Some Recurrence Properties in a Tachibana R-Recurrent Space

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ABSTRACT

The present paper is devoted to the study of some recurrence properties in a Tachibana r-recurrent space, wherein we have defined and studied Weyl-Tachibana projective r-recurrent and Weyl-Tachibana conformal r-recurrent spaces and several theorems have been established. The necessary and sufficient condition for a Weyl-Tachibana projective r-recurrent space to be a Tachibana r-recurrent space has been derived therein.

1. FUNDAMENTAL FORMULAE

Mathai [2] and Walker [6] have studied Kaehlerian spaces and Ruse's spaces of recurrent curvature tensors respectively. Singh and Nautiyal [3] have defined and studied some recurrence properties in a Kaehler space and several theorems have been investigated.

Further, Singh and Kumar [4] have defined and studied some recurrence properties in a Tachibana space and several interesting results have been obtained.

Here, we shall firstly define Tachibana space and give some preliminary formulae, which are pre-requisites to understand such a space.

An almost Tachibana space is an almost Hermite space ($F^h{}_i$, g_{ij}), where $F^h{}_i$ is an almost complex structure and g_{ij} is a Hermite metric, such that

$$F^h{}_{i,j} + F^h{}_{j,i} = 0,$$

.... (1.1)

where the comma (,) followed by indices denotes the operation of covariant differentiations with respect to the symmetric connection Γ^h_{ij}.

In an almost Tachibana space, we have (Yano [8])

$$N^h_{ji} = -4(F^a_{i,j}) F^h_a , \qquad\qquad .... (1.2)$$

where $F^h_{i,j}$ is pure in i and j and N^h_{ji} is the *Nijenhuis tensor* (Yano [8]). When the Nijenhuis tensor vanishes, the almost Tachibana space is called a Tachibana space and it will be denoted in brief by 'T_n' – space.

A Tachibana space is called recurrent Tachibana space, if it's curvature tensor 'R^h_{ijk}' satisfies the condition (Lal and Singh [1]):

$$R^i_{jkm,a} = \lambda_a R^i_{jkm}, \qquad\qquad .... (1.3)$$

Or,

$$R^i_{jkm,a} - \lambda_a R^i_{jkm} = 0,$$

where λ_a is a non-zero recurrence vector field and the Riemannian curvature tensor, which we have denoted by R^h_{ijk} , is defined as

$$R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^m_{ik} \Gamma^k_{mj} - \Gamma^m_{ij} \Gamma^h_{mk}, \qquad\qquad .... (1.4)$$

where $\partial_j \equiv \partial/\partial x^j$ and $\{x^i\}$ denotes the real local co-ordinates.

The Ricci tensor and the scalar curvature tensor are respectively given by

$$R_{ij} = R^h_{ijh} \text{ and } R = R_{ij} g^{ij} .$$

If the curvature tensor satisfies the conditions :

$$R^h_{ijk,a_1....a_r} - \lambda a_1 .... a_r R^h_{ijk} = 0, \qquad\qquad .... (1.5)$$

for some non-zero recurrence tensor field $\lambda a_1 ... a_r$, then the space is called Tachibana r-recurrent space.

We shall call such a space an ^rT_n – space.

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The space T_n is said to be Tachibana Ricci – r recurrent, if it satisfies the condition:

$$R_{ij}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R_{ij} = 0, \qquad \ldots(1.6)$$

for some non-zero recurrence tensor field $\lambda a_1 \ldots a_r$ and is denoted in brief by an $R - {}^r T_n -$ space.

Multiplying equation (1.6) by g^{ij} and using the fact that

$g^{ij}, a_1 \ldots a_r = 0$, we obtain

$$R, a_1 \ldots a_r - \lambda a_1 \ldots a_r R = 0. \qquad \ldots(1.7)$$

The Weyl projective curvature tensor and Weyl conformal curvature tensor in a T_n-space are respectively given by

$$W^h_{ijk} = R^h_{ijuk} + \frac{1}{n-1} (R_{ik} \delta^h_j - R_{ij} \delta^h_k) \qquad \ldots(1.8)$$

and

$$C^h_{ijk} = R^h_{ijk} + \frac{1}{n-2} (R_{ik} \delta^h_j - R_{ij} \delta^h_k + g_{ik} R^h_j - g_{ij} R^h_k) - \frac{R}{(n-1)(n-2)} (g_{ik} \delta^h_j - g_{ij}\delta^h_k).$$

$$\ldots(1.9)$$

In view of (1.8) and (1.9), we have

$$C^h_{ijk} = W^h_{ijk} + \frac{1}{(n-1)(n-2)} (R_{ik} \delta^h_j - R_{ij} \delta^h_k) + \frac{1}{n-2} (g_{ik} R^h_j - g_{ij} R^h_k)$$

$$- \frac{R}{(n-1)(n-2)} (g_{ik} \delta^h_j - g_{ij} \delta^h_k). \qquad \ldots(1.10)$$

Remark 1.1. From (1.5), it follows that every ${}^r T_n$-space is an $R - {}^r T_n -$ space, but the converse is not necessarily true.

2. WEYL–TACHIBANA PROJECTIVE r-RECURRENT AND WEYL-TACHIBANA CONFORMAL r-RECURRENT SPACES

Definition 2.1. A Tachibana space T_n satisfying the condition:

$$W^h_{ijk}, a_1 \ldots a_r - \lambda a_1 \ldots a_r W^h_{ijk} = 0, \qquad \ldots(2.1)$$

for some non-zero recurrence tensor $\lambda a_1 \ldots a_r$ is called Weyl-Tachibana projective r-recurrent space and is denoted by W-^rT_n – space.

Definition 2.2. A Tachibana space T_n satisfying the condition:

$$C^h_{ijk}, a_1 \ldots a_r - \lambda a_1 \ldots a_r \, C^h_{ijk} = 0. \qquad \ldots(2.2)$$

for some non-zero recurrence tensor $\lambda a_1 \ldots a_r$ is called a Weyl-Tachibana conformal r-recurrent space and is denoted by an C-^rT_n-space.

We, now, have the following theorems:

Theorem 2.1 . Every ^rT_n-space is W-^rT_n-space.
Proof. Differentiating (1.8), we have

$$W^h_{ijk},a_1\ldots a_r = R^h_{ijk},a_1\ldots a_r + \frac{1}{n-1} (\delta^h_j \, R_{ik},a_1\ldots a_r - \delta^h_k \, R_{ij},a_1\ldots a_r). \qquad \ldots(2.3)$$

Multiplying (1.8) by $\lambda a_1 \ldots a_r$ and subtracting the result thus obtained from (2.3), we get

$$W^h_{ijk},a_1\ldots a_r - \lambda a_1 \ldots a_r \, W^h_{ijk} = R^h_{ijk},a_1\ldots a_r - \lambda a_1 \ldots a_r \, R^h_{ijk} + \frac{1}{n-1} \{\delta^h_j \, (R_{ik},a_1 \ldots a_r$$

$$- \lambda a_1 \ldots a_r \, R_{ik}) - \delta^h_k \, (R_{ij}, a_1 \ldots a_r - \lambda a_1 \ldots a_r \, R_{ij})\} \qquad \ldots(2.4)$$

If the space is ^rT_n – space, then (1.5) and (1.6) are satisfied and (2.4), in view of (1.5) and (1.6), gives

$$W^h_{ijk}, a_1 \ldots a_r - \lambda a_1 \ldots a_r \, W^h_{ijk} = 0,$$

which shows that the space is W-^rT_n – space.
This completes the proof of the theorem.

Theorem 2.2. Every ^rT_n-space is C – ^rT_n – space.
Proof. Differentiating (1.9), we get

$$C^h_{ijk}, a_1\ldots a_r = R^h_{ijk}, a_1 \ldots a_r + \frac{1}{n-2} (\delta^h_j \, R_{ik}, a_1 \ldots a_r - \delta^h_k \, R_{ij}, a_1 \ldots a_r + g_{ik} \, R^h_j,a_1\ldots a_r$$

$$R, a_1 \dots a_r$$
$$- g_{ij} R^h{}_k, a_1 \dots a_r) - \frac{}{(n-1)(n-2)} (g_{ik} \delta^h{}_j - g_{ij} \delta^h{}_k). \qquad \dots(2.5)$$

Multiplying (1.9) by $\lambda a_1 \dots a_r$ and subtracting the result thus obtained from (2.5), we get

$$C^h{}_{ijk}, a_1 \dots a_r - \lambda a_1 \dots a_r C^h{}_{ijk} = R^h{}_{ijk}, a_1 \dots a_r - \lambda a_1 \dots a_r R^h{}_{ijk} + \frac{1}{n-2} \{ \delta^h{}_j (R_{ik}, a_1 \dots a_r$$

$$- \lambda a_1 \dots a_r R_{ik}) - \delta^h{}_k (R_{ij}, a_1 \dots a_r - \lambda a_1 \dots a_r R_{ij}) + g_{ik} (R^h{}_j, a_1 \dots a_r - \lambda a_1 \dots a_r R^h{}_j)$$

$$- g_{ij} (R^h{}_k, a_1 \dots a_r - \lambda a_1 \dots a_r R^h{}_k) \} - \frac{(R, a_1 \dots a_r - \lambda a_1 \dots a_r R)}{(n-1)(n-2)} (g_{ik} \delta^h{}_j - g_{ij} \delta^h{}_k) \} \qquad \dots(2.6)$$

If the space is ^rT_n-space, then (1.5), (1.6) and (1.7) are satisfied and (2.6), in view of (1.5), (1.6) and (1.7), becomes

$$C^h{}_{ijk}, a_1 \dots a_r - \lambda a_1 \dots a_r C^h{}_{ijk} = 0,$$
which shows that the space is C- ^rT_n – space.

This completes the proof of the theorem.

Theorem 2.3. If in a T_n – space any two of the following properties are satisfied:
(i)   the space is R – ^rT_n,
(ii)   the space is W – ^rT_n,
(iii)   the space is C – ^rT_n,
then the third is also satisfied.

Proof. Differentiating (1.10), we have

$$C^h{}_{ijk}, a_1 \dots a_r = W^h{}_{ijk}, a_1 \dots a_r + \frac{1}{(n-1)(n-2)} (\delta^h{}_j. R_{ik}, a_1 \dots a_r - \delta^h{}_k R_{ij}, a_1 \dots a_r)$$

$$+ \frac{1}{(n-2)} (g_{ik} R^h{}_j, a_1 \dots a_r - g_{ij} R^h{}_k, a_1 \dots a_r) - \frac{R, a_1 \dots a_r}{(n-1)(n-2)} (g_{ik} \delta^h{}_j - g_{ij} \delta^h{}_k)$$

$$\dots (2.7)$$

Multiplying (1.10) by $\lambda\, a_1 .. a_r$ and subtracting the result thus obtained from (2.7), we get,

$$C^h{}_{ijk},\, a_1 .. a_r - \lambda\, a_1 .. a_r\, C^h{}_{ijk} = W^h{}_{ijk},\, a_1 .. a_r - \lambda\, a_1 .. a_r\, W^h{}_{ijk} + \frac{1}{(n-1)(n-2)} \{\delta^h{}_j\, (R_{ik},\, a_1 .. a_r$$

$$- \lambda\, a_1 .. a_r\, R_{ik}) - \delta^h{}_k\, (R_{ij},\, a_1 .. a_r - \lambda\, a_1 .. a_r\, R_{ij})\} + \frac{1}{n-2} \{g_{ik}\, (R^h{}_j,\, a_1 .. a_r - \lambda\, a_1 .. a_r\, R^h{}_j)$$

$$- g_{ij}\, (R^h{}_k,\, a_1 .. a_r - \lambda\, a_1 .. a_r\, R^h{}_k)\} - \frac{R,\, a_1 .. a_r - \lambda\, a_1 .. a_r\, R}{(n-1)(n-2)}\, (g_{ik}\, \delta^h{}_j - g_{ji}\, \delta^h{}_k) \qquad \dots (2.8)$$

Making use of equations (1.6), (1.7), (2.1), (2.2) and (2.8), we obtain the proof of the above theorem.

Theorem 2.4. The necessary and sufficient condition for a $W - {}^r T_n -$ space to be ${}^r T_n -$ is that the space be $R - {}^r T_n$ one.

Proof. Let the $W - {}^r T_n -$ space be ${}^r T_n -$ space, so that equations (1.5) and (2.1) are satisfied and (2.4), in view of (1.5) and (2.1), reduces to

$$\delta^h{}_j\, (R_{ik},\, a_1 .. a_r - \lambda\, a_1 .. a_r\, R_{ik}) - \delta^h{}_k\, (R_{ij},\, a_1 .. a_r - \lambda\, a_1 .. a_r\, R_{ij}), \text{ which after some}$$

simplification and further calculation shows that the space is $R - {}^r T_n$.

Conversely, let the $W - {}^r T_n -$ space be $R - {}^r T_n$, so that (1.6) and (1.7) are satisfied. Then (2.4), in view of (1.6) and (2.1), reduces to

$$R^h{}_{ijk},\, a_1 .. a_r - \lambda\, a_1 .. a_r\, R^h{}_{ijk} = 0,$$

which shows that the spae is ${}^r T_n -$ space.

Hence the theorem is completed.

Theorem 2.5. The necessary and sufficient condition for a $C - {}^r T_n -$ space to be ${}^r T_n -$ space is that the space be $R - {}^r T_n$.

Proof. Let the $C - {}^r T_n -$ space be ${}^r T_n -$ space, so that (1.5) and (2.2) are satisfied and (2.6), in view of (1.5) and (2.2), reduces to

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$$\frac{1}{(n-2)} \{ \delta^h_j (R_{ik}, a_1 .. a_r - \lambda a_1 .. a_r R_{ik}) - \delta^h_k (R_{ij}, a_1 .. a_r - \lambda a_1 .. a_r R_{ij}) + g_{ik} (R^h_j, a_1 ..$$

$$a_r - \lambda a_1 .. a_r R^h_j) - g_{ij} (R^h_k, a_1 .. a_r - \lambda a_1 .. a_r R^h_k) \} - \frac{(R, a_1 .. a_r - \lambda a_1 .. a_r R)}{(n-1)(n-2)} (g_{ik} \delta^h_j$$

$$- g_{ji} \delta^h_k) = 0.$$

or,

$$(n-1) \{ \delta^h_j (R_{ik}, a_1 .. a_r - \lambda a_1 .. a_r R_{ik}) - \delta^h_k (R_{ij}, a_1 .. a_r - \lambda a_1 .. a_r R_{ij}) + g_{ik} (R^h_j, a_1 .. a_r$$

$$- \lambda a_1 .. a_r R^h_j) - g_{ij} (R^h_k, a1 .. a_r R^h_k - \lambda a_1 .. a_r R^h_k) \}$$

$$- (R, a_1 .. a_r - \lambda a_1 .. a_r R) (g_{ik} \delta^h_j - g_{ij} \delta^h_k) = 0,$$

which after some simplification shows that the space in $R - {}^rT_n$.

Conversely, let the $C - {}^rT_n -$ space be $R- {}^rT_n$, so that (1.6), (1.7) and (2.2) reduces to

$$R^h_{ijk}, a_1 .. a_r - \lambda a_1 .. a_r R^h_{ijk} = 0,$$

which shows that the space is ${}^rT_n -$ space.

This completes the proof of the theorem.

In view of equations (1.5), (1.6) and (2.4), the following theorem can be proved easily:

Theorem 2.6. If in a $T_n -$ space, any two of the following properties are satisfied:

(i)   the space is ${}^rT_n$,
(ii)   the space is $R - {}^rT_n$,
(iii)   the space is $W - {}^rT_n$,

then it must also satisfy the third.

Similarly, making use of (1.5), (1.6), (1.7) and (2.6), we may immediately prove the following:

Theorem 2.7. If in a $T_n -$ space, any two of the following properties are satisfied:

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(i)   the space is ${}^{r}T_n$,
(ii)   the space is $R - {}^{r}T_n$,
(iii)   the space is $C - {}^{r}T_n$,

then it must satisfy the third also.

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