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Weighted composition transformations and Lie group representations

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Abstract

In this article we present an application of weighted composition transformations in multiplier representations of Lie groups.

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1 Introduction and Preliminaries

Let U be an open subset of $\mathbb{C}^n$ and let $H(U)$ denote the algebra of all complex-valued analytic functions on U. Let $\varphi : U \to U$ be an analytic map. Then $f\circ\varphi \in H(U)$ for all $f \in H(U)$. The map $f \to f\circ\varphi$ is a linear transformation on $H(U)$ and we denote it by C_φ. Since $C_\varphi(f_1 \cdot f_2) = f_1 \circ \varphi \cdot f_2 \circ \varphi = C_\varphi f_1 \cdot C_\varphi f_2$, C_φ is an algebra homomorphism with $C_\varphi 1 = 1$, where 1 is the constant one function on U. If φ is a diffeomorphism, then C_φ is invertible and $C_\varphi^{-1} = C_{\varphi^{-1}}$. Since $C_{\varphi\circ\psi} f = f\circ\varphi\circ\psi = C_\psi C_\varphi f$; $C_{\varphi\circ\psi} = C_\psi C_\varphi$ and hence $\varphi \to C_\varphi$ is not multiplicative. Let $\theta : U \to \mathbb{C}$ be a complex-valued analytic map and $\varphi : U \to U$ be an analytic map. Then define the map $\theta C_\varphi : H(U) \to H(U)$ as $\theta C_\varphi f = \theta \cdot f \circ \varphi$. Then certainly θC_φ is a linear transformation on $H(U)$ and it is called the weighted composition transformation on $H(U)$ induced by θ and φ. If $\varphi(x) = x$, then θC_φ is the multiplication transformation $M_\theta f = \theta \cdot f$ and if $\theta = 1$, then θC_φ is the composition transformation on C_φ. The class of multiplication transformations and the class of composition transformations on $H(U)$ are contained in the class of all weighted composition transformations. If $H(U)$ has a topology on it and C_φ and M_θ are continuous, then they are called composition operators and multiplication operators respectively. These operators have been studied extensively during last five decades or so and play significant roles in study of dynamical systems, semigroups of operators, Cauchy problems and wavelet theory. For details we refer to [2, 5, 6]

In this article we present an application of weighted composition transformations in multiplier representations of Lie groups. This establishes a connection between classical mathematics and modern mathematics.

Definition: By an $n-$dimensional differentiable (real) manifold, we mean a Hausdorff topological space which is connected and each point has a neighbourhood homeomorphic to some open subset of $\mathbb{R}^n$.Similarly, we can define $n-$dimensional differentiable complex manifold.

Definition: Let M be an $n-$dimensional differentiable manifold and let $C^\infty(M)$ be the algebra of all $C^\infty-$ functions on M. Then a tangent vector at a point $p \in M$ is a linear map $T_p : C^\infty(M) \to \mathbb{R}$ suchthat $T_p(f_1 f_2) = f_1(p)T_p f_2 + f_2(p)T_p f_1$, for all $f_1, f_2 \in C^\infty(M)$.

The set $T_p(M)$ of all tangent vectors at p is called the tangent space at p and the disjoint union of the tangent spaces of M i.e $TM = \bigcup_{p\in M} T_pM$, is called the tangent bundle of the differentiable manifold M.

A vector field on M is a smooth map $V : M \to TM$ such that the image of p, denoted by V_p, lies in T_pM, the tangent space at p. The vector field is smooth if for every $f \in C^\infty(M)$, the function $Vf : M \to \mathbb{R}$ defined by $Vf(p) = V_p(f)$ is smooth on M. The set of all smooth vector fields on M is denoted by $\chi(M)$, which is also a vector space.

Definition: Let M and N be two differentiable manifolds and $\varphi : M \to N$ be a smooth map. Then the composition transformation $C_\varphi : C^\infty(N) \to C^\infty(M)$ is given by $C_\varphi f = f \circ \varphi$. It is clear that for every $f \in C^\infty(N)$, $f \circ \varphi \in C^\infty(M)$ and C_φ is algebra homomorphism. This homomorphism is some times called pull-back map and it is denoted as φ^*. Suppose $p \in M$ and $\varphi(p) = q \in N$. Then define the map $\Psi_p : T_p(M) \to T_q(N)$ as $\Psi_p f = T_p(C_\varphi f) = T_p(\varphi^* f)$. This map Ψ_p is called differential map and denoted as $d\varphi_p$.

The differential map $d\varphi_p$ is some times called the tangential map induced by φ or a push-forward map from the tangent space at $q = \varphi(p)$. Given a vector field V on M, a map $\hat{\Phi} : \mathbb{R} \times M \to M$ suchthat $\lim\limits_{t\to 0} \frac{f(\hat{\Phi})-f(p)}{t} = Vf(p)$ (for $f \in C^\infty(M)$), is called the flow or exponential map of V and also written as $\exp(tV) = \hat{\Phi}$. This $\exp(tV)$ satisfies the condition $\exp((t+s)V) = \exp(tV).\exp(sV)$. So $\{\exp(tV), t \in \mathbb{R}\}$ is a one-parameter group of diffeomorphisms whose derivative is vector field V. This is the solution of the differential equation $\frac{d\hat{\Phi}}{dt} = V(p)$. For further details of above concepts we refer to [1, 3].

Definition: A Lie group is a non empty set G with a binary operation satisfying the following conditions-

1. G is a group (with identity element e)
2. G is a smooth manifold
3. G is a topological Group: In particular, the group operation $G \times G \to G$ and the inverse map $G \to G$ are smooth.

By a Lie transformation group, we mean the triple (M, G, π), where G is a Lie group, M is a differentiable manifold and $\pi : M \times G \to M$ is a map satisfying the following conditions-

1. $\pi(x, e) = x$ for all $x \in M$.
2. $\pi(x, gh) = \pi(\pi(x, g), h)$.
3. π is analytic in x and g. The map π is called an action of G on M or a motion on M induced by G.

We say that G acts on M effectively if for every $g \in G, g \neq e$, there exists $p \in M$ such that $\pi(p, g) \neq p$ and π is said to act freely on M if for every $g \in G, g \neq e$ and for every $p \in M$, $\pi(p, g) \neq p$.

2 Weighted composition transformations induced by a Lie transformation group

Let (U, G, π) be a Lie transformation group. For $g \in G$, let the map $\pi_g : U \to U$ be defined as $\pi_g(x) = \pi(x, g)$, for every $x \in U$. The map π_g is a diffeomorphism and $\pi_g^{-1} = \pi_{g^{-1}}$. This π_g induces the composition transformation C_{π_g} on $H(U)$. We denote this transformation by C_g. Clearly $C_{gh} = C_g C_h$. Let $C_G = \{C_g : g \in G\}$. Then C_G is a group under composition.The map $g \to C_g$ is a homomorphism from G to C_G. Define $\pi_G : B(H(U)) \times C_G \to B(H(U))$ as $\pi_G(A, C_g) = AC_g$, where $B(H(U))$ is the vector space of all linear transformations on $H(U)$. Then π_G is an action of C_G on $B(H(U))$. Similarly G acts on $H(U)$ with action π' induced by composition transformations i.e $\pi' : H(U) \times G \to H(U)$ is defined as $\pi'(f, g) = C_g f$. Thus a Lie group action on U gives rise to an action of G on $H(U)$ and an action on $B(H(U))$ induced by the composition transformations.

For $x \in U$, define the map $\pi^x : G \to U$ as $\pi^x(g) = \pi(x, g)$, for every $g \in G$. The range of $\pi^x = \{\pi(x, g) : g \in G\}$ and it is called the orbit of x. We denote it by O^x and the collection $\{O^x : x \in U\}$ is called the orbit space.

Let $H_b(U)$ denote the space of all bounded analytic functions and $H_0(U)$ denote the space of all analytic functions vanishing at infinity. Then $H_b(U)$ and $H_0(U)$ are Banach spaces of analytic functions with norm defined as $\|f\| = \sup_{x \in U} |f(x)|$. Let $\nu : U \times G \to \mathbb{C}$ be an analytic map and let $\nu_h(x) = \nu(x, h)$ for $x \in U$. For $h, g \in G$, define the map $\nu_h C_g : H(U) \to H(U)$ as $\nu_h C_g f = \nu_h \cdot f \circ \pi_g$. Then $\nu_h C_g$ is the weighted composition transformation on $H(U)$ induced by the pair (h, g). Consider the map $\hat{T} : G \to B(H(U))$ defined by $\hat{T}(g) = \nu_g C_g$. We see later that $\hat{T}$ is a homomorphism under certain condition on ν. Thus every element g of G gives rise to a weighted composition transformation with weight function ν_g.

In general, every element $(h, g) \in G \times G$ gives rise to weighted composition transformation $\nu_h C_g$. Thus we have the mapping $T' : G \times G \to B(H(U))$ defined as $T'(h, g) = \nu_h C_g$.

Clearly $T'(g, g) = \nu_g C_g = \hat{T}(g)$ and $T'(h, e) = M_{\nu_h}$(multiplication transformation induced by ν_h).

THEOREM 2.1 *Let (U, G, π) be a Lie transformation group and $\nu : U \times G \to \mathbb{C}$ be an analytic map. Then*

1. *C_g is a continuous operator on $H_b(U)$ and $C_{gh} = C_g C_h$.*

2. *$\nu_h C_g$ is continuous on $H_b(U)$ if ν_h is bounded.*

3. *C_g is invertible for every $g \in G$.*

4. *$\nu_h C_g$ is invertible if $\nu_h \neq 0$.*

5. *The map $\varphi : G \to B(H(U))$ defined by $\varphi(g) = C_g$ is a homomorphism, and hence it is a representation of G on $H(U)$.*

Outline of proof:

1. Since $\|C_g f\| = \sup\limits_{x \in U} |(f \circ \pi_g)(x)| \leq \sup\limits_{x \in U} |f(x)| = \|f\|$, therefore C_g is continuous.

2. $\|\nu_h C_g f\| = \sup\limits_{x \in U} |\nu_h(x)(f \circ \pi_g)(x)| = \sup |\nu_h(x)| . \sup |(f \circ \pi_g)(x)| \leq c \|f\|$, where c is a positive constant.

3. $C_g^{-1} = C_{g^{-1}}$ since $C_{g^{-1}} C_g f = C_{g^{-1}}(f \circ \pi_g) = f \circ \pi_g \circ \pi_{g^{-1}} = f$.

4. $\ker \nu_h C_g = (f : \nu_h C_g f = 0) = \{0\}$. For, $\nu_h C_g f = 0 \Rightarrow \nu_h (f \circ \pi_g)(x) = 0 \Rightarrow f = 0$ since $\nu_h(x) \neq 0$ for every $x \in U$. Let $f \in H(U)$ and let $f' = \frac{f \circ \pi_{g^{-1}}}{\nu_h \circ \pi_g^{-1}}$. Then $f' \in H(U)$ and $\nu_h C_g f' = f$. Hence $\nu_h C_g$ is invertible.

5. $\varphi(gh) f = C_{gh} f = f \circ \pi_{gh} = f \circ \pi_h \circ \pi_g = C_g C_h f = \varphi(g) \varphi(h) f$.

Note : If $A(U)$ denote the vector space of all complex-valued functions on U which are analytic in some neighbourhood of 0, then most of results reported so far are true in case of $A(U)$. Clearly $H(U)$ is contained in $A(U)$.

Definition: An analytic mapping $\nu : U \times G \to \mathbb{C}$ is called a co-cycle over G if $\nu(x, e) = 1$ and $\nu_{gh} = \nu_g \cdot \nu_h \circ \pi_g$ for every g and h in G. A co-cycle ν is called co-boundary if $\nu_g(x) = \frac{\beta(\pi_g(x))}{\beta(x)}$ i.e $\nu_g = \frac{\beta o \pi_g}{\beta}$ for some $0 \neq \beta \in H(U)$. In case ν is a co-boundary, the weighted composition transformation $\nu_g C_g$ is given by $\nu_g C_g f = \beta^{-1} C_g(\beta f)$. [2].

In the following theorem we present a representation of Lie group G in terms of the weighted composition transformations.

THEOREM 2.2 *Let (U, G, π) be a Lie transformation group and let ν be a co-cycle over G. Then $\nu_g C_g$ is a weighted composition transformation on $H(U)$ and the mapping $\hat{T} : G \to B(H(U))$ given by $\hat{T}(g) = \nu_g C_g$ is a multiplier representation of G, where $B(H(U))$ is the algebra of all linear transformations on $H(U)$.*

Outline of the proof:

We have already seen that $\nu_h C_g$ is a weighted composition transformation on $H(U)$. We shall show that the mapping $\hat{T}$ is a representation.

1. Since $[\hat{T}(g)f](x) = [\nu_g C_g f](x)$ for every $f \in H(U), g \in G$ and $x \in U$, we have $[\hat{T}(e)f](x) = \nu_e(x)(C_e f)(x) = \nu(x,e)f(\pi(x,e)) = f(x)$. Hence $\hat{T}(e) = 1$.

2. Let g and h be in G. Then for $f \in H(U)$,

$$\begin{aligned} \hat{T}(gh)(f) &= \nu_{gh} C_{gh} f \\ &= \nu_g . \nu_h \circ \pi_g \cdot f \circ \pi_h \circ \pi_g \\ &= \nu_g . (\nu_h . f \circ \pi_h) \circ \pi_g \\ &= \nu_g . (\hat{T}(h) f) \circ \pi_g \\ &= \hat{T}(g)\hat{T}(h) f. \end{aligned}$$

Thus $\hat{T}(gh) = \hat{T}(g)\hat{T}(h)$. This shows that $\hat{T}$ is a homomorphism and hence a multiplier representation of G induced by co-cycle ν.

Note: If ν is a co-boundary over G, then representation $\hat{T}$ is given by $\hat{T}(g) = \frac{\beta \circ \pi_g}{\beta} . C_g$, which is also a weighted composition transformation.

Definition: Let G be a local Lie group and U be an open subset of $\mathbb{C}^n$. Suppose π is a mapping from $U \times G \to \mathbb{C}^n$. Then G acts on U as a local Lie transformation group if

1. $\pi(x, g)$ is analytic in x and g.
2. $\pi(x, e) = x$ for all $x \in U$.
3. $\pi(x, gh) = \pi(\pi(x, g), h)$ if $\pi(x, g) \in U$.

It is evident that π_g is locally an injection for g in small neighbourhood of e. We denote this local Lie transformation group by triple $(U, G, \pi)_l$

Let $(U, G, \pi)_l$ be a (local) Lie transformation group acting on an open neighbourhood U of $\mathbb{C}^n, 0 \in U$ and let $A(U)$ be the vector space of all complex-valued functions on U analytic in a neighbourhood of 0. Then a (local) multiplier representation $\hat{T}$ of G on $A(U)$ with multiplier ν, consists of a mapping $\hat{T}_g$ on $A(U)$ defined for $g \in G, f \in A(U)$ by

$$(\hat{T}_g f)(x) = \nu_g(x) f(\pi_g(x)); \; x \in U \; , \pi_g(x) \in U \text{and } \pi_g(x) = \pi(x, g).$$

In the following theorem we generalise Theorem 2 to give a multiplier representation of local Lie group in terms of weighted composition transformations.

THEOREM 2.3 *Let $(U, G, \pi)_l$ be a local Lie transformation group and let $\nu : U \times G \to \mathbb{C}$ be a co-cycle over G. Let $A(U)$ be the vector space of all complex-valued functions on U which are analytic on a neighbourhood of 0. Then $f \to \nu_g \cdot C_g f$ is a weighted composition transformation on $A(U)$ and $g \to \nu_g C_g$ is a (local) multiplier representation of G on $A(U)$.*

Proof. The proof runs parallel to the proof of Theorem 2.

Note: $(C_g f)(x) = f(\pi_g(x))$, whenever $\pi_g(x) \in U$, since π_g is locally an injection , $f \circ \pi_g \in A(U)$.

Definition: Let G be a local Lie group. Then the Lie algebra $L(G)$ of the local Lie group G is the set of all tangent vectors at e equipped with the operations of vector addition and Lie product. Let $\hat{T}$ be a multiplier representation of G and let $f \in A(U)$. If $\alpha \in L(G)$, then $\hat{T}_{\exp \alpha t} f \in A(U)$ for sufficiently small values of $|t|$.

The generalised Lie derivative $D_\alpha f$ of an analytic function f under 1−parameter group $\exp \alpha t, t \in \mathbb{R}$ is the analytic function $D_\alpha f(x) = \frac{d}{dt}[\hat{T}_{\exp \alpha t} f](x)$ at $t = 0$.

In case $\nu = 1$, the generalised Lie derivative becomes the ordinary Lie derivative.

We shall cite the following example from [4] to illustrate Theorem 3.

Example 1 *Let $G = SL(2)$, the Lie group of 2×2 matrices with determinant 1. Let U be an open subset of $\mathbb{C}$ and ν be the cocycle over G defined by $\nu(z, g) = (bz + d)^{2u}$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$ and $2u$ is a not a non negative integer. If π is the map from $U \times G \to \mathbb{C}$ defined* *by $\pi(z,g) = \frac{az+c}{bz+d}$, then the multiplier representation of $SL(2)$ with ν as multiplier is given by the following weighted composition transformation*

$$\begin{aligned}
[\hat{T}_g(f)](z) &= \nu_g(z) f(\pi_g(z)) \\
&= \nu(z,g)(C_g f)(z) \\
&= \nu(z,g) f(\pi(z,g)) \\
&= (bz+d)^{2u} f\left(\frac{az+c}{bz+d}\right).
\end{aligned}$$

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