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Additional Characterizations of Separation Axioms Using T_0-Identification Spaces and Subspaces

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Abstract

Within this paper recent characterizations of T_i; $i = 1, 2$, R_i; $i = 0, 1$, and Urysohn spaces using proper subspaces are combined with T_0-identification spaces to further characterize each of R_0, R_1, and weakly Urysohn spaces.

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1 Introduction

When studying a property of topological spaces the following question often arises: "Does the space have the property if and only if each subspace of the space has the property?, i.e., is the property a subspace property? In a recent paper [2], it was observed that the proof of the converse statement in subspace theorems is quick and easy simply citing that a space is a subspace of itself and that the property itself plays no role in the proof. In response proper subspaces were examined for subspace properties and other properties giving the properties themselves a new, central role in subspace questions [2].

Definition 1.1. Let (X, T) be a space and let P be a property of topological spaces. If the fact that every proper subspace of (X, T) has property P implies (X, T) has property P, then P is called a proper subspace inherited property (psip) [2].

In 1943 [8] T_1 spaces were generalized to R_0 spaces.

Definition 1.2. A space (X,T) is R_0 if and only if for each closed set C and each $x \notin C$, $C \cap Cl(\{x\}) = \phi$ [8].

In 1961 [1] A. Davis was searching for properties, which together with T_{i-1}, would be equivalent to T_i; $i = 1, 2$, which led to the rediscovery of R_0 spaces and the introduction and investigation of R_1 spaces. Within the 1961 paper [1], it was proven that a space is T_i if and only if it is T_{i-1} and R_{i-1}; $i = 1, 2$.

Definition 1.3. A space (X,T) is R_1 if and only if for each $x, y \in X$ such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $Cl(\{x\}) \subseteq U$ and $Cl(\{y\}) \subseteq V$ [1].

Within the 1961 paper [1], the R_0 separation axiom was equivalently defined by: A space (X,T) is R_0 if and only if one of the following equivalent statements are true: (a) for each $O \in T$ and each $x \in O$, $Cl(\{x\}) \subseteq O$ and (b) $\{Cl(\{x\}) \mid x \in X\}$ is a decomposition of X.

In 1975 [7] R_1 spaces were further studied under the name weakly Hausdorff spaces. In that paper [7] it was proven that a space is weakly Hausdorff if and only if its T_0-identification space is T_2.

Definition 1.4. Let R be the equivalence relation on a space (X,T) defined by xRy if and only if $Cl(\{x\}) = Cl(\{y\})$. The T_0-identification space of (X,T) is $(X_0, Q(X,T))$, where X_0 is the set of equivalence classes of R and $Q(X,T)$ is the decomposition topology on X_0 [9]. For each $x \in X$, let C_x denote the R equivalence class containing x and let $P_X : (X,T) \to (X_0, Q(X,T))$ be the natural map.

In 1977 [3] R_0 spaces were further characterized using T_0-identification spaces.

Theorem 1.1. A space is R_0 if and only if its T_0-identification is T_1 [3].

In 1988 [4] Urysohn spaces were generalized to weakly Urysohn spaces and it was proven that a space is weakly Urysohn if and only if its T_0-identification space is Urysohn.

Definition 1.5. A space (X,T) is weakly Urysohn if and only if for $x, y \in X$ such that $Cl(\{x\}) \neq Cl(\{y\})$ there exist open sets U and V such that $Cl(\{x\}) \subseteq U$, $Cl(\{y\}) \subseteq V$ and $Cl(U) \cap Cl(V) = \phi$.

Within the paper [2] it was proven that each of the subspace properties T_i and R_{i-1}, $i = 1, 2,$, weakly Urysohn, and Urysohn are psip and the following results were obtained.

Theorem 1.2. Let (X,T) be a space and let P be any of the properties T_i, $i = 0,1,2$, R_i; $i = 0,1$, weakly Urysohn, or Urysohn. Then (X,T) has property P if and only if every proper subspace of (X,T) has property P [2].

Within a follow up paper [5] the results above were used to further characterize T_i; $i = 0,1,2$, and Urysohn spaces, which will be used below along with T_0-identification spaces to obtain additional characterizations of R_i; $i = 0,1$, and weakly Urysohn spaces.

Since singleton sets satisfy many properties, in the works cited above and here, only spaces with three or more elements are considered. Below, for those theorems in which T_0-identification spaces are used, only spaces (X,T) for which X_0 has three or more elements are considered.

2 New Characterizations of R_0 Spaces.

Theorem 2.1. Let (X,T) be a space. Then the following are equivalent:

(a) (X,T) is R_0,

(b) for each $x \in X$ and $Y = X \setminus \{x\}$, (Y,T_Y) is R_0,

(c) for each $Y \subseteq X$, for each $O \in T_Y$ and for each $x \in O$, $Cl_{T_Y}(\{x\}) \subseteq O$,

(d) for each $Y \subseteq X$, $\{Cl_{T_Y}(\{x\}) \mid x \in Y\}$ is a decomposition of Y,

(e) for each subset $\mathcal{Z}$ of X_0, $(\mathcal{Z}, Q(X,T)_{\mathcal{Z}})$ is T_1, and

(f) for each proper subset $\mathcal{Z}$ of X_0, $(\mathcal{Z}, Q(X,T)_{\mathcal{Z}})$ is T_1.

Proof: By the results above, (a) implies (b).

(b) implies (c): Let Z be a proper subset of X. Let $x \in X \setminus Z$ and let $Y = X \setminus \{x\}$. Then (Z, T_{Y_Z}) is a subspace of the R_0 space (Y,T_Y) and is R_0. Since $T_{Y_Z} = T_Z$, (Z,T_Z) is R_0. Hence every proper subspace of (X,T) is R_0 and (X,T) is R_0.

Let $Y \subseteq X$. Since (X,T) is R_0, then (Y,T_Y) is R_0 and for each $O \in T_Y$ and each $x \in O$, $Cl_{T_Y}(\{x\}) \subseteq O$.

By the results above, (c) implies (d).

(d) implies (e): By the results above, for each $Y \subseteq X$, (Y,T_Y) is R_0, which implies that (X,T) is R_0. Thus $(X_0, Q(X,T))$ is T_1 and every subspace of $(X_0, Q(X,T))$ is T_1.

Clearly (e) implies (f).

(f) implies (a): Since every proper subspace of $(X_0, Q(X,T))$ is T_1, $(X_0, Q(X,T))$ is T_1 and (X,T) is R_0.

Corollary 2.1. Let (X,T) be a space. Then the following are equivalent:

(a) (X,T) is R_0,

(b) for each proper subset Y of X, for each $O \in T_Y$ and for each $x \in O$, $Cl_{T_Y}(\{x\}) \subseteq O$, and

(c) for each proper subset Y of X, $\{Cl_{T_Y}(\{x\}) \mid x \in X\}$ is a decomposition of Y.

Theorem 2.2. Let (X,T) be a space. Then the following are equivalent:

(a) (X,T) is R_0,

(b) for each subset Y of X, $(P_X(Y), Q(X,T)_{P_X(Y)})$ is T_1, and

(c) for each proper subset Y of X, $(P_X(Y), (Q(X,T)_{P_X(Y)})$ is T_1.

Proof: (a) implies (b): Let Y be a subset of X. Then $P_X(Y)$ is a subset of the T_1 space $(X_0, Q(X,T))$ and $(P_X(Y), Q(X,T)_{P_X(Y)})$ is T_1.

Clearly (b) implies (c).

(c) implies (a): Let Z be a proper subset of X_0 and let $Y = P_X^{-1}(\mathcal{Z})$. Then Y is a proper subset of X, $\mathcal{Z} = P_X(Y)$, and $(\mathcal{Z}, Q(X,T)_{\mathcal{Z}})$ is T_1. Thus, by Theorem 2.1, (X,T) is R_0.

Theorem 2.3. Let (X,T) be a space. Then the following are equivalent:

(a) (X,T) is R_0,

(b) for each subset Y of X, $(Y_0, Q(Y,T_Y))$ is T_1, and

(c) for each proper subset Y of X, $(Y_0, Q(Y,T_Y))$ is T_1.

Proof: (a) implies (b): Let Y be a subset of X. Then (Y,T_Y) is R_0 and $(Y_0, Q(Y,T_Y))$ is T_1.

Clearly (b) implies (c).

(c) implies (a): Let Y be a proper subset of X. Since $(Y_0, Q(Y,T_Y))$ is T_1, (Y,T_Y) is R_0. Hence (X,T) is R_0.

Since a space is R_0 if and only if its T_0-identification space is R_0 [3], additional characterizations of R_0 spaces can be obtained by replacing T_1 in Theorems 2.1, 2.2, and 2.3 above by R_0.

Theorem 2.4. Let (X,T) be a space. Then the following are equivalent:

(a) (X,T) is R_0,

(b) for each finite subset $\mathcal{Y} = \{Ci \mid i = 1, \cdots, n\}$ of X_0, $Q(X,T)_{\mathcal{Y}}$ is the discrete topology on $\mathcal{Y}$,

(c) for distinct elements C and D in X_0 and $\mathcal{Y} = \{C, D\}$, $Q(X,T)_{\mathcal{Y}}$ is the discrete topology on $\mathcal{Y}$,

(d) for each finite proper subset $Y = \{x_i \mid i = 1, \cdots, n\}$ of X such that $Cl(\{x_i\}) = Cl(\{x_j\})$ if and only if $i = j$, $n \geq 2$, T_Y is the discrete topology on Y, and

(e) for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$ and $Y = \{x, y\}$, T_Y is the discrete topology on Y.

Proof: (a) implies (b): Since (X,T) is R_0, $(X_0, Q(X,T))$ is T_1. Then, by Theorem 2.4 in the paper [5], (b) follows.

Clearly (b) implies (c).

(c) implies (d): By Theorem 2.4 in [3], $(X_0, Q(X,T))$ is T_1. Thus (X,T) is R_0 and (d) follows from Theorem 2.4 in the paper [6].

Clearly (e) follows from (d).

(e) implies (a): Let $O \in T$ and let $x \in O$. Let $y \notin O$. Then $Cl(\{x\}) \neq Cl(\{y\}$ and there exists a T-open set U such that $y \in U$ and $x \notin U$. Hence $y \notin Cl(\{x\})$ and $Cl(\{x\}) \subseteq O$. Thus (X,T) is R_0.

Corollary 2.2. Let (X,T) be a space. Then the following are equivalent:

(a) (X,T) is R_0,

(b) for each proper subset Z of X and finite subset $Y = \{x_i \mid i = 1, \cdots, n\}$ of Z such that $Cl_{T_Z}(\{x_i\}) = Cl_{T_Z}(\{x_j\})$ if and only if $i = j$, T_{ZY} is the discrete topology on Y, and

(c) for each proper subset Z of X and subset $Y = \{x, y\}$ of Z such that $Cl_{T_Z}(\{x\}) \neq Cl_{T_Z}(\{y\})$, T_{ZY} is the discrete topology on Y.

T_0-identification spaces are a clever mathematical creation that has been greatly utilized. With the focus in this paper on subspaces, below the relationship between the spaces $(P_X(Y), Q(X,T)_{P_X(Y)})$ and $(Y_0, Q(Y, T_Y))$ for a space (X,T) and subset Y of X is resolved. The proof uses results from category theory as required by the referee.

Theorem 2.5. Let (X,T) be a space and let Y be a subset of X. Then $(P_X(Y), Q(X,T)_{P_X(Y)})$ and $(Y_0, Q(Y,T_Y))$ are homeomorphic.

Proof: Given a subspace (Y, T_Y) of (X,T),

$(P_X)_Y : (Y, T_Y) \to (P_X(Y), Q(X,T)_{P_X(Y)})$ is a T_0-epireflection of (Y, T_Y) since, given any continuous function $f : (X,T) \to (Z,W)$, (Z,W) being T_0, the function $f^* : (P_X(Y), Q(X,T)_{P_X(Y)}) \to (Z,W)$, defined by $f^*(P_X(y)) = f(y)$, $y \in Y$, is verified to be well-defined and unique with the property $f^* \circ P_X = f$.

3 New Characterizations of R_1 and Weakly Urysohn Spaces.

Theorem 3.1. Let (X,T) be a space. Then the following are equivalent:

(a) (X,T) is R_1,

(b) for each $x \in X$ and $Y = X\backslash\{x\}$, (Y,T_Y) is R_1,

(c) for each subset $\mathcal{Z}$ of X_0, $(\mathcal{Z}, Q(X,T)_{\mathcal{Z}})$ is T_2,

(d) for each proper subset $\mathcal{Z}$ of X_0, $(\mathcal{Z}, Q(X,T)_{\mathcal{Z}})$ is T_2,

(e) for each subset Y of X, $(Y_0, Q(Y,T_Y))$ is T_2,

(f) for each proper subset Y of X, $(Y_0, Q(Y,T_Y))$ is T_2,

(g) for each subset $\mathcal{Z}$ of X_0 and each finite set $\mathcal{Y} = \{C_i \mid i = 1,\cdots,n\}$; $n \geq 2$, of distinct elements of $\mathcal{Z}$, there exist disjoint $Q(X,T)_{\mathcal{Z}}$-open sets $\mathcal{O}_i$; $i = 1,\cdots,n\}$, such that $C_i \in \mathcal{O}_i$ for each i,

(h) for each subset $\mathcal{Z}$ of X_0, for distinct elements C_i; $i = 1,2$, of $\mathcal{Z}$, there exist disjoint $Q(X,Y)_{\mathcal{Z}}$-open sets $\mathcal{O}_i$, $i = 1,2$, such that $C_i \in \mathcal{O}_i$; $i = 1,2$,

(i) for each proper subset $\mathcal{Z}$ of X_0 and distinct elements C_i; $i = 1,2$, of $\mathcal{Z}$, there exist disjoint $Q(X,T)_{\mathcal{Z}}$-open sets $\mathcal{O}_i$; $i = 1,2$, such that $C_i \in \mathcal{O}_i$; $i = 1,2$,

(j) for each subset Z of X and each finite set $Y = \{x_i \mid i = 1,\cdots,n\}$; $n \geq 2$, of Z such that $Cl_T(\{x_i\}) = Cl_T(\{x_j\})$ if and only if $i = j$, there exist disjoint T_Z-open sets O_i; $i = 1,\cdots,n\}$, such that $x_i \in O_i$ for each i, and

(k) for each proper subset Z of X and elements x_i; $i = 1,2$, in Z such that $Cl_T(\{x_1\}) \neq Cl_T(\{x_2\})$, there exist disjoint T_Z-open sets O_i; $i = 1,2$, such that $x_i \in O_i$; $i = 1,2$.

Proof: By the results in the introduction, (a) implies (b).

(b) implies (c): Let Z be a proper subset of X. Let $x \in X\backslash Z$ and let $Y = X\backslash\{x\}$. Then (Z,T_Z) is a subspace of the R_1 space (Y,T_Y), which implies (Z,T_Z) is R_1. Hence, by the results above, (X,T) is R_1, $(X_0, Q(X,Y))$ is T_2, and for each subset $\mathcal{Z}$ of X_0, $(\mathcal{Z}, Q(X,T)_{\mathcal{Z}}))$ is T_2.

Clearly (c) implies (d).

(d) implies (e): By the results above, $(X_0, Q(X,T))$ is T_2 and every subspace of $(X_0, Q(X.T))$ is T_2. Let $Y \subseteq X$. Then $(P_X(Y), Q(X,T)_{P_X(Y)})$ is T_2, and, by Theorem 2.6 above, $(Y_0, Q(Y,T_Y))$ is T_2.

Clearly (e) implies (f).

(f) implies (g): Let Y be a proper subset of X. Since $(Y_0, Q(Y,T_Y))$

is T_2, then (Y, T_Y) is R_1. Thus every proper subspace of (X,T) is R_1, which implies (X,T) is R_1. Then $(X_0, Q(X,Y))$ is T_2, which implies every subspace of $(X_0, Q(X,T))$ is T_2, and (g) follows immediately by Theorem 2.5 in [5].

Clearly (g) implies (h) and (h) implies (i).

(i) implies (j): By Theorem 2.5 in [5], $(X_0, Q(X,T))$ is T_2. Thus (X,T) is R_1. Let Z be a proper subset of X. Then (Z, T_Z) is R_1 and (j) follows immediately by Theorem 2.6 in [6].

Clearly (j) implies (k).

(k) implies (a): Let $\mathcal{Z}$ be a proper subset of X_0 and let C_{x_i}; $i = 1, 2$, be distinct element of X_0. Then $Z = P_X^{-1}(\mathcal{Z})$ is a proper subset of X and $Cl_T(\{x_1\}) \neq Cl_T(\{x_2\})$. Let O_i; $i = 1, 2$ be disjoint T_Z-open sets such that $x_i \in O_i$; $i = 1, 2$. Then $P_X(O_i)$; $i = 1, 2$, are disjoint $Q(X,T)_{P_X(Y)}$-open sets containing C_i; $i = 1, 2$, respectively, and by the arguments above (X,T) is R_1.

Theorem 3.2. Let (X,T) be a space. Then the following are equivalent:

(a) (X,T) is weakly Urysohn,

(b) for each $x \in X$ and $Y = X \backslash \{x\}$, (Y, T_Y) is weakly Urysohn,

(c) for each subset $\mathcal{Z}$ of X_0, $(\mathcal{Z}, Q(X, T_{\mathcal{Z}})$ Urysohn,

(d) for each proper subset $\mathcal{Z}$ of X_0, $(\mathcal{Z}, Q(X,T)_{\mathcal{Z}})$ is Urysohn,

(e) for each subset Y of X, $(Y_0, Q(Y, T_Y))$ is Urysohn,

(f) for each proper subset Y of X, $(Y_0, Q(Y, T_Y))$ is Urysohn,

(g) for each subset $\mathcal{Z}$ of X_0 and each finite set $\mathcal{Y} = \{C_i \mid i = 1, \cdots, n\}$; $n \geq 2$, of distinct elements of $\mathcal{Z}$, there exist $Q(X,T)_{\mathcal{Z}}$-open sets $\mathcal{O}_i$; $i = 1, \cdots, n$, such that $C_i \in \mathcal{O}_i$ for each i and $Cl_{Q(X,T)_{\mathcal{Z}}}(\mathcal{O}_i) \cap Cl_{Q(X,T)_{\mathcal{Z}}}(\mathcal{O}_j) \neq \phi$ if and only if $i = j$,

(h) for each subset $\mathcal{Z}$ of X_0 and distinct elements C_i; $i = 1, 2$, in $\mathcal{Z}$, there exist $Q(X,T)_{\mathcal{Z}}$-open sets $\mathcal{O}_i$; $i = 1, 2$, such that $C_i \in \mathcal{O}_i$, $i = 1, 2$, and $Cl_{Q(X,T)_{\mathcal{Z}}}(\mathcal{O}_1) \cap Cl_{Q(X,T)_{\mathcal{Z}}}(\mathcal{O}_2) = \phi$,

(i) for each proper subset $\mathcal{Z}$ of X_0 and distinct elements C_i; $i = 1, 2$, of $\mathcal{Z}$, there exist $Q(X,T)_{\mathcal{Z}}$-open sets $\mathcal{O}_i$; $i = 1, 2$, such that $C_i \in \mathcal{O}_i$; $i = 1, 2$, and $Cl_{Q(X,T)_{calZ}}(\mathcal{O}_1) \cap Cl_{Q(X,T)_{\mathcal{Z}}}(\mathcal{O}_2) = \phi$,

(j) for each subset Z of X and each finite subset $Y = \{x_i \mid 1 = 1, \cdots, n\}$; $n \geq 2$, of Z such that $Cl_{T_Y}(\{x_i\}) \cap Cl_{T_Y}(\{x_j\}) \neq \phi$ if and only if $i = j$, there exist T_Y-open sets O_i; $i = 1, \cdots, n$, such that $x_i \in O_i$ for each i and $Cl_{T_Y}(O_i) \cap Cl_{T_Y}(O_j) \neq \phi$ if and only if $i = j$, and

(k) for each proper subset Z of X and elements x_i; $i = 1, 2$, of Z

such that $Cl_{T_Y}(\{x_1\}) \cap Cl_{T_Y}(\{x_2\}) = \phi$, there exist T_Y-open sets O_i; $i = 1, 2$, such that $x_i \in O_i$; $i = 1, 2$, and $Cl_{T_Y}(O_1) \cap Cl_{T_Y}(O_2) = \phi$.

The proof is similar to that of Theorem 3.1 using the results above, Theorem 2.8 in [4], and Theorem 2.6 in [5] and is omitted.

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