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Curvature Collineations in a Recurrent Hermitian Space

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ABSTRACT

The purpose of the present paper is to introduce notions of curvature collineations and various kinds of motions in a recurrent Hermitian space. The necessary and sufficient conditions for the curvature collineations in recurrent Hermitian space will be investigated. Further, relations between curvature collineations and other symmetries will be established and several theorems will be proved.

1. FUNDAMENTAL FORMULAE

Katzin, Levine and Davis [1] have defined and studied curvature collineations: a fundamental symmetry property of space time of general relativity defined by vanishing Lie-derivative of Riemannian curvature tensor. Later, Singh and Singh [3] introduced the notions of affine motion and curvature collineations in a recurrent Riemannian space and several properties of this space have been studied. Further, Singh and Kothari [4] have introduced the notions of curvature collineation and various kinds of motions in a recurrent – Tachibana space. The relations between curvature collineations and other symmetries have been established by them and several theorems have been derived.

Here, we shall firstly define the Hermitian space and give some preliminary formulae, which are pre-requisites to understand such a space.

Let us assume, that there is a self-conjugate positive definite Riemannian metric

$$ds^2 = g_{ij}\, dz^i\, dz^j \;{}^{2)} \qquad \ldots (1.1)$$

in the complex manifold C_n of dimension n.

If the fundamental metric tensor g_{ij} is hybrid, then we evoke such a metric a 'Hermite metric' and the complex manifold equipped with this metric is said to be a Hermitian manifold (Yano and Bochner [6]), which will be denoted by H_n.

Since the fundamental tensor g_{ij} is hybrid, therefore its contravariant components will satisfy the relation:

$$g^{ij} = \begin{pmatrix} 0 & g^{\lambda\mu} \\ g^{\lambda\mu} & 0 \end{pmatrix} \qquad \ldots\ldots (1.2)$$

or

$$F^i_h\, F^j_k\, g^{kh} = g^{ij}, \qquad \ldots. (1.3)$$

where F^i_h is an almost complex structure.

Moreover, a Hermitian space H_n is said to be a recurrent Hermitian space, if its curvature tensor field R^h_{ijk} satisfies the condition: (Lal and Singh [2]) :

$$R^h_{ijk,a} - \lambda_a R^h_{ijk} = 0, \qquad \ldots. (1.4)$$

where λ_a is a non-zero recurrence vector field and

$$R^h_{ijk} \overset{def}{=} \partial_j \{{}_i{}^h{}_k\} - \partial_k \{{}_i{}^h{}_j\} + \{{}_i{}^m{}_k\} \{{}_m{}^h{}_j\} - \{{}_i{}^m{}_j\} \{{}_m{}^h{}_k\},$$

$$(\partial_j \equiv \partial/\partial x^j).$$

We shall denote such a space by H^*_n – space.

In the present paper, we are concerned with a symmetry property of space time, which we call curvature collineation (CC).

A H^*_n – space is said to admit a CC, if there exists an infinitesimal transformation:

$$x^i = x^i + v^i(x)\, \delta t, \qquad \ldots\ldots (1.5)$$

where δt is a positive infinitesimal parameter.

The transformation (1.5) is called an affine motion iff

$$\pounds_v \{_j{}^i{}_k\} = 0, \qquad \text{.... (1.6)}$$

where $\pounds_v$ denotes the Lie-derivative (Yano [8]) with respect to the vector space v^i of transformation (1.5).

Clearly, the investigation of the symmetry property, which is

$$\pounds_v R^h{}_{ijk} = 0, \qquad \text{...(1.7)}$$

is strongly motivated by all important roles of the curvature tensor field $R^h{}_{ijk}$ in the general theory of relativity.

Throughout this paper, we need to refer to the equations describing motions, conformal motions, affine motions, affine collineations, projective collineations, homothetic collineations and conformal collineations.

We, therefore, need to outline a symmetry of these well known space time symmetries.

MOTION (M) : A $H^*{}_n$ – space is said to admit a M, if there exists a killing vector v^i, such that

$$h_{ij} = \pounds_v g_{ij} = v_{i,j} + v_{j,i} = 0. \qquad \text{...(1.8)}$$

AFFINE COLLINEATIONS (AC): A $H^*{}_n$ – space is said to admit an AC, if there exists a vector v^i, such that

$$\pounds_v \{_i{}^k{}_j\} = v^k{}_{,ji} + v^m R^k{}_{jmi} . = \tfrac{1}{2} g^{kl} (h_{li,j} + h_{lj,i} - h_{ij,l}) = 0, \qquad \text{...(1.9)}$$

where $\{_i{}^k{}_j\}$ is the christoffel symbol of the second kind.

Alternatively, the necessary and sufficient condition (1.9) for an AC may be expressed in the form:

$$h_{ij,k} = 0. \qquad \text{....(1.10)}$$

Obviously, every M is an AC. We use the terminology proper AC (prop AC) to denote those AC, which are not M.

PROJECTIVE COLLINEATION (PC) : A H^*_n – space is said to admit a PC, if there exists a vector v_i, such that :

$$\pounds_v \pi^i_{jk} = 0, \quad \text{.....(1.11)}$$

where the projective connection

$$\pi^i_{jk} = \{j\,{}^i\,k\} - (n+1)^{-1} [\delta^i_j \{h\,{}^h\,k\} + \delta^i_k \{h\,{}^h\,j\}].$$

Alternatively, we may express (1.6) in the form:

$$\pounds_v \{j\,{}^i\,k\} = \delta^i_j\ \psi,_k + \delta^i_k\ \psi,_j\ , \quad \text{.... (1.12)}$$

where

$$\psi,_j = (n+1)^{-1}\ v^m\ ,_{mj}\ . \quad \text{.... (1.13)}$$

It follows from (1.12) that for a PC, we get

$$h_{ij,k} = 2g_{ij}\ \psi,_k + g_{ik}\ \psi,_j + g_{ji}\ \psi,_i \quad \text{.... (1.14)}$$

In addition, we find that for every PC, we have

$$\pounds_v W^i_{jkl} = 0, \quad \text{..... (1.15)}$$

where W^i_{jkl} is Weyl projective curvature defined as follows:

$$W^i_{jkl} = R^i_{jkl} - (n+1)^{-1}\ (\delta^i_l R_{jk} - \delta^i_k R_{jl}) \quad \text{.... (1.16)}$$

By observation, every AC is a PC (i.e., a PC with $\psi,_k = 0$).

We shall use the terminology proper PC (prop. PC) to denote those PC, which are not AC.

CONFORMAL MOTION (Conf. M): A H^*_n- space is said to admit a conformal motion, i.e. conf. M, if there exists a vector, such that

$$\pounds_v (g^{-1/n}\ g_{ij}) = 0, \quad \text{.... (1.17)}$$

Where

$g \equiv {}_{ij}$.

Equivalently, we have

$$h_{ij} = 2\ \rho\ g_{ij}\ , \quad \text{.... (1.18)}$$

where ρ is a scalar expressible in the form:

$$\rho = n^{-1}\ v^k,_k\ . \quad \text{... (1.19)}$$

It follows that every conf. M must satisfy

$£_v \{ _j{}^i{}_k \} = \delta^i_j \rho_{,k} + \delta^i_k \rho_{,j} - g_{jk} g^{im} \rho_{,m}$ ..... (1.20)

It can also be shown that every conf. M satisfies

$£_v K^i_{jk} = 0,$

where the conformal connection K^i_{jk} is formed with the relative tensor $(g^{-1/n} g_{ij})$ in the same manner as the Christoffel symbol $\{ _j{}^i{}_k \}$ is constructed with the metric tensor g_{ij}.

Alternatively, K^i_{jk} may be expressible in the form:

$K^i_{jk} = \{ _j{}^i{}_k \} - n^{-1} (\delta^i_j \{ _m{}^m{}_k \} \delta^i_k \{ _m{}^m{}_j \} - g_{ik} g^{im} \{ _h{}^h{}_m \}).$

We use the concept proper conf. M with $\rho \neq$ constant.

HOMOTHETIC MOTION (HM): A H^*_n – space is said to admit, HM, if there exists a vector v^i, such that (1.18) holds with ρ a non-zero constant.

CONFORMAL COLLINEATIONS (conf. C): A H^*_n – space is said to admit a conf. C., if there exists a vector for which (1.20) holds.

It follows that every conf. M. is a conf. C, but the converse is not necessarily true. It can be shown that the necessary and sufficient condition (1.20) for a conf. C may be expressed in the equivalent form:

$h_{ij,k} = 2\rho_{,k} g_{ij}$ .....(1.21)

and that every conf. C must satisfy

$£_v C^h_{ijk} = 0,$ ..... (1.22)

where the conformal curvature tensor C^h_{ijk} is defined by

$C^h_{ijk} \equiv R^h_{ijk} + (n-2)^{-1} (\delta^h_j R_{ik} - \delta^h_k R_{ij} + g_{ik} R^h_j - g_{ij} R^h_k)$

$+ R[(n-1)(n-2)]^{-1} (\delta^h_k g_{ij} - \delta^h_j g_{ik})$ ..... (1.23)

We define the Ricci tensor by $R_{ij} = R^h_{ijh}$ and the scalar curvature $R = R^i_i = R_{ij} g^{ij}$.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR CURVATURE COLLINEATIONS IN A H^*_n - SPACE

The infinitesimal transformation:

$$x^{i} = x^{i} + v^{i}(x)\,\delta t, \qquad \ldots(2.1)$$

where δt is positive infinitesimal, defines a curvature collineation (CC), if the curvature tensor of H^{*}_{n} – space admits a vector field $v^{i}(x)$, such that

$$£_{v}\, C^{k}_{jhi} = 0. \qquad \ldots(2.2)$$

In general the solution of (2.2) consists of a set of vectors $v^{i}_{(\alpha)}$, $\alpha = 1, 2, \ldots, r$, which define an r-parameter invariance group. However, in this paper, we shall not investigate the group property of CC.

We have the following:

$$£_{v}\, R^{k}_{jhi} = R^{k}_{jhi,m}\, v^{m} + R^{k}_{mhi}\, v^{m}_{,j} + R^{k}_{jmi}\, v^{m}_{,jh} + R^{k}_{jhm}\, v^{m}_{,i} - R^{m}_{jhi}\, v^{m}_{,m}. \qquad \ldots(2.3)$$

By the use of covariant differentiation, we get

$$£_{v}\, R^{h}_{jhi} = R^{k}_{jhi,m}\, v^{m} + R^{k}_{mhi}\, v^{m}_{,j} + R^{k}_{jmi}\, v^{m}_{,h} + R^{k}_{jhm}\, v^{m}_{,i} - R^{m}_{jhi}\, v^{k}_{,m}. \qquad \ldots(2.4)$$

If we introduce the Bianchi and Ricci identities and use (1.9), we find that (2.4) can be expressed in the form:

$$£_{v}\, R^{k}_{jhi} = (£_{v}\, \{_{i}{}^{k}{}_{j}\})_{,h} - (£_{v}\, \{_{h}{}^{k}{}_{j}\})_{,i} \qquad \ldots(2.5)$$

and

$$£_{v}\, R^{k}_{jhi} = \tfrac{1}{2}\, g^{km}\, [(h_{im,j} + h_{mj,i} - h_{ij,m})_{,h} - (h_{hm,j} + h_{mj,h} - h_{hj,m} \qquad \ldots(2.6)$$

By substitution of $£_{v}R^{k}_{jhi}$ as given by (2.6) into (2.2) and multiplying the resulting equaion by g_{kl} to lower the index k, we get, the following theorem.

Theorem 2.1 : A necessary and sufficient condition for a H^{*}_{n} – space to admit a CC is that there exists a transformation of the form (2.1), such that the vector v^{i} satisfies:

$$(h_{im,j} + h_{mj,i} - h_{ij,m})_{,h} - (h_{hm,j} + h_{mj,h} - h_{hj,m})_{,i} = 0 \qquad \ldots(2.7)$$

We may express (2.7) in an equivalent, but simpler form, by returning to (2.2) and substituting (2.5) into (2.2) and using the first expression for $£_{v}\{_{j}{}^{i}{}_{k}\}$ given by (1.9) alongwith the Ricci identity to obtain

$$(v_{i,mj} + v_{m,ji} - v_{i,jm})_{,h} - (v_{h,mj} + v_{m,jh} - v_{h,jm})_{,i} = 0. \qquad \ldots(2.8)$$

Although(2.8) is a simpler equation than (2.7), we find (2.7) to be more useful for most of our considerations.

From (2.2), we observe, by contracting on the indices k and i, that every CC, vector v^i satisfies:

$$£_v R_{jh} = 0. \qquad \ldots(2.9)$$

In general, if a H^*_n- space admits a vector v^i such that (2.9) holds, we say that the H^*_n – space admits "Ricci-Collineation" (RC).

Thus, we have the following:

Theorem 2.2: In a H^*_n – space, every CC is an RC.

In (2.7), if we interchange the indices j and m and add the resulting equation to (2.7), we get

Theorem 2.3 : A necessary condition for a transformation of the form (2.1) to define CC is that

$$h_{jm,ih} - h_{jm,hi} = 0. \qquad \ldots. (2.10)$$

It is of interest to note that (2.10) could also be obtained by starting with

$$g_{ia} R^a{}_{jkm} + g_{ja} R^a{}_{ikm} = 0. \qquad \ldots. (2.11)$$

Taking the Lie-derivative of (2.11), it follows that if (2.2) holds, we have

$$h_{ia} R^a{}_{jkm} + h_{ja} R^a{}_{ikm} = 0, \qquad \ldots. (2.12)$$

which by means of the Ricci identity reduces to (2.10).

The necessary condition (2.10) of a CC leads directly to an identity that has been of special interest in the formulation of the conservation laws of general relativity.

In particular, if the conditio (2.10) is multiplied by $g^{1/2}\, g^{jh}\, g^{mi}$, where $g = {}_{ij}$, we get

$$[g^{1/2} (v^i{}_{,j} - v^j{}_{,i})]_{,ji} = \{[g^{1/2} (v^i{}_{,j} - v^j{}_{,i})]_{,j}\}_{,i} = 0, \qquad \ldots(2.13)$$

which is covariant identity.

Since, this tensor expression is obviously a vanishing identity for all v^i, it follows that this necessary condition for a CC places no restriction on v^i.

3. RELATIONS BETWEEN CC AND OTHER SYMMETRIES

From the condition (1.8) of a M in a H^*_n – space, it is immediate that we may state the following:

Theorem 3.1 : In a H^*_n- space, every M is a CC.

Similarly, from the condition (1.9) of an AC, it follows that we may state the following:

Theorem 3.2 : In a H^*_n – space, every AC is a CC.

Also, it follows immediately from the definition of HM that from (1.18), (1.10) is satisfied and hence as a consequence of theorem (3.2), we state the following:

Theorem 3.3 : In a H^*_n- space every HM is a CC.

From Yano [7] (p. 167), it is known that if a transformation is both a conf. M and PC, then it is a HM. Hence, we have the following as a consequence of theorem (3.3):

Theorem 3.4 : In a H^*_n-space, if a transformation in both a conf. M and a PC, then it is a CC.

Next, let us consider under what conditions a PC is a CC.

We, therefore, require that $\pounds_v \{_j{}^i{}_k\}$ be given by (1.7) and substitute for $\pounds_v\{_j{}^i{}_k\}$ in (2.5), if we then demand that

$$\pounds_v R^k{}_{ijh} = 0,$$

we have

$$\delta^k{}_i \psi_{,jh} - \delta^k{}_h \psi_{,ji} = 0. \qquad \text{.....(3.1)}$$

We set k= 1 and sum in (3.1) to get $\psi_{,jh} = 0$. We call a projective collineation with $\psi_{,jh} = 0$, a special projective collineation (SPC). It follows immediately by a covariant differentiation of (1.12) that an SPC satisfies

$$\pounds_v \{_i{}^k{}_j\}_{,h} = 0. \qquad \text{....(3.2)}$$

In general, if a H^*_n – space admits a vector v^i, such that (3.2) holds, we say that the H^*_n – space admits a special curvature collination (SCC). Thus, every SPC is a SCC. We summarise the above by stating the following:

Theorem 3.5 : The necessary and sufficient condition for a PC to be a CC is that

$$\psi_{,jh} = 0, \qquad \text{....(3.3)}$$

where

$$\psi_{,jh} = (n+1)^{-1} v^i_{,ijh} ,$$

i.e., a PC must be an SPC.

Corollary 3.1. If a H^*_n – space admits a SPC, then it admits a parallel field of vectors

$$\psi_j = (n+1)^{-1} v^i_{,ij} ,$$

where v^i defines the SPC.

We, now, turn our attention to the condition for a conf. C to be a CC. We thus, assume that the H^*_n – space admits a conf. C, i.e., (1.21) holds. Now, we use (1.20) to evaluate $£_v \{_i{}^k{}_j\}$, in (2.5) and require that $£_v \{_i{}^h{}_j\} = 0$. We immediately obtain

$$\delta^k_i \rho_{,jh} - \delta^k_h \rho_{,ji} - g_{ij} g^{km} \rho_{,mj} + g_{hj} g^{km} \rho_{,mi} = 0. \qquad \text{...(3.4)}$$

We set $k = i$ and sum in (3.4) to obtain

$$(n-2) \rho_{,jh} + g_{hj} g^{im} \rho_{,mi} = 0. \qquad \text{....(3.5)}$$

In (3.5), we multiply by g^{jh} and sum to obtain

$$g^{jh} \rho_{,jh} = 0. \qquad \text{....(3.6)}$$

It follows from (3.5) and (3.6) that $\rho_{,ij} = 0$. We call a conformal collineation with $\rho_{,ij} = 0$ a special conformal collineation (S conf. C). It follows immediately by covariant differentiation of (1.15) that an S conf. C satisfies (3.2). Thus, every S conf. C is a SCC.

We, now, summarise the above by stating the following :

Theorem 3.6: The necessary and sufficient condition for a conf. C to be a CC is that

$$\rho_{,ijh} = 0, \qquad \text{... (3.7)}$$

where

$\rho_{,jh} = n^{-1} v^{i}_{,ijh}$,

i.e., the conf. C must be a S conf. C.

Corollary 3.2: If a H^{*}_{n}-space, admits a S conf. C, then it admits a parallel vector field

$\rho_j = n^{-1} v^{i}_{,ij}$,

where v^i defines the S conf. C.

We define special conformal motion (S conf. M) as a conf. M with

$\rho_{,ij} = 0$.

Hence, we have the following:

Theorem 3.7 : Every S conf. M is a S conf. C.

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