we are discussing the similar type of results by using the theory of fractional Fourier transform.
In this section, we studied the characterization of W^p-type spaces by using the fractional Fourier transformation.
Theorem 3.1 Let $M(x)$ and $\Omega(y)$ be a pair of functions which are dual in the sense of Young . Then

$$F_\alpha\left[W^p_{M,a}\right] \subset W^{\Omega,\frac{1}{a},r}, \qquad p,r \geq 1. \tag{3.1}$$

Proof. The proof of above theorem is given in [24].

Theorem 3.2 Let $M(x)$ and $\Omega(y)$ be a pair of function which are dual in sense of Young. Then

$$F_\alpha\left[W^{\Omega,b,p}\right] \subset W^r_{M,\frac{1}{b}}, \quad p,r \geq 1.$$

Proof. The proof of above theorem is given in [24].

Theorem 3.3 Let $\Omega_0(y)$ and $M_0(x)$ be the functions which are dual in sense of Young to the functions $M(x)$ and $\Omega(y)$ respectively. Then

$$F_\alpha\left[W^{\Omega,b,p}_{M,a,}\right] \subset W^{\Omega_0,\frac{1}{a},r}_{M_0,\frac{1}{b}}, \quad p,r \geq 1.$$

Proof. The proof of above theorem is given in [24].

The relations between W and W^p type of spaces are given below as:

Theorem 3.4 Let $M(x), \Omega(y)$ be the pair of functions which are dual in sense of Young. Then

$$W^p_{M,a} = W_{M,a}, \quad 1 \leq p < \infty.$$

Proof. Now, for showing the above theorem we shall prove the following Lemma.

Lemma 3.5 Let $1 \leq p < \infty$. Then $W^p_{M,a} \subset W_{M,a}$.
Proof. The proof of above Lemma is given in [24].

Lemma 3.6 Let $1 \leq p < \infty$. Then $W_{M,a} \subset W^p_{M,a}$.

Proof. The proof of above Lemma is given in [24].

Theorem 3.7 Let $M(x)$ and $\Omega(y)$ be the same functions as in Theorem 3.2 Then

$$W^{\Omega,b,p} = W^{\Omega,b}, \quad 1 \leq p < \infty. \tag{3.2}$$

Proof. The proof of above theorem is given in [24, 26].

Theorem 3.8 Let $\Omega_0(y)$ and $M_0(x)$ be the functions which are dual in sense of Young to the functions $M(x)$ and $\Omega(y)$ respectively. Then

$$W^{\Omega,b,p}_{M,a} = W^{\Omega_0,b}_{M_0,a}, \quad 1 \leq p < \infty. \tag{3.3}$$

Proof. The proof of above theorem is given in [24].

We apply the aforesaid characterizations of W-type spaces for discussing the uniqueness theorem of Cauchy problem by exploiting the fractional Fourier transform technique:

$$\frac{\partial u(x,t)}{\partial t} = P(i\triangle_x)u(x,t), \quad \forall\,(x,t) \in \mathbb{R}^n \times [0,T], \tag{3.4}$$

$$u(x,0) = u_0(x), \tag{3.5}$$

where

$$\triangle_x^k = \triangle_{x_1}^{k_1} \cdots \triangle_{x_n}^{k_n}, \tag{3.6}$$

$$= \left(\frac{\partial}{\partial x_1} - ix_1 \cot\alpha\right)^{k_1} \cdots \left(\frac{\partial}{\partial x_n} - ix_n \cot\alpha\right)^{k_n}, \tag{3.7}$$

is a differential operator and u(x, t) is an $N \times 1$ column vector. Here P is an $N \times N$ polynomial matrix with constant coefficients of order k. A similar problem has been investigated by Gelfand and Shilov [5], Friedman [4] by exploiting the theory of Fourier transform. Also, Upadhyay [27] studied the uniqueness of Cauchy problem by using the theory of Hankel transform.

Theorem 3.9 The Cauchy problem (3.4) and (3.5) possesses a unique solution u(x, t) in the space $(W^{\Omega_0, \frac{1}{a-\theta}}_{M_0, \frac{1}{b-\theta}})'$ for the interval $0 \le t \le T$,
$T < (2cp_0)^{-1}(\theta/2)^{p_0}, \quad \theta < a,$
and for any initial function $u_0(x)$ belonging to the same space, where p_0 is the reduced order of the system (3.4) and (3.5) with $i\triangle_x$ replaced by $i\frac{\partial}{\partial x}$ and c is a constant depending on P.

Proof. The proof of above theorem is given in [24].

4 Asymptotic series of general symbol of pseudo-differential operator

An asymptotic series of symbols and of general pseudo-differential operators has been studied by Zaidman [29]. Zayed [28], Bhosale and Chaudhary [39], studied the fractional Fourier transform on distributions with compact support. Recently, the properties of asymptotic series of general symbol of pseudo-differential operator has been studied by Upadhyay et. al [22] by using the fractional Fourier transform.

In this section, the important facts about the pseudo-differential operator associated with class of symbol and useful definitions are given below:

Let $a(x,\xi) \in \mathcal{G}$ be a class of all measurable complex-valued functions which are defined on $\mathbb{R} \times \mathbb{R} - \{0\}$. Then we assume the following properties:

(i) $\lim_{x\to\infty} a(x,\xi) = a(\infty,\xi)$ exists for all $\xi \in \mathbb{R} - \{0\}$ and is bounded measurable function.

(ii) We define $a'(x,\xi) = a(x,\xi) - a(\infty,\xi)$ then, we have

$$a'(x,\xi) = \frac{C'_\alpha}{2\pi}\int_{\mathbb{R}} e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}(F_\alpha a')(\eta,\xi)d\eta, \qquad \forall\,(x,\xi)\in\mathbb{R}\times\mathbb{R}-\{0\}\,.$$

where $(F_\alpha a')(\eta,\xi)$ is complex-valued function defined on $\mathbb{R}\times\mathbb{R}-\{0\}$, which is measurable in η and ξ for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}-\{0\}$ and satisfies the estimate:

$$|(F_\alpha a')(\eta,\xi)| \leq k(\eta), \quad \forall\,\eta\in\mathbb{R}. \tag{4.1}$$

where $(1+|\eta|^2)^l k(\eta)\in L^1(\mathbb{R}), \quad \forall\, l = 0,1,2,3,\ldots.$

Let $(r_j)_0^\infty$ be a strictly decreasing sequence that is $r_0 > r_1 > r_2 > \cdots > r_j \to -\infty$ as $j\to\infty$ and $\psi\in C^\infty(\mathbb{R})$ such that $0\leq\psi(\xi)<\infty$ for all $\xi\in\mathbb{R}-\{0\}$,

$$\psi(\xi) = \begin{cases} 0 & \text{if } 0<\xi\leq\frac{1}{2} \\ 1 & \text{if } \xi\geq 1. \end{cases} \tag{4.2}$$

Let $\{a_j(x,\xi)\}_0^\infty$ be an infinite sequence of functions in $\mathcal{G}$ defined on $\mathbb{R}\times\mathbb{R}-\{0\}$.
Then we take the symbol

$$a(x,\xi) = \sum_{j=0}^{\infty}\psi(\frac{\xi}{t_j})\ |\xi|^{r_j}a_j(x,\xi), \tag{4.3}$$

where $(t_j)_0^\infty$ be a sequence of positive real numbers such that $t_j\to\infty$ as $j\to\infty$.

From equation (4.3), it is clear that $a(x,\xi) = 0$ for $|\xi|\leq\frac{t_j}{2}$, $x\in\mathbb{R}$. Then $a(x,\xi)$ is infinitely differential with respect to x, for any $\xi\in\mathbb{R}-\{0\}$.

The global estimate of the above defined symbol $a(x,\xi)$ and of remainders of order N are given as

$$\begin{aligned} b_N(x,\xi) &= a(x,\xi) - \sum_{j=0}^{N-1}\psi(\frac{\xi}{t_j})\ |\xi|^{r_j}a_j(x,\xi) \\ &= \sum_{j=N}^{\infty}\psi(\frac{\xi}{t_j})\ |\xi|^{r_j}a_j(x,\xi), \quad \forall\,\xi\in\mathbb{R}-\{0\}. \end{aligned} \tag{4.4}$$

Theorem 4.1 Let $\{t_j\}_0^\infty$ be a sequence of positive real numbers such that the following inequalities:

$$|a(x,\xi)|\leq C|\xi|^{r_0}, |b_N(x,\xi)|\leq C|\xi|^{r_N}, \quad \forall\,(x,\xi)\in\mathbb{R}\times\mathbb{R}-\{0\}\,. \tag{4.5}$$

are satisfied for $N = 1, 2, 3, \ldots$. In particular the estimates,

$$|a(\infty, \xi)| \leq C|\xi|^{r_0},\ |b_N(\infty, \xi)| \leq C|\xi|^{r_N}, \quad \forall \xi \in \mathbb{R} - \{0\}\,. \tag{4.6}$$

Proof. The proof of above theorem is given in Zaidman [29, pp. 233-234] .

Theorem 4.2 Let $\{t_j\}_0^\infty$ be a sequence of positive real numbers such that it follows the following estimates:

$$|(F_\alpha a')(x, \xi)| \leq K(\lambda)|\xi|^{r_0}, |(F_\alpha b'_N)(x, \xi)| \leq K_N(\lambda)|\xi|^{r_N}, \quad \forall (\lambda, \xi) \in \mathbb{R} \times \mathbb{R} - \{0\}\,. \tag{4.7}$$

are satisfied for $N = 1, 2, 3, \ldots$. In particular the estimates,

$$|(F_\alpha a')(\infty, \xi)| \leq C|\xi|^{r_0}, |(F_\alpha b'_N)(\infty, \xi)| \leq C|\xi|^{r_N}, \quad \forall \xi \in \mathbb{R} - \{0\}\,. \tag{4.8}$$

where $(1+|\lambda|^2)^p K(\lambda) \in L^1(\mathbb{R})$, $(1+|\lambda|^2)^p K_N(\lambda) \in L^1(\mathbb{R})$, $\forall\, p = 0, 1, 2, 3, \ldots$ and $N = 1, 2, 3, \ldots$.
Definition 5.1.1. Let $a(x, \xi)$ be a general symbol belonging to $\mathcal{G}$. Then pseudo-differential operator

$a(x, D) = A_{a,\alpha}$ associated with symbol $a(x, \xi)$ is defined by

$$(a(x, D)\phi)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{K_\alpha(x, \xi)}\, a(x, \xi)\ \hat{\phi}_\alpha(\xi)\, d\xi, \tag{4.9}$$

where $\hat{\phi}_\alpha(\xi)$ is defined in equation (1.4) $\forall\, (x, \xi) \in \mathbb{R} \times \mathbb{R} - \{0\}$.

Definition 4.3 Fractional Fourier transform of pseudo-differential operator $a(x, D)$ associated with symbol $a(x, \xi) \in \mathcal{G}$ is given by

$$F_\alpha[e^{\frac{ix^2 \cot\alpha}{2}} a(x, D)\phi(x)](\xi) = \int_{\mathbb{R}} e^{\frac{i(x^2+\xi^2)\cot\alpha}{2} - ix\xi \csc\alpha} a(x, \xi)\phi(x) dx, \quad \xi \in \mathbb{R}, \tag{4.10}$$

where $a(x, D) = a(\infty, D) + a'(x, D)$ and $a(x, \xi) = a(\infty, \xi) + a'(x, \xi)$.

Definition 4.4 The generalized Sobolev space $\mathcal{G}_\alpha^{s,p}(\mathbb{R})$ involving fractional Fourier transform is defined by

$$\|\phi\|_{s,p} = \left[\int_{\mathbb{R}} (1 + |\xi|^2)^{\frac{sp}{2}} |\hat{\phi}_\alpha(\xi)|^p\ d\xi\right]^{\frac{1}{p}}, \tag{4.11}$$

$s \in \mathbb{R}$ and $\phi \in S'(\mathbb{R})$.

Definition 4.5 An infinite differential complex-valued function $\phi(x)$ member of $S(\mathbb{R})$ iff for every choice of μ and ν of non-negative integers it satisfies

$$\gamma_{\mu,\nu}(\phi) = \sup_{x\in\mathbb{R}} |x^\mu D^\nu \phi(x)| < \infty. \tag{4.12}$$

Lemma 4.6 A function $\phi \in C^{\infty}(\mathbb{R})$ satisfies (4.12) if and only if

$$\tau_{m,\beta}(\phi) = \sup_{x\in\mathbb{R}} |(1+|x|^2)^{\frac{m}{2}} D^{\beta}\phi(x)| < \infty, \qquad \forall\ m,\beta \in \mathbb{Z}_+. \tag{4.13}$$

Lemma 4.7 (Peetre's inequality) For any real number t and for all $\xi, \eta \in \mathbb{R}$, the estimate

$$\left(\frac{(1+|\xi|^2)}{(1+|\eta|^2)}\right)^t \leq 2^{|t|}(1+|\xi-\eta|)^{|t|}, \tag{4.14}$$

is satisfied.

Definition 4.8 The convolution of two functions $\phi \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, is defined by

$$(\phi * g)(x) = \int_{\mathbb{R}} \phi(y)g(x-y)\, dy. \tag{4.15}$$

The properties of asymptotic expansion of symbols of pseudo-differential operator associated with fractional Fourier transform are discussed as given below:

Theorem 4.9 Let $a'(x,\xi) \in \mathcal{G}$ then we have the following relation

$$F_\alpha[e^{\frac{ix^2\cot\alpha}{2}} a'(x,D)\phi(x)](\xi) = \frac{C'_\alpha}{2\pi}\int_{\mathbb{R}} e^{-i(\eta^2-\xi\eta)\cot\alpha}\ \hat{a}'_\alpha(\xi-\eta,\xi)\hat{\phi}_\alpha(\eta)d\eta, \tag{4.16}$$

where $\phi \in S(\mathbb{R})$, $x \in \mathbb{R}$.
Proof. The proof of above theorem is given in [22].

Theorem 4.10 If $a(x,\xi) \in \mathcal{G}$ is a symbol and $a(x,D) = a(\infty,D) + a'(x,D)$ is the associated operator. Then we have the following relation:

$$\begin{aligned} F_\alpha[e^{\frac{ix^2\cot\alpha}{2}}\ a(x,D)\phi(x)](\xi) &= a(\infty,\xi)F_\alpha[e^{\frac{ix^2\cot\alpha}{2}}\ \phi(x)](\xi) \\ &\quad + \frac{C'_\alpha}{2\pi}\int_{\mathbb{R}} e^{-i(\eta^2-\xi\eta)\cot\alpha}\ \hat{a}'_\alpha(\xi-\eta,\xi)\hat{\phi}_\alpha(\eta)d\eta, \end{aligned}$$

where $\phi \in S(\mathbb{R})$, $x \in \mathbb{R}$.
Proof. The proof of above theorem is given in [22].

Theorem 4.11 Let $a(x,\xi) \in \mathcal{G}$ be a symbol and $a(x,D)$ be the associated pseudo-differential operator then we have the following relation

$$\|(a(x,D)\phi)\|_{s,p} \leq C_{s,p}\, \|\phi\|_{s+r_0,p}\,, \tag{4.17}$$

where $\phi \in S(\mathbb{R})$, $s \in \mathbb{R}$.
Proof. The proof of above theorem is given in [22].

Theorem 4.12 We have the following estimates,

$$\left\|\psi_r\left(\frac{1}{t}D\right)\phi\right\|_{s,p} \le C\,\|\phi\|_{s+r,p}, \quad \forall\,\phi \in S(\mathbb{R}) \text{ and } s, r \in \mathbb{R}. \tag{4.18}$$

Proof. The proof of above theorem is given in [22].

Definition 4.13 A linear operator L with $r \ge 0$ and $\forall s \in \mathbb{R}$, there exists a constant $C_s > 0$ such that

$$\|L\phi\|_{s,p} \le C_s\,\|\phi\|_{s+r,p}, \qquad \forall\phi \in \mathcal{G}_\alpha^\infty. \tag{4.19}$$

Then r is called the order of L and infimum of all orders r is called true order of L.

Definition 4.14 Let $\psi_r\left(\left(\frac{1}{t}D\right)a(x,D)\right)$ be a linear operator from $\forall\,\phi \in \mathcal{G}_\alpha^\infty$ into itself and satisfies the following inequality:

$$\left\|\psi_r\left(\frac{1}{t}D\right)a(x,D)\phi\right\|_{s,p} \le C\,\|\phi\|_{s+r,p}, \qquad \forall\,s \in \mathbb{R},\ \phi \in \mathcal{G}_\alpha^\infty(\mathbb{R}).$$

Then $\psi_r\left(\frac{1}{t}D\right)a(x,D)$ is said to be a canonical operator of degree r where $r \in \mathbb{R}$.

Definition 4.15 Let $(r_j)_0^\infty$ be a strictly decreasing sequence of real numbers and $\left\{\psi_{r_j}\left(\frac{1}{t}D\right)a_j(x,D)\right\}$ be a sequence of canonical operators of degree r_j corresponding to a sequence of positive real numbers $(t_j)_0^\infty$ and to a sequence of symbols $\left\{a_j(x,\xi)\right\}_0^\infty \in \mathcal{G}$. A linear operator M : $\mathcal{G}_\alpha^\infty \to \mathcal{G}_\alpha^\infty$ is asymptotically expanded into the series $\sum_{j=0}^\infty \psi_{r_j}\left(\frac{1}{t}D\right)a_j(x,D)$ if it is satisfies the following inequality:

$$t.o.\left[M - \sum_{j=0}^{N}\psi_{r_j}\left(\frac{1}{t_j}D\right)a_j(x,D)\right] < r_N.$$

Theorem 4.16 Let $\left\{a_j(x,\xi)\right\}_0^\infty$ be a sequence of symbols belonging in $\mathcal{G}$ and $(r_j)_0^\infty$ be strictly decreasing sequence of real numbers tend to $-\infty$. Then there exists a sequence of canonical operators $K_{j,\alpha}$ of degree r_j and a linear operator $A_{a,\alpha}$ in $\mathcal{G}_\alpha^\infty$ such that

1. $t.o.(A_{a,\alpha}) \le r_0$.
2. $A_{a,\alpha} \sim \sum_{j=o}^\infty K_{j,\alpha} \quad i.e \quad t.o.\left[A_{a,\alpha} - \sum_{j=o}^N K_{j,\alpha}\right] < r_N$.

Proof. The proof of above theorem is given in [22].

5 The continuous fractional wavelet transform

The continuous wavelet transform on the space of type -W has been studied by Pathak and Pandey [19], Upadhyay et al. [21] and found many important consequences.

Novel fractional wavelet transform was given by Jun et al. [20]. They applied novel fractional wavelet transform on $L^2(\mathbb{R})$ - space and got many important properties. Prasad et al. [16], Prasad and Mahato [17] studied the generalized continuous wavelet transform associated with fractional Fourier transform on the $L^2(\mathbb{R})$ and W-type spaces, respectively and obtained important results.

In this section, the various characterization of *n*-dimensional fractional wavelet and wavelet transform are studied.
From [16, 20], we define *n*-dimensional fractional wavelet as given below

$$\psi^{\alpha}_{\sigma,a}(t) = a^{-n}\psi\left(\frac{t-\sigma}{a}\right) e^{\frac{-i(|t|^2-|\sigma|^2)\cot\alpha}{2}}, \tag{5.1}$$

where $a \in \mathbb{R}_+$ is called the scaling parameter which measure the degree of compression or scale and $\sigma \in \mathbb{R}^n$ is a translation parameter which determines the time location of the wavelet.
Proposition 5.1 If $\psi(t) \in L^2(\mathbb{R}^n)$, then $\psi^{\alpha}_{\sigma,a}(t) \in L^2(\mathbb{R}^n)$ for $a \in \mathbb{R}_+$ and $\sigma \in \mathbb{R}^n$.

Proof. The proof of above Proposition is given in [25].
Proposition 5.2 If $\psi \in L^2(\mathbb{R}^n)$, then the fractional Fourier transform of $\psi^{\alpha}_{\sigma,a}(t)$ is given as

$$\begin{aligned} \hat{\psi}^{\alpha}_{\sigma,a,\alpha}(x) &= e^{\frac{i(|x|^2+|\sigma|^2)\cot\alpha}{2} - i\langle x,\sigma\rangle\csc\alpha - \frac{ia^2|x|^2\cot\alpha}{2}} \\ &\quad \times F_{\alpha}\left(e^{\frac{-i|.|^2\cot\alpha}{2}}\psi\right)(ax), \end{aligned} \tag{5.2}$$

where $\hat{\psi}^{\alpha}_{\sigma,a,\alpha}$ is the fractional Fourier transform of $\psi^{\alpha}_{\sigma,a}$.
Proof. The proof of above Proposition is given in [25].
Lemma 5.3 (Parseval's Identity). If $\phi, \psi \in L^2(\mathbb{R}^n)$, we have the following equalities

$$\int_{\mathbb{R}^n} \phi(t)\overline{\psi(t)}dt = K(\alpha)\int_{\mathbb{R}^n} \widehat{\phi}_{\alpha}(x)\overline{\widehat{\psi}_{\alpha}(x)}dx. \tag{5.3}$$

If $\phi(t) = \psi(t)$, then (5.3) becomes

$$\int_{\mathbb{R}^n} |\phi(t)|^2 dt = K(\alpha)\int_{\mathbb{R}^n} |\hat{\phi}_{\alpha}(x)|^2 dx, \tag{5.4}$$

where $K(\alpha) = [\pi(1 - e^{-2i\alpha})]^n$.

Proof. The proof of above Lemma is given in [25].

Lemma 5.4 If $f, \psi \in {}^{\vee 2}(\mathbb{R}^n)$, then the continuous fractional wavelet transform W^{α}_{ψ} associated with fractional Fourier transform is defined by

$$(W_\psi^\alpha \phi)(\sigma, a) = K(\alpha)\int_{R^n} \hat{\phi}_\alpha(x) \overline{e^{\frac{i(|x|^2+|\sigma|^2)\cot\alpha}{2} - i\langle x,\sigma\rangle \csc\alpha - \frac{ia^2|x|^2\cot\alpha}{2}}} \times \Psi_\alpha(ax)\,dx, \tag{5.5}$$

where

$$\Psi_\alpha(ax) = \overline{F_\alpha\left(e^{\frac{-i|.|^2\cot\alpha}{2}}\psi\right)}(ax). \tag{5.6}$$

Proof. The proof of above Lemma is given in [25].

Lemma 5.5 Let $\overline{e^{\frac{-i|t|^2\cot\alpha}{2}}\psi(t)} \in W_M(\mathbb{R}^n)$. Then $\Psi_\alpha(az) \in W^\Omega(\mathbb{C}^n)$ where $\Psi_\alpha(az) = F_\alpha\left(e^{\frac{-i|.|^2\cot\alpha}{2}}\psi\right)(az)$.

Proof. The proof of above Lemma is given in [25].

Theorem 5.6 Let $M(x)$ and $\Omega(y)$ be the functions, which are dual in sence of Young and $\hat{\phi}_\alpha, \Psi_\alpha \in W^\Omega(\mathbb{C}^n)$. Then the fractional wavelet transformation $W_\psi^\alpha : W_M(\mathbb{R}^n) \to W_M(\mathbb{R}^n \times \mathbb{R}_+)$, is continuous and linear.

Proof. The proof of above Theorem is given in [25].

Lemma 5.7 Let $\phi \in W_M(\mathbb{R}^n)$. Then for $\beta \in \mathbb{N}_0^n$ there exists $C > 0$, one has the following relation

$$|D_x^\beta \hat{\phi}_\alpha(x)| \leq C\ e^{-M\left[\left(\frac{\csc\alpha}{\rho}\right)x\right]}, \quad \rho > 0. \tag{5.7}$$

Proof. The proof of above Lemma is given in [25].

Theorem 5.8 Let $M(x)$ and $\Omega(y)$ be the functions which are dual in the sense of Young and $\hat{\phi}_\alpha(x), \Psi_\alpha(ax) \in W_M(\mathbb{R}^n)$ then the fractional wavelet transformation $W_\psi^\alpha \phi$ is a continuous linear mapping from $W^\Omega(\mathbb{C}^n)$ into $W^\Omega(\mathbb{C}^n \times \mathbb{R}_+)$.

Proof. The proof of above Theorem is given in [25].

Lemma 5.9 Let $\hat{\phi}_\alpha(z) \in W_{M_0}^{\Omega_0}(\mathbb{C}^n)$ and $\Psi_\alpha(az) \in W_{M_0}^{\Omega_0}(\mathbb{C}^n)$. Then one has the following relation,

$$|\Phi_\alpha(z, a)| \leq Ce^{-M_0[(a-1)\rho x] + \Omega_0[(a+1)\delta y]}, \tag{5.8}$$

where $\Phi_\alpha(z, a) = \hat{\phi}_\alpha(z)\Psi_\alpha(az)$.

Proof. See [21, p. 247].

Theorem 5.10 Let $\Omega_0(y)$ and $M_0(x)$ be the functions which are dual in the sense of Young to the functions $M(x)$ and $\Omega(y)$ respectively. Suppose $\phi \in W_M^\Omega(\mathbb{C}^n)$ and $e^{\frac{-i|.|^2\cot\alpha}{2}}\psi \in W_M^\Omega(\mathbb{C}^n)$, then the fractional wavelet transformation W_{ψ}^{α} is a continuously linear mapping from $W_M^{\Omega}(\mathbb{C}^n)$ into $W_M^{\Omega}(\mathbb{C}^n \times \mathbb{R}_+)$.

Proof. The proof of above Theorem is given in [25].

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