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EUCLIDEAN SEMIRINGS AND SUBTRACTIVE NOETHERIAN SEMIMODULES

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Abstract: We study a multiplicative Euclidean norm on a semiring R to have unique factorization of nonassociates of 1_R in terms of prime elements of R and find the conditions under which a multiplicative Euclidean norm defined on a semiring makes all its principal ideals subtractive. Further, subtractive noetherian semimodules over semirings are also studied.

Keywords: Semirings; Multiplicative Euclidean Norms; Subtractive Noetherian Semimodules.

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1. Introduction

A semiring is a nonempty set R on which operations of addition and multiplication have been defined such that (i) $(R;+)$ is a commutative monoid with identity element 0; (ii) $(R;\cdot)$ is a monoid with identity element 1_R; (iii) Multiplication distributes over addition from either side; (iv) $a\cdot 0 = 0 = 0\cdot a$; for all $a \in R$ and (v) $1_R = 0$. A multiplicative Euclidean norm δ on R is a function $\delta : R \to \mathbb{N}$ satisfying the conditions: (i) $\delta(r) = 0$ if and only if $r = 0$, (ii)$\delta(rs) = \delta(r)\delta(s)$for all $r, s \in R$ and (iii) If r and s are elements of R with $s \neq 0$, then there exist elements u and v of R such that $r = us + v$ with $\delta(v) \leq \delta(s)$. Obviously the maps $\delta : n \mapsto n$ and $\delta : n \mapsto n^2$ are multiplicative Euclidean norms on the semiring $\mathbb{N}$.

If R has a multiplicative Euclidean norm, then it follows from (i) and (ii) that $\delta(1_R) = 1 = \delta(r)$ for all units r in R and R is entire. Two elements $r, s \in R$ are said to be associates of each other if their δ values are same; i.e., $\delta(r) = \delta(s)$. Every unit in R is an associate of 1_R. An element r of R is said to be prime if r is not an associate of 1_R and whenever $r = st$, then one of r and t is an associate of 1_R. We prove that nonzero nonassociates of 1_R can be written as the product of a finite number of prime elements in R. This factorization is unique if δ satisfies an additional property (iv) $\delta(r+s) \geq \delta(r) + \delta(s)$ for all $r, s \in R$ and R is yoked (for $r, s \in R$ there exists $t \in R$ such that $r + t = s$ or $s + t = r$).

It is proved in [1] that every subtractive ideal ($r+s, s \in I$ and $r \in R$ implies that $r \in I$) of a semiring R with multiplicative Euclidean norm δ defined on it is principal. We give an example (Example 2.6) to show that the converse is not true in general. However if R is yoked and a multiplicative Euclidean norm δ has an additional property (iv), then converse also holds. Using this we prove that ideals generated by prime elements of R are precisely the maximal subtractive ideals of R.

A noetherian semimodule over a semiring is defined in [1]. Here noetherian semimodules mean left noetherian semimodules. Further noetherian semimodules are studied by Katsov et. al. [3] with the restriction that all subsemimodules of the semimodules considered are subtractive. More generally, we define a subtractive noetherian semimodule considering the chains of subtractive subsemimodules of a semimodule and prove that if M is subtractive noetherian, then both K and M/K are subtractive noetherian, where K is a subtractive subsemimodule of M.

2. Multiplicative Euclidean Norms on a Semiring

A useful tool in the theory of semirings is $R^{\Delta} = \{a - b : a, b \in R\}$, the ring of differences of R which exists when R is additively cancellative. In R^{Δ}, we have $a - b = c - d$ if and only if there exist $r, r' \in R$ such that $a + r = c + r'$ and $b + r' = d + r'$. The set R^{Δ} becomes a ring under componentwise addition and multiplication given by $(a-b)(c-d) = (ac+bd) - (ad+bc)$. Clearly R^{Δ} contains R by way of embedding $a \mapsto a - 0$. The zero element of R^{Δ} is $a - a$, denoted by 0 and multiplicative identity is 1_R.Throughout this section, we assume that R is an additively cancellative commutative strict semiring ($a + b = 0$ implies $a = 0$ and $b = 0$) with multiplicative identity 1_R.

In this section, we repeatedly use a property of yoked semirings observed in [5]; that is, for $x, y \in R$ either $y - x \in R$ or $x - y \in R$.

Lemma 2.1. Let $D(r) = \{s \in R : r \in Rs\}$ be the set of divisors of r in R. Then we have
(a) $r \in D(r)$,
(b) if $s \in D(r)$ then $D(s) \subseteq D(r)$,

(c) $U(R) = D(1_R) \subseteq D(r)$ for all $r \in R$,
(d) $D(r) \subseteq D(rs)$ for all $s \in R$,
(e) $u \in D(r) \cap D(s)$ gives $u \in D(ar + bs)$ for all $a, b \in R$.
(f) For $r \in R$, $D(r) \subseteq D_{R^\Delta}(r) = \{s \in R^\Delta : r \in R^\Delta s\}$.
(g) If R is a yoked semiring, then for $r \in R$, we have $D_{R^\Delta}(r) \cap R = D(r)$.
(h) Let R has a multiplicative Euclidean norm δ and $r, s \in R$ be such that $D(r) = D(s)$. Then r is an associate of s.
(i) If r is prime and $y \in D(r)$, then y is an associate of 1_R or r.

Proof. (a)-(f) are obvious.

(g) Let $u \in D_{R^\Delta}(r) \cap R$, then there exist $x, y \in R$ such that $r = (x - y)u$. Since R is yoked so either $x-y \in R$ or $y-x \in R$. If $y-x \in R$ then $-r = (y - x)u \in R$ which is not possible, because R is strict. Therefore $x-y \in R$ but then $u \in D(r)$ and hence $D_{R^\Delta}(r) \cap R = D(r)$.

(h) $D(r) = D(s)$ implies $s \in D(r)$ and $r \in D(s)$. So there exist some $t, t' \in R$ such that $r = st$ and $s = rt'$. Therefore $r = st = rt't$ which gives $\delta(t) = 1$. Hence $\delta(r) = \delta(s)$.

(i) Since $y \in D(r)$, there exists $x \in R$ such that $r = xy$. As r is prime, either x is an associate of 1_R or y is an associate of 1_R. If x is an associate of 1_R, then $\delta(r) = \delta(xy) = \delta(y)$ gives y is an associate of r.

Let A be any finite subset of a yoked semiring R, then there is an element $y \in CD(A) = \cap\{D(r) : r \in A\}$, the set of common divisors of A, which can be expressed as a linear combination of elements of A in R^Δ and any two such elements are associated.

Lemma 2.2. If R is a yoked semiring having a multiplicative Euclidean norm δ and A any finite subset of R, then there exists $y \in CD(A)$ such that $D(y) = CD(A)$. Further any two such elements are associated.

Proof. Let $A = \{a_1, a_2, ..., a_m\}$, define $J = \{\lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_m a_m : \lambda_i \in R^\Delta, 1 \le i \le m\}$. Then clearly J is an ideal of R^Δ. Thus $I = J \cap R$ is a subtractive ideal of R by Lemma 3.2.(v) of [6] and hence I is principal. Let $I = \langle y \rangle$ for $y \in I = J \cap R$, so there exist $\lambda_i \in R^\Delta$ $(1 \le i \le m)$ such that $y = \lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_m a_m$. Now each $a_i \in J \cap R = I$, so there exists some $x_i \in R$ such that $a_i = x_i y$ which implies that $y \in CD(A)$. Let $u \in CD(A)$, then $u \in D(a_i)$ for all i and which on using Lemma 2.1 gives $u \in D_{R^\Delta}(\lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_m a_m) \cap R = D(y)$. Hence $D(y) = CD(A)$. The rest of the proof follows from

Lemma 2.1.(h).

Lemma 2.3. Let R be a yoked semiring with multiplicative Euclidean norm δ and $r, s, t \in R$ such that $r \in D(st)$ and $D(1_R) = CD(\{r, s\})$. Then $r \in D(t)$.

Proof. Since $D(1_R) = CD(\{r, s\})$, by Lemma 2.2 there exists $y \in CD(\{r, s\})$ of the type $y = \lambda r + \mu s$; $\lambda, \mu \in R^{\Delta}$ such that $D(y) = D(1_R)$ and so there exists $x \in R$ such that $1_R = xy = x\lambda r + x\mu s$. Now $r \in D(st)$ so there exists $z \in R$ such that $st = zr$. Thus $t = t1_R = t(x\lambda r + x\mu s) = (tx\lambda r + x\mu z)r = r'r$ with $r' = tx\lambda + x\mu z \in R^{\Delta}$. Let $r' = a - b$ for $a, b \in R$. Since R is yoked, either $a - b \in R$ or $b - a \in R$. If $b - a \in R$, then $-t = -r'r \in R$ which is a contradiction as R is strict. Hence $r' = a - b \in R$ gives $r \in D(t)$.

Lemma 2.4. Let r be a prime element of a yoked semiring R having a multiplicative Euclidean norm δ satisfying property (iv).

(a) The function δ is one-one.

(b) Let $s, y \in R$ with $D(y) = CD(\{r, s\})$, then $y = 1_R$ or $y = r$.

(c) For $s \in R$, either $r \in D(s)$ or $CD(\{r, s\}) = D(1_R)$.

(d) Let $s, t \in R$ such that $r \in D(st)$, then $r \in D(s)$ or $r \in D(t)$.

Proof. (a) Let $r, s \in R$ be such that $\delta(r) = \delta(s)$. Now R is yoked so either $r - s \in R$ or $s - r \in R$. By symmetry we assume that $r - s \in R$, then $r = r - s + s$. Therefore $\delta(r) = \delta(r - s + s) \geq \delta(r - s) + \delta(s) = \delta(r - s) + \delta(r)$. Thus by property (i), $\delta(r - s) = 0$ implying that $r = s$.

(b) Follows directly using Lemma 2.1.(i) as δ is one-one.

(c) By (b) either $CD(\{r, s\}) = D(1_R)$ or $CD(\{r, s\}) = D(r)$. If $CD(\{r, s\}) = D(r)$ then $r \in D(s)$.

(d) By (c) either $r \in D(s)$ or $CD(\{r, s\}) = D(1_R)$. If $CD(\{r, s\}) = D(1_R)$, then using Lemma 2.3 we get $r \in D(t)$.

Now we prove the Unique Factorization Theorem for semirings.

Theorem 2.5. Let R be a semiring with multiplicative Euclidean norm δ defined on it.

(a) Every nonzero element of R is either an associate of 1_R or it can be written as the product of a finite number of prime elements in R.

(b) If δ satisfies property (iv) and R is yoked, then the prime factorization given in part (a) is unique.

Proof. (a) Let r be a nonzero element in R. We prove this by induction on $\delta(r)$. The result is obvious if $\delta(r) = 1 = \delta(1_R)$. So suppose that $\delta(r) > 1$ and $r = st$ with $\delta(s) > 1$ and $\delta(r) > 1$. Then the result follows by induction hypothesis as $\delta(s) < \delta(s)\delta(t) = \delta(st) = \delta(r)$ and $\delta(t) < \delta(s)\delta(t) = \delta(r)$.

(b) Let r be a nonzero nonassociate element of R and $r = p_1p_2...p_n = q_1q_2...q_m$ where p_i's and q_i's are prime elements in R. Without loss of generality we suppose that $m \leq n$. Then $q_1 = D(p_1p_2...p_n)$ so by Lemma 2.4.(d) $q_1 \in D(p_i)$ for some $i, 1 \leq i \leq n$. By renumbering p_i's take $i = 1$. So by definition of $D(p_1)$, $p_1 = sq_1$ for some $s \in R$. Hence $\delta(p_1) = \delta(s)\delta(q_1)$. Since p_1,q_1 are primes and δ is one-one, so $s = 1_R$ and hence $p_1 = q_1$. Now $p_1p_2...p_n = q_1q_2...q_m$ so that $\delta(p_1)\delta(p_2...p_n) = \delta(q_1)\delta(q_2...q_m)$ implying $\delta(p_2...p_n) = \delta(q_2...q_m)$. Since δ is one-one, we get $p_2...p_n = q_2...q_m$. Repeating the above process for $m-1$ times, we get $p_i = q_i$ for $i = 1, 2, ...m$. Suppose $m < n$, then $p_{m+1}...p_n = 1_R$ which is not possible as δ is one-one and p_i's are prime elements in R. Hence $n = m$.

Let δ be a multiplicative Euclidean norm defined on R, then every subtractive ideal in R is principal [1]. The converse is not true, in general (as observed in the following example).

Example 2.6. Let $R = (\mathbb{N} \cup \{-\infty\}, \max, +)$ with $\delta : R \to \mathbb{N}$ given by $\delta(-\infty) = 0$ and $\delta(i) = c^i$ for $i \in \mathbb{N}$ and $1 < c \in \mathbb{N}$. Then δ is a multiplicative Euclidean norm on R. Let A be a principal ideal of R generated by $r\,(r > 0)$. Then A is not a subtractive ideal of R as $\max(r, 0) = r \in A$ but $0 \notin A$.

Now we prove converse for a yoked semiring with multiplicative Euclidean norm on it that satisfies property (iv).

Lemma 2.7. If R is yoked having a multiplicative Euclidean norm δ satisfying property (iv), then an ideal in R is a subtractive ideal if and only if it is princial.

Proof. Let $A = \langle r \rangle$ be an ideal of R with $a \neq 0$ and $A \neq R$. Let $xa + y \in A$, $y \in R$. Then there exists some $z \in R$ such that $xa + y = za$. Therefore $\delta(za) = \delta(xa + y) \geq \delta(xa) + \delta(y)$ implying that $\delta(za) \geq \delta(xa)$. So by property (ii), $\delta(z) \geq \delta(x)$. Now $z, x \in R$ and R is yoked so either $z - x \in R$ or $x - z \in R$. If $x - z \in R$ then $x = z + (x - z)$ implies that $\delta(x) = \delta(z + (x - z)) \geq \delta(z) + \delta(x - z)$.

Thus $\delta(x) > \delta(z)$ which is not possible. Therefore $z - x \in R$, but then $y = (z - x)\,a \in A$. Hence A is a subtractive ideal of R.

Note. As an application of the above result it can be seen that the subtractive ideals of semiring $\mathbb{N}$ are precisely those which are generated by the non-negative integers.

Definition 2.8. An ideal $A\,(\neq R)$ of R is said to be a maximal subtractive ideal if A is a subtractive ideal of R and whenever there is subtractive ideal B of R such that $A \subseteq B \subseteq R$, then either $B = A$ or $B = R$.

Now we prove that ideals generated by prime elements of R are precisely the maximal subtractive ideals of R, if R is yoked and δ satisfies property (iv).

Lemma 2.9. Let R be a yoked semiring having a multiplicative Euclidean norm δ satisfying (iv). Then an ideal of R is a maximal subtractive ideal if and only if it is generated by a prime element of R.

Proof. Let $A = \langle r \rangle$ be a maximal subtractive ideal of R. Suppose $r = st$ for some $s, t \in R$ with $\delta(s) \neq 1$ and $\delta(t) \neq 1$. Let $B = \langle s \rangle$. Then B is a subtractive ideal of R containing A. If $B = R$, then $\delta(s) = 1$ which is not possible. If $B = A$, then $\delta(t) = 1$ which is not possible. Hence we have a contradiction to the maximality of subtractive ideal A. Therefore must r be prime.

Conversely suppose that $A = \langle r \rangle$, where r is a prime element of R. Then by Lemma 2.7, A is a subtractive ideal of R. Let $A \subseteq B \subseteq R$ with B a subtractive ideal of R. By Lemma 2.7, $B = \langle s \rangle$ for some $s \in R$. Now $r \in A \subseteq B = \langle s \rangle$ so there exists some $t \in R$ such that $r = ts$. Since r is prime and δ is one-one, either $s = 1_R$ or $t = 1_R$ implying either $B = R$ or $B = A$.

3. Subtractive Noetherian Semimodules

A subsemimodule (nonempty subset) A of a semimodule M over a semiring R is strong if and only if $a + b \in A$ implies that $a, b \in A$. Clearly, every strong subset (subsemimodule) of a semimodule M is subtractive.

An R-semimodule M is said to be subtractive(strong) noetherian if every ascending chain of subtractive(strong) subsemimodules of M is stationary after a finite number steps. Clearly M is strong noetherian if it is subtractive noetherian.

We observe that every ascending chain of subtractive ideals of a semiring R with a multiplicative Euclidean norm becomes stationary after a finite number of steps and hence R becomes subtractive noetherian.

Lemma 3.1. Let R be a semiring having multiplicative Euclidean norm δ. Let $\{I_n : n = 1, 2, ...\}$ be a chain of subtractive ideals of R i.e $I_1 \subseteq I_2 \subseteq ....$ Then there exists an integer N_0 such that $I_k = I_{N_0}$ for all $k \geq N_0$.

Proof. We know that every subtractive ideal of a semiring R having a multiplicative Euclidean norm is principal, so let $I_n = \langle a_n \rangle$ for some $a_n \in R$. Let $I = \bigcup_{n=1}^{\infty} I_n$. Since $I_k \subseteq I_l$ for all $k \leq l$ therefore I is an ideal of R. Next we show that I is a subtractive ideal of R. For this, let $a, a+b \in I$, then there exist integers $k.l$ such that $a \in I_k$ and $a+b \in I_l$. Now we have either $I_k \subseteq I_l$ or $I_l \subseteq I_k$ but then either $a, a+b \in I_l$ or $a, a+b \in I_k$. As both I_k and I_l are subtractive ideals therefore either $b \in I_l \subseteq I$ or $b \in I_k \subseteq I$. So I is a subtractive ideal of R and by the result stated earlier there exists some $a \in R$ such that $I = \langle a \rangle$. Now $a \in I = \bigcup_{n=1}^{\infty} I_n$ so there exists some integer N_0 such that $a \in I_{N_0}$. But then $I_k = I_{N_0} = I$ for all $k \geq N_0$.

The ascending chain condition and maximal condition on subtractive subsemimodules are connected to each other as observed below:

Proposition 3.2. An R-semimodule M is subtractive noetherian if and only if any collection of subtractive subsemimodules of M has a maximal element.

Proof. First suppose that M is subtractive noetherian R-semimodule. Let $\tau = \{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of subtractive subsemimodules of M. Define a inclusion relation on τ. Let $\omega \subseteq \tau$ be any chain. Then ω becomes stationary after a finite number of steps. Therefore ω has an upper bound in τ. But then by Zorn's Lemma τ has a maximal element.

Conversely, let $A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq ...$ be an ascending chain of subtractive ideals of M. Then $\{A_n\}_{n \in \mathbb{N}}$ has a maximal element say A. Then there exists some $n_0 \in \mathbb{N}$ such that $A = A_{n_0}$. But then $A_n = A_{n_0} = A$ for all $n \geq n_0$. Hence M is subtractive noetherian.

In the following proposition we show that if every subtractive subsemimodule of a semimodule M is finitely generated then the above two equivalent conditions hold.

Proposition 3.3. Let M be an R-semimodule having every subtractive subsemimodule finitely generated. Then M is subtractive noetherian.

Proof. Let $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$ be an ascending chain of subtractive subsemimodules of M and $A = \bigcup_{n\in\mathbb{N}} A_n$. Clearly A is subtractive subsemimodule of M. So there exist $a_1, a_2, .., a_m \in M$ such that $A = Ra_1 + Ra_2 + \ldots + Ra_m$. Now $a_i \in A = \bigcup_{n\in\mathbb{N}} A_n$ so there is some $n_i \in \mathbb{N}_0$ such that $a_i \in A_{n_i}$ for all $i = 1, 2, .., m$. By renumbering these n_i's assume that $A_{n_1} \subseteq A_{n_2} \subseteq \ldots \subseteq A_{n_m}$. Therefore $a_1, a_2, .., a_m \in A_{n_m}$. But then $A = A_{n_m}$ so for $A_i = A$ all $i \geq n_m$. Hence M is subtractive noetherian.

For any subsemimodule A of an R-semimodule M we have an R-congruence relation $\cong_A$ on M, known as the Bourne relation, by setting $b \cong_A b_0$ if and only if there exist elements $a, a_0 \in A$ such that $b + a = b_0 + a_0$ and M/A denotes the factor R-semimodule $M/\cong_A$.

Proposition 3.4. Let K be a fixed subsemimodule of the R-semimodule M and A a subtractive subsemimodule of M such that $K \subseteq A \subseteq M$. Then A/K is a subtractive subsemimodule of M/K. Furthermore, if B is a subtractive subsemimodule of M/K then there is one and only one subtractive subsemimodule A of M such that $B = A/K$.

Proof. Now $A/K = \{\overline{x} : x \in A\} \subseteq M/K$. Let $\overline{x}, \overline{y} \in A/K$ then $\overline{x} + \overline{y} = \overline{x+y} \in A/K$ and for $r \in R$ we have $r\overline{x} = \overline{rx} \in A/K$ as A is a semimodule of M. Therefore A/K is a subsemimodule of M/K. Let $\overline{x}, \overline{x} + \overline{y} \in A/K$ for some $\overline{y} \in M/K$. Then there exists some $z \in A$ such that $\overline{x+y} = \overline{x} + \overline{y} = \overline{z} \in A/K$. Therefore there exist $n, n_1 \in K$ such that $x + y + n = z + n_1$. Now $x + n, z + n_1 \in A$ and A is subtractive so $y \in A$ and hence $\overline{y} \in A/K$ proves that A/K is a subtractive subsemimodule of M/K.

Next suppose that B is a subtractive subsemimodule of M/K. Define $A = \{x \in M : \overline{x} \in B\}$. For $x \in K$ we have $\overline{x} = \overline{0} \in B \Longrightarrow x \in A$ so $K \subseteq A$. Let $x, y \in A$. Then $\overline{x}, \overline{y} \in B$ gives $\overline{x+y} = \overline{x} + \overline{y} \in B$ and therefore $x, y \in A$. Also for $r \in R$ we have $\overline{rx} = r\overline{x} \in B$ implying that $rx \in A$. Let $x, x + y \in A$ for some $y \in M$. Then $\overline{x}, \overline{x+y} = \overline{x} + \overline{y} \in B$ gives $\overline{y} \in B$ and therefore $y \in A$. Hence A is a subtractive subsemimodule of M. Clearly, by defining of A we have $B = A/K$. If A_1 is a subtractive subsemimodule of M such that $B = A_1/K$ then

$x \in A \Longrightarrow \overline{x} \in A/K$ so there exists some $y \in A_1$ such that $\overline{x} = \overline{y}$. But then $x + n = y + n_1$ for some $n, n_1 \in K$. Now $x + n, n \in A_1$ and A_1 is a subtractive so $x \in A_1$ and therefore $A \subseteq A_1$. Similarly we have $A_1 \subseteq A$. Hence $A = A_1$.

Theorem 3.5. Let K be a subtractive subsemimodule of an R-semimodule M. If M is subtractive noetherian then both K and M/K are subtractive noetherian.

Proof. First note that if K is a subtractive subsemimodule of a R-semimodule M and A is a subtractive subsemimodule of K, then A is a subtractive subsemimodule of M. For, let $x, x+y \in A$ for some $y \in M$. Then $x, x+y \in K$ and K is a subtractive subsemimodule of M implies that $y \in K$. But then $y \in A$ as A is a subtractive subsemimodule of K.

Now M is subtractive noetherian and K is a subtractive subsemimodule of M therefore any chain $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$ of subtractive subsemimodules of K is a chain of subtractive subsemimodules of M. So it becomes stationary after a finite number of steps showing that K is subtractive noetherian. By the Proposition 3.4 there is one-to-one correspondence between the subtractive subsemimodules of M and those of M/K. Also this correspondence preserves the inclusion relation therefore M/K must be subtractive noetherian as M is subtractive noetherian.

For a partial converse of the above result, we need

Lemma 3.6. Let A, B, K be subsemimodule of an R-semimodule M. If $A \subseteq B$, $A + K = B + K$, $A \cap K = B \cap K$ and B a subtractive then $A = B$.

Proof. Let $b \in B$. Then $b = b + 0 \in B + K = A + K$ so there exist some $a \in A$ and $k \in K$ such that $b = a + k$. Now $a \in A \subseteq B$, $b \in B$ and B a subtractive therefore $k \in B$. But then $k \in B \cap K = A \cap K$ implying that $k \in A$. Hence $b = a + k \in A$ gives $A = B$.

Lemma 3.7. Let $\{A_i\}_{i \in \Lambda}$ be a family of strong subsemimodules of a semimodule M over a semiring R. Then

(i) $\bigcup\limits_{i \in \Lambda} A_i$ is a strong subsemimodule of M; if M is yoked;

(ii) $\bigcup\limits_{i \in \Lambda} A_i = \sum\limits_{i \in \Lambda} A_i$; if M is yoked.

Proof. (i) First we prove that $\bigcup\limits_{i \in \Lambda} A_i$ is a subsemimodule of M, if M

is yoked. Let $a, b \in \underset{i\in\Lambda}{\cup} A_i$. Then $a \in A_i$ and $b \in A_j$ for some $i, j \in \Lambda$. Since M is yoked, there exists an element $r \in M$ such that either $a + r = b$ or $b + r = a$. Suppose $a + r = b \in A_j$ then and the strong character of A_j will yield $a \in A_j$ and $r \in A_j$. Hence $a + b \in A_j \subseteq \underset{i\in\Lambda}{\cup} A_i$. If $b + r = a$ then we get $a + b \in A_i \subseteq \underset{i\in\Lambda}{\cup} A_i$. Obviously for $r \in R$ and $a \in \underset{i\in\Lambda}{\cup} A_i$, we have $ra, ar \in \underset{i\in\Lambda}{\cup} A_i$. Clearly $\underset{i\in\Lambda}{\cup} A_i$ is strong as each A_i is strong.

(ii) Let $a \in \sum_{i\in\Lambda} A_i$. Then $a = \sum_{i\in\Lambda} x_i$ where $x_i \in A_i$. This implies $x_i \in \underset{i\in\Lambda}{\cup} A_i$ for all i and since $\underset{i\in\Lambda}{\cup} A_i$ is a subsemimodule of M; $a = \sum_{i\in\Lambda} x_i \in \underset{i\in\Lambda}{\cup} A_i$. Hence $\sum_{i\in\Lambda} A_i \subseteq \underset{i\in\Lambda}{\cup} A_i$. Converse follows from the fact that $A_i \subseteq \sum_{i\in\Lambda} A_i$ for all i and therefore $\underset{i\in\Lambda}{\cup} A_i = \sum_{i\in\Lambda} A_i$.

Theorem 3.8. If K is a strong subsemimodule of a yoked R-semimodule M such that both K and M/K are subtractive noetherian, then M is strong noetherian.

Proof. Let $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$ be an ascending chain of strong subsemimodules of M. Then $A_1 \cap K \subseteq A_2 \cap K \subseteq \ldots \subseteq A_n \cap K \subseteq \ldots$ is a chain of strong (and hence subtractive) subsemimodules of K. Moreover by Lemma 3.7, $\frac{A_1+K}{K} \subseteq \frac{A_2+K}{K} \subseteq \ldots \subseteq \frac{A_n+K}{k} \subseteq \ldots$ is a chain of subtractive subsemimodules of M/K. Since both K and M/K are subtractive noetherian so there exist some $n_1, n_2 \in \mathbb{N}$ such that $A_m \cap K = A_{n_1} \cap K$ for all $m \geq n_1$ and $\frac{A_m+K}{k} = \frac{A_{n_2}+K}{k}$ for all $m \geq n_2$. Let $\eta = \max\{n_1, n_2\}$ then $A_m \cap K = A_\eta \cap K$ and $\frac{A_m+K}{k} = \frac{A_\eta+K}{k}$ for all $m \geq \eta$. Now $\frac{A_m+K}{k} = \frac{A_\eta+K}{k}$ for all $m \geq \eta$ implies that $A_m + K = A_\eta + K$ for all $m \geq \eta$ so by the Lemma 3.6, $A_m = A_\eta$ for all $m \geq \eta$. Hence M is strong noetherian.

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