Ganita, Vol. 62, 2011, 1-13

On Matrix Transformations and Multipliers

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Abstract

In this paper, we characterize matrix transformations between weighted modular sequence spaces ℓ_α^M and the space ℓ^∞ of all bounded sequences, and also a general Fréchet $K-$space λ. In the final section, we identify the class of diagonal operators or multipliers between two modular sequence spaces as a modular sequence space.

Keywords : Orlicz functions, sequence spaces, matrix transformations, multipliers.

AMS Classification : 46A04, 46A45, 47B37.

1 Introduction

This paper deals with two aspects of studies on particular type of sequence spaces, namely, the matrix transformations and the multipliers which are, indeed, diagonal transformations. In order to understand the subject matter of this paper, let us consider a few notations, definitions and results on locally convex spaces, general sequence spaces, and in particular, Orlicz and modular sequence spaces from [1], [2], [4], [5], [6], [7] and [8].

We denote by (X, T), a locally convex space with X as a vector space over the field $\mathbb{K}$ of real or complex numbers and D_T as the family of semi-norms generating the topology T on X. In case, D_T is countable (X, T) is metrizable and if (X, T) is also complete, it is called a **Fréchet space**. In this paper, we consider vector spaces consisting of scalar sequences, namely, the sequence spaces.

To be precise, let us denote by ω the family of all real or complex sequences, which is a vector space with the usual pointwise addition and scalar multiplication. We write e^n $(n \geq 1)$ for the n^{th} unit vector in ω, i.e $e^n = \{\delta_{nj}\}_{j=1}^{\infty}$ where δ_{nj} is the Kronecker delta, and ϕ for the subspace of ω, generated by the e^n's, $n \geq 1$, i.e ϕ is the vector space of all finitely non-zero sequences. For a sequence $x = \{x_n\}$ in ω, we write its n^{th} **section** as $x^{(n)}$ which is defined as the sequence $(x_1, x_2, x_3, \ldots x_n, 0, 0, \ldots)$. Clearly, $x^{(n)} \in \phi$, for each $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers. For the sequences $x = \{x_i\}$ and $y = \{y_i\}$ in ω, we write xy for the sequence $\{x_i y_i\}$ in ω. A **sequence space** λ is a subspace of ω containing ϕ. A sequence space λ equipped with a linear topology T is called a $K-$**space** if its topology is finer than the co-ordinate wise convergence topology; and an $AK-$**space** if it is a $K-$space and $x^{(n)} \to x$ in T as $n \to \infty$, for each $x \in \lambda$. A $K-$space (λ, T) is an $FK-$**space**(resp. **Fréchet** $K-$**space**) if (λ, T) is also a Fréchet space(resp. if (λ, T) is complete and metrizable). For any sequence space λ, its β-**dual** λ^β is given by

$$\lambda^\beta = \{y = \{y_i\} \in \omega \ : \ \sum x_i y_i \text{ converges for all } \{x_i\} \in \lambda\}$$

Let λ and μ be two sequence spaces. An infinite matrix $A = (a_{ij})$ is said to be a **matrix transformation** from λ to μ if for each $x \in \lambda$, the series $\sum_{j \geq 1} a_{ij} x_j$ converges and if $y_i = \sum_{j \geq 1} a_{ij} x_j$, then $Ax = \{y_i\} \in \mu$. The collection of all matrix transformations from λ to μ is denoted by (λ, μ).

For an infinite matrix A, the set d_A defined by

$$d_A = \{ x : x \in \omega, \ Ax \text{ exists} \}$$

is called the domain of the matrix A. In case Ax also belongs to λ for a given sequence space λ, write λ_A for d_A and call it the **summability domain** of A; in particular, for $\lambda = c, c_0$ and ℓ^∞ the spaces $c_A, (c_0)_A$ and ℓ^∞_A are known as the **convergence domain, null domain** and **bounded domain** of A respectively.

Topologizing the space λ_A, we have the following result due to Zeller [9] (cf. also [2], p.209)

Theorem 1.1 *Let $A = (a_{ij})$ be an infinite matrix and (λ, T) be an $FK-$ space with $D_T = \{r_i\}$. Suppose S is the topology on λ_A generated by $\{p_i\}$, $\{q_i\}$ and $\{r_i \circ A\}$, where $p_i(x) = |x_i|$, $q_i(x) = sup_n |\sum_{j=1}^n a_{ij}x_j|$ and $r_i \circ A(x) = r_i(A(x))$ for $i \geq 1$. Then (λ_A, S) is an $FK-$space and the map $A : (\lambda_A, S) \to (\lambda, T)$ is continuous.*

It is known that if (λ, T) and (μ, S) are two Fréchet $K-$spaces such that $\lambda \subset \mu$, then the identity map $I : (\lambda, T) \to (\mu, S)$ is continuous, cf.[2], p.204.

Corresponding to a given sequence space λ, a series $\sum_{i\geq 1} x_i$ in a locally convex TVS (X, T) is said to be **weakly $\lambda-$unconditionally Cauchy** (w. $\lambda-$u.C) provided $\sum_{i\geq 1} \alpha_i x_i$ converges in X for each $\alpha \in \lambda$, cf.[2], p.185.

We make use of the following characterization of a w. $\lambda-$u.C. series.

Theorem 1.2 *Let λ be a monotone $FK - AK$ space and (X, T) be a sequentially complete locally convex space. Then the series $\sum_{i\geq 1} x_i$ in X is w. $\lambda-$u. C. if and only if $\{f(x_i)\} \in \lambda^\beta$ for all $f \in X^*$.*

An **Orlicz function** is a continuous, convex, non-decreasing function defined from $[0, \infty)$ to itself such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Such function M always has the following integral representation $M(x) = \int_0^x p(t)dt$, where p, known as the **kernel** of M, is right continuous for $t > 0, p(0) = 0, p(t) > 0$ for

$t > 0$, p is non- decreasing and $p(x) \to \infty$ as $x \to \infty$.

Given an Orlicz function M with kernel p, define $q(s) = \sup\{t : p(t) \leq s\}, s \geq 0$. Then q possesses the same properties as p and the function N defined as $N(x) = \int_0^x q(t)dt$, is an Orlicz function. The functions M and N are called **mutually complementary Orlicz functions.**

An Orlicz function M is said to satisfy the **Δ_2-condition** for small x or at '0' if for each $k > 1$, there exist $R_k > 1$ and $x_k > 0$ such that

$$M(kx) \leq R_k M(x)\ , \quad \forall\, x \in (0, x_k].$$

For a sequence $\{M_n\}$ of Orlicz functions, the modular sequence space $\ell\{M_n\}$ is defined as

$$\ell\{M_n\} = \{\ x \in \omega\ :\ \sum_{n \geq 1} M_n(\frac{|x_n|}{k}) < \infty, \quad \text{for some } k > 0\}.$$

The space $\ell\{M_n\}$ is a Banach space with respect to the norm $\|.\|_{\{M_n\}}$ defined as

$$\|x\|_{\{M_n\}} = \inf\{k > 0 : \sum_{n \geq 1} M_n(\frac{|x_n|}{k}) \leq 1\}.$$

An important subspace of $\ell\{M_n\}$, which is an AK-apace is the space $h\{M_n\}$ defined by

$$h\{M_n\} = \{x \in \ell\{M_n\} : \sum_{n \geq 1} M_n(\frac{|x_n|}{k}) < \infty, \quad \text{for each } k > 0\}.$$

In case, $M_n = M$ for each n, we write ℓ_M for $\ell\{M_n\}$, h_M for $h\{M_n\}$ and $\|x\|_{(M)}$ for $\|x\|_{\{M_n\}}$. Further, $h_M = \ell_M$ iff M satisfies Δ_2-condition at '0'. If N is an Orlicz function complementary to M, then an equivalent definition of ℓ_M is also given as

$$\ell_M = \{x \in \omega : \sum_{i \geq 1} x_i y_i \text{ converges for each } \{y_i\} \text{ with } \delta(y, N) = \sum_{i \geq 1} N(|y_i|) < \infty\}$$

and an equivalent norm $\|.\|_M$ is defined as

$$\|x\|_M = \sup\{|\sum_{i \geq 1} x_i y_i| : \delta(y, N) \leq 1\}$$

A sequence $\{M_n\}$ of Orlicz functions is said to satisfy **uniform Δ_2-condition** at '0' if there exists $p \geq 1$ and $n_0 \in \mathbb{N}$ such that for all $x \in (0,1)$ and $n > n_0$ we have $\frac{xM_n'(x)}{M_n(x)} \leq p$. The subspace $h\{M_n\}$ is a closed subspace of $\ell\{M_n\}$ with $\{e^n\}$ as its Schauder basis. Further, $h\{M_n\} = \ell\{M_n\}$ if and only if $\{M_n\}$ satisfies uniform Δ_2-condition at '0'; cf. [2], p.299.

A sequence $\sigma = \{\sigma_n\}$ is called a **multiplier between the modular sequence spaces** $\ell\{M_n\}$ and $\ell\{N_n\}$ if for each $x = \{x_n\} \in \ell\{M_n\}$ we have $\sigma x = \{\sigma_n x_n\} \in \ell\{N_n\}$, where $\{M_n\}$ and $\{N_n\}$ are sequences of Orlicz functions.

We now recall from [3], the spaces ℓ_α^M and ℓ_N^α, where $\alpha = \{\alpha_i\}$ is a fixed sequence of strictly positive reals and M and N are complementary Orlicz functions. Indeed, we have

$$\ell_\alpha^M = \{ x \in \omega : \sum_{i\geq 1} M(\frac{|x_i|}{\alpha_i \rho}) < \infty \text{ for some } \rho > 0\}$$

and

$$\ell_N^\alpha = \{ x \in \omega : \sum_{i\geq 1} N(\frac{\alpha_i |x_i|}{\rho}) < \infty \text{ for some } \rho > 0\}.$$

The functions $\|.\|_\alpha^M : \ell_\alpha^M \to \mathbb{R}^+$ and $\|.\|_N^\alpha : \ell_N^\alpha \to \mathbb{R}^+$ defined by

$$\|x\|_\alpha^M = \inf \{\rho > 0 : \sum_{n\geq 1} M(\frac{|x_n|}{\rho\alpha_n}) \leq 1\}$$

$$\|x\|_N^\alpha = \inf \{\rho > 0 : \sum_{n\geq 1} N(\frac{\alpha_n |x_n|}{\rho}) \leq 1\}$$

are norms on ℓ_α^M and ℓ_N^α respectively. Note that $x \in \ell_\alpha^M$ iff $\frac{x}{\alpha} = \{\frac{x_i}{\alpha_i}\} \in \ell_M$. Also if M satisfies Δ_2-condition at '0' $(\ell_\alpha^M, \|\cdot\|_\alpha^M)$ is an $AK - BK$ space and $(\ell_\alpha^M)^\beta = \ell_N^\alpha$.

2 Matrix Transformation

Let M and N be complementary Orlicz functions where M satisfies Δ_2-condition at '0', and λ be any sequence space equipped with a

Fréchet topology. In this section, we characterize members of the space $(\ell_\alpha^M, \ell^\infty)$and (ℓ_α^M, λ).

Let us begin with a simple

Lemma 2.1 *The sets* $\{x \in \ell_\alpha^M : \|x\|_\alpha^M \le 1\}$ *and* $\{x \in \ell_\alpha^M : \delta(\frac{x}{\alpha}, M) \le 1\}$ *in the space* ℓ_α^M *are the same.*

Proof : Omitted.

We now prove

Theorem 2.2 *Let* $a^i = \{a_{ij}\}_{j=1}^\infty$ *denote the* i^{th} *row of an infinite matrix* $A = (a_{ij})$. *Then* $A \in (\ell_\alpha^M, \ell^\infty)$ *iff (I)* $a^i \in \ell_N^\alpha$, *(II)* $K = sup\|\alpha a^i\|_N < \infty$; *and* $\|A\| \le K$.

Proof : Let $A \in (\ell_\alpha^M, \ell^\infty)$. Then for $x = \{x_i\} \in \ell_\alpha^M$, $Ax = \{y_i\} \in \ell^\infty$, where $y_i = \sum_{j=1}^\infty a_{ij}x_j$. Since $\sum_{j=1}^\infty a_{ij}x_j$ converges for all $x \in \ell_\alpha^M$, $\alpha a^i \in (\ell_M)^\beta = \ell_N$; cf. [2], p.311. Hence $a^i \in \ell_N^\alpha$. For proving (II), define $F_i : \ell_\alpha^M \to \mathbb{K}$ by $F_i(x) = \sum_{j=1}^\infty a_{ij}x_j$, $i \in \mathbb{N}$. Then F_i's are pointwise bouneded as $\{y_i\} \in \ell^\infty$ and by Lemma 2.1,

$$\begin{aligned} \|F_i\| &= \sup\{\ |F_i(x)| \ : \ \|x\|_\alpha^M \le 1\} \\ &= \sup\{\ |\sum \alpha_j a_{ij} z_j| \ : \ \delta(z, M) \le 1\} \\ &= \sup\|\alpha a^i\|_N. \end{aligned}$$

Now apply uniform boundedness principle to conclude (II).

Conversely, suppose (I) and (II) are true. Then for $x \in \ell_\alpha^M$,

$$\sum_j |a_{ij}x_j| \le \sum N(\frac{|\alpha_j a_{ij}|}{\rho}) + \sum M(\rho\frac{|x_j|}{\alpha_j}) < \infty.$$

where $\rho > 0$ is such that $\sum N(\frac{|\alpha_j a_{ij}|}{\rho}) < \infty$. Thus $\sum_j a_{ij}x_j$ converges and if $y_i = \sum_j a_{ij}x_j$, $i \in \mathbb{N}$, then $\{y_i\} \in \ell^\infty$ by (II). Indeed,

$$|\sum_{i\ge1} a_{ij}x_i| \le \begin{cases} \|\alpha a^i\|_N, & \text{if } \delta(\frac{x}{\alpha}, M) \le 1 \\ \delta(\frac{x}{\alpha}, M)\|\alpha a^i\|_N, & \text{if } \delta(\frac{x}{\alpha}, M) > 1 \end{cases}$$

cf. [2], p.299. Hence

$$\|Ax\|_\infty = \sup_{i\ge1} |y_i| \le \max\{K, \delta(\frac{x}{\alpha}, M)K\} < \infty$$

$\Rightarrow \|A\| \leq K$, by Lemma 2.1.

Note 2.3 It is natural to ask in the above theorem whether we can replace ℓ^∞ by a general sequence space λ equipped with a suitable linear topology. This is answered in the last part of this section, namely Theorem 2.9. However, we make some preparation for proving this result.

Let us begin with

Propostion 2.4 *Let (λ, T) be a sequentially complete $K-$space such that the Banach space ℓ_α^M is continuously embedded in λ, i.e. $\ell_\alpha^M \subset \lambda$ and the inclusion map from ℓ_α^M to λ is continuous. Then $\{f(e^i)\} \in \ell_N^\alpha$ for all $f \in \lambda^*$.*

Proof : Since $\ell_\alpha^M = h_\alpha^M$ and h_α^M is an $AK-BK$ space with $\{e^i\}$ as a Schauder basis; for $\{e^i\} \subset \lambda$ and $x = \{x_n\} \in \ell_\alpha^M$, the series $\sum x_i e^i$ converges to x in ℓ_α^M. Hence $\{f(e^i)\} \in \ell_N^\alpha$ for all $f \in \lambda^*$.

Propostion 2.5 *Let (λ, T) be a sequentially complete $K-$space such that $\{f(e^i)\} \in \ell_N^\alpha$ for each $f \in \lambda^*$. Then $\ell_\alpha^M \subset \lambda$*

Proof : Let $x = \{x_n\} \in \ell_\alpha^M$. Since $x = \sum_{i \geq 1} x_i e^i$ in ℓ_α^M and $\{f(e^i)\} \in (\ell_\alpha^M)^\beta = \ell_N^\alpha$ for each $f \in \lambda^*$, the series $\sum_{i \geq 1} x_i e^i$ converges in λ by Theorem 1.2. Hence $\ell_\alpha^M \subset \lambda$.

Combining the above propositions and the result mentioned after Theorem 1.1, we conclude

Theorem 2.6 *Let (λ, T) be a Fréchet $K-$space. Then $\ell_\alpha^M \subset \lambda$ iff $\{f(e^i)\} \in \ell_N^\alpha$ for each $f \in \lambda^*$.*

For the final result of this section, we now prove a variation of Theorem 1.1, which is also of independent interest

Theorem 2.7 *Let (λ, T) be a sequentially complete $K-$space with D_T as the family of semi-norms generating the topology T; and $A = (a_{ij})$ be an infinite matrix. If S is the topology generated by $\{p_i\}$, $\{q_i\}$ and $\{r.A : r \in D_T\}$ on λ_A, then (λ_A, S) is a sequentially complete $K-$space and $A : (\lambda_A, S) \to (\lambda, T)$ is continuous. (Here p_i, q_i are the same as in Theorem 1.1).*

Proof : (λ_A, S) is already a $K-$space. For proving sequential completeness of (λ_A, S), consider a Cauchy sequence $\{u^n\}$ in λ_A. Then for each $i \in \mathbb{N}$, $\{u_i^n\}$ is Cauchy in the field $\mathbb{K}$ and so for each i, let $u = \{u_i\}$. Further for $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that

$$q_i(u^n - u^m) = \sup_k |\sum_{j=1}^{k} a_{ij}(u_j^n - u_j^m)| < \varepsilon$$

and

$$r.A(u^n - u^m) = r(Au^n - Au^m) < \varepsilon, \quad \text{for all} \quad n, m \geq n_0.$$

Thus

$$q_i(u^n - u) = \sup_k |\sum_{j=1}^{k} a_{ij}(u_j^n - u_j)| \leq \varepsilon, \quad \text{for all} \quad n \geq n_0 \qquad (1)$$

and $\{Au^n\}$ is Cauchy in λ. Since (λ, T) is sequentially complete, there exists $z = \{z_i\}$ in λ such that $Au^n \to z \Rightarrow \sum_{j=1}^{\infty} a_{ij}u_j^n \to z_i$ as $n \to \infty$ for each $i \in \mathbb{N}$.

We now show that $z_i = \sum_{j=1}^{\infty} a_{ij}u_j$ for each $i \in \mathbb{N}$. Note that the series $\sum_{j=1}^{\infty} a_{ij}u_j$ is convergent for each $i \in \mathbb{N}$ in view of (1). Also

$$\begin{aligned} |\sum_{j=1}^{\infty} a_{ij}u_j - z_i| &\leq \sup_m |\sum_{j=1}^{m} a_{ij}(u_j - u_j^n)| + |\sum_{j=1}^{\infty} a_{ij}u_j^n - z_i| \\ &\leq \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

Hence $z_i = \sum_{j=1}^{\infty} a_{ij}u_j$ and so $Au = z$. Consequently, $u \in \lambda_A$ and $u^n \to u$ in λ_A. This completes the proof.

Note 2.8 If S^* is the topology on λ_A, generated by the family of semi-norms $\{r.A : r \in D_T\}$, then one can easily check that (λ_A, S^*) is sequentially complete if (λ, T) is sequentially complete space and A is a bijective map from λ_A to λ. However, if (λ, T) is also metrizable, then both the topologies S and S^* are metrizable and in this case $S = S^*$. Further in this case the inductive topology T_I on λ for which the bijective map $A : \lambda_A \to \lambda$, is continuous would coincide with the original topology T of λ.

Finally. we prove

Theorem 2.9 *Let $A = (a_{ij})$ be an infinite matrix and (λ, T) be a sequentially complete $K-$space. Consider the following conditions*

(i) $A \in (\ell_\alpha^M, \lambda)$

(ii) $\{f(a^i)\} \in \ell_N^\alpha$ *for all* $f \in \lambda^*$, *where* $a^i = \{a_{ij}\}_{j=1}^\infty$

Then (i) $\Rightarrow$ (ii) if ℓ_α^M is continuously embedded in λ; and (ii) $\Rightarrow$ (i) if A is a bijective map from λ_A to λ and (λ, T) is also metrizable.

Proof : *(i)* $\Rightarrow$ *(ii)* Let $A \in (\ell_\alpha^M, \lambda)$ then $\ell_\alpha^M \subset \lambda_A$. So by using Proposition 2.4 $\{g(e^i)\} \in (\ell_\alpha^M)^\beta = \ell_N^\alpha$ for all $g \in (\lambda_A)^*$. Now $A : \lambda_A \to \lambda$ is continuous and if $f \in \lambda^*$ then $f \circ A \in (\lambda_A)^*$ so we have $f \circ A(e^i) = g(a^i) \in \ell_N^\alpha$ for all $f \in \lambda^*$.

(ii) $\Rightarrow$ *(i)* Let us assume that $\{f(a^i)\} \in \ell_N^\alpha$ for all $f \in \lambda^*$. For proving $A \in (\ell_\alpha^M, \lambda)$ it suffices to prove that $\{g(e^i)\} \in \ell_N^\alpha$ for all $g \in (\lambda_A)^*$. So, consider $g \in (\lambda_A)^*$ and define $f : \lambda \to \mathbb{K}$ such that $f \circ A = g$. As A is bijective, f is well-defined and is continuous in view of the above note; Indeed, $T = T_I$ and g is continuous. Hence by hypothesis, $\{g(a^i)\} = \{f(a^i)\} \in \ell_N^\alpha$. This completes the proof.

The above result, yields

Corollary 2.10 *Let $(\lambda.T)$ be a Fréchet $K-$spaces and $A = (a_{ij})$ be a bijective map from λ_A to λ. Then A is a matrix transformation from ℓ_α^M to λ iff $\{f(a^i)\} \in \ell_N^\alpha$ for all $f \in \lambda^*$, where a^i is the i^{th} row of the matrix A.*

3 Multipliers Between Modular Sequence Spaces

In this section, we identify the collection of multipliers or diagonal maps between two modular sequence spaces as a modular sequence space. To begin with, we have

Propostion 3.1 *A sequence $\sigma = \{\sigma_n\}$ is a multiplier between $\ell\{M_n\}$ and $\ell\{N_n\}$ iff it defines a continuous diagonal operator T_σ to $\ell\{M_n\}$ and $\ell\{N_n\}$, $T_\sigma(x) = \{x_n\sigma_n\} = \sigma x$*

Proof : If σ is a multiplier between $\ell\{M_n\}$ and $\ell\{N_n\}$, then T_σ is continuous from the closed graph theorem. Converse is immediate.

Note 3.2 *We can thus identify multipliers with diagonal operators between $\ell\{M_n\}$ and $\ell\{N_n\}$. If $D(\ell\{M_n\},\ell\{N_n\})$ denotes the space of all multipliers between $\ell\{M_n\}$ and $\ell\{N_n\}$, it becomes a Banach space with the usual operator norm $\|.\|_0$, defined by $\|\sigma\|_0 = sup\{\|\sigma x\|_{\{N_n\}} : \|x\|_{\{M_n\}} \leq 1\}$.*

Next, we have

Propostion 3.3 *For $x \in \ell\{M_n\}$. we have*

(i) $\|x\|_{\{M_n\}} \leq \sum_{n=1}^{\infty} M_n(|x_n|)$ *if* $\|x\|_{\{M_n\}} > 1$;

(ii) $\sum_{n=1}^{\infty} M_n(|x_n|) \leq \|x\|_{\{M_n\}}$ *if* $\|x\|_{\{M_n\}} < 1$.

Proof : (i) If $\|x\|_{\{M_n\}} > 1$, choose $\beta > 0$ such that $\|x\|_{\{M_n\}} > \beta > 1$. Hence $\sum_{n\geq 1} M_n(\frac{|x_n|}{\beta}) \geq 1$ and so $\beta \leq \sum_{n\geq 1} M_n(|x_n|) \Rightarrow \|x\|_{\{M_n\}} \leq \sum_{n=1}^{\infty} M_n(|x_n|)$.

(ii) Let $\|x\|_{\{M_n\}} < 1$. Choose $\alpha > 0$ such that $\|x\|_{\{M_n\}} < \alpha < 1$. Then $\sum_{n\geq 1} M_n(\frac{x_n}{\alpha}) \leq 1$ and so $\sum_{n\geq 1} M_n(|x_n|) \leq \alpha \sum_{n\geq 1} M_n(\frac{|x_n|}{\alpha}) \leq \alpha \Rightarrow \sum_{n\geq 1} M_n(|x_n|) \leq \|x\|_{\{M_n\}}$

Corresponding to two sequences $\{M_n\}$ and $\{N_n\}$ of Orlicz functions, a new sequence $\{P_n\}$ of Orlicz functions is defined as

$$P_n(s) = \max(0, \sup_{0\leq t\leq 1}\{N_n(st) - M_n(t)\}), s \geq 0.$$

As mentioned in the beginning of this section we now proceed to show that the space $D(\ell\{M_n\},\ell\{N_n\})$ of multipliers between $\ell\{M_n\}$ and $\ell\{N_n\}$ coincides with the modular sequence spaces $\ell\{P_n\}$. Let us first prove

Propostion 3.4 *$\ell\{P_n\} \subset D(\ell\{M_n\},\ell\{N_n\})$ and for $\sigma \in \ell\{P_n\}$, $x \in \ell\{M_n\}$;*

$$\|\sigma x\|_{\{N_n\}} \leq 2\|\sigma\|_{\{P_n\}}\|x\|_{\{M_n\}}$$

Proof : Let $\sigma \in \ell\{P_n\}$ and $x \in \ell\{M_n\}$. Choose ρ, r such that $\rho > \|\sigma\|_{\{P_n\}}, r > \|x\|_{\{M_n\}}$. Then

$$\sum_{n\geq 1} P_n(\frac{|\sigma_n|}{\rho}) \leq 1 \quad \text{and} \quad \sum_{n\geq 1} M_n(\frac{|x_n|}{r}) \leq 1.$$

Hence for the sequences $\{\tilde{\sigma}_n\} = \{\frac{\sigma_n}{\rho}\}$ and $\{\tilde{x_n}\} = \{\frac{x_n}{r}\}$, we have

$$\sum_{n\geq 1} N_n(\frac{|\tilde{\sigma}_n\tilde{x}_n|}{2}) \leq 1$$

Consequently, $\sigma x \in \ell\{N_n\}$ and $\|\sigma x\|_{\{N_n\}} \leq 2\rho r$. Thus $\|\sigma x\|_{\{N_n\}} \leq 2\|\sigma\|_{\{P_n\}}\|x\|_{\{M_n\}}$ for all $\sigma \in \ell\{P_n\}$ and $x \in \ell\{M_n\}$. This completes the proof.

Finally, we prove

Theorem 3.5 *For any two sequences of Orlicz functions $\{M_n\}$ and $\{N_n\}$ with $M_n(1) = 1 = N_n(1)$ for all $n \geq 1$, the sequence spaces $D(\ell\{M_n\}, \ell\{N_n\})$ and $\ell\{P_n\}$ are the same. Further the topologies generated by the norms $\|.\|_0$ and $\|.\|_{\{P_n\}}$ on these spaces are equivalent.*

Proof : In view of Proposition 3.4, it suffices to prove that

$$D(\ell\{M_n\}, \ell\{N_n\}) \subset \ell\{P_n\}$$

and

$$\|\sigma\|_{\{P_n\}} \leq 2\ \|\sigma\|_0 \quad \text{for all} \quad \sigma \in D(\ell\{M_n\}, \ell\{N_n\}).$$

Since $M_n(1) = 1 = N_n(1)$ for all $n \in \mathbb{N}$, we have $\|e^n\|_{\{M_n\}} = 1$, $\|e^n\|_{\{N_n\}} = 1$. Now consider $\sigma \in D(\ell\{M_n\}, \ell\{N_n\})$ such that $\|\sigma\|_0 = \frac{1}{2}$. Then

$$|\sigma_n| = \|\sigma e^n\|_{\{N_n\}} \leq \|\sigma\|_0\|e^n\|_{\{M_n\}} = \frac{1}{2}.$$

If $P_n(|\sigma_n|) \neq 0$, by definition of P_n there exists $x_n \in [0, 1], n \geq 1$ such that

$$P_n(|\sigma_n|) = N_n(|\sigma_n|x_n) - M_n(x_n)$$

Then now

$$\|\sigma_n x_n e^n\|_{\{N_n\}} \leq \|\sigma\|_0\|x_n e^n\|_{\{M_n\}} = \frac{1}{2}\|x_n e^n\|_{\{M_n\}} \leq \frac{1}{2}$$

$$\Rightarrow \sum_{j\geq 1} N_j(|\sigma_n|x_n e^n) = N_n(|\sigma_n|x_n) \leq \|\sigma_n x_n e^n\| \leq \frac{1}{2}$$

by Proposition 3.3 (ii). Hence for each $n \in \mathbb{N}$, for which $P_n(|\sigma_n|) \neq 0$, we get

$$M_n(x_n) = N_n(|\sigma_n|x_n) - P_n(|\sigma_n|) < \frac{1}{2}$$

We now prove by induction that $\sum_{n=1}^{k} M_n(x_n) \leq \frac{1}{2}$ for $k = 1, 2, 3, \ldots$. Clearly, it is true for $k = 1$. Consider the sequence $\xi^k = \sum_1^k x_n e^n$ and assume that our claim is true upto k. Then

$$\sum_{n=1}^{k+1} M_n(x_n) = \sum_{1}^{k} M_n(x_n) + M_{n+1}(x_{n+1}) \leq \frac{1}{2} + \frac{1}{2} = 1$$

Hence $\|\xi^{k+1}\|_{\{M_n\}} \leq 1$ and so

$$\|\sigma\xi^{k+1}\| \leq \|\sigma\|_0 \|\xi^{k+1}\|_{\{M_n\}} \leq \frac{1}{2}$$

$$\Rightarrow \sum_{n=1}^{k+1} M_n(x_n) \leq \sum_{n=1}^{k+1} N_n(|\sigma_n|x_n) \leq \|\sigma\xi^{k+1}\|_{\{N_n\}} \leq \frac{1}{2}$$

by Proposition 3.3 (ii) . Consequently, $\sum_n M_n(|x_n|) \leq 1$. Thus $x \in \ell\{M_n\}$ and $\|x\|_{\{M_n\}} \leq 1$. Also, using Proposition 3.3 (ii) and $\|\sigma\|_0 = \frac{1}{2}$, we get

$$\sum_n N_n(|\sigma_n|x_n) \leq \|\sigma x\|_{\{N_n\}} \leq \frac{1}{2}\|x\|_{\{M_n\}} \leq \frac{1}{2}$$

Hence $\sum_n P_n(|\sigma_n|) \leq \frac{1}{2}$ and so $\sigma \in \ell\{P_n\}$ and $\|\sigma\|_{\{P_n\}} \leq 1$.

Now let $\mu \in D(\ell\{M_n\}, \ell\{N_n\})$ be an arbitrary multiplier. Consider $\sigma = \frac{\mu}{\rho}$ where $\rho = 2\|\mu\|_0$, then $\sigma \in \ell\{P_n\}$ and $\|\sigma\|_{\{P_n\}} = \|\frac{\mu}{\rho}\|_{\{P_n\}} \leq 1$, hence $\mu \in \ell\{P_n\}$ and $\|\mu\|_{\{P_n\}} \leq 2\|\mu\|_0$.

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