Some Recurrence Properties in a Tachibana R-Recurrent Space

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ABSTRACT
The present paper is devoted to the study of some recurrence properties in a Tachibana r-recurrent space, wherein we have defined and studied Weyl-Tachibana projective r-recurrent and Weyl-Tachibana conformal r-recurrent spaces and several theorems have been established. The necessary and sufficient condition for a Weyl-Tachibana projective r-recurrent space to be a Tachibana r-recurrent space has been derived therein.

1. FUNDAMENTAL FORMULAE

Mathai [2] and Walker [6] have studied Kaehlerian spaces and Ruse’s spaces of recurrent curvature tensors respectively. Singh and Nautilyal [3] have defined and studied some recurrence properties in a Kaehler space and several theorems have been investigated.

Further, Singh and Kumar [4] have defined and studied some recurrence properties in a Tachibana space and several interesting results have been obtained.

Here, we shall firstly define Tachibana space and give some preliminary formulae, which are pre-requisities to understand such a space.

An almost Tachibana space is an almost Hermite space \((F^h_{ij}, g_{ij})\), where \(F^h_i\) is an almost complex structure and \(g_{ij}\) is a Hermite metric, such that

\[ F^h_{ij} + F^h_{ji} = 0, \]

.... (1.1)
where the comma (,) followed by indices denotes the operation of covariant differentiations with respect to the symmetric connection $\Gamma^h_{ij}$.

In an almost Tachibana space, we have (Yano [8])

$$N^h_{ji} = -4(F^a_{ij}) F^h_a, \quad \ldots \quad (1.2)$$

where $F^h_{ij}$ is pure in $i$ and $j$ and $N^h_{ji}$ is the Nijenhuis tensor (Yano [8]). When the Nijenhuis tensor vanishes, the almost Tachibana space is called a Tachibana space and it will be denoted in brief by $'T_n'$ – space.

A Tachibana space is called recurrent Tachibana space, if it’s curvature tensor $'R^h_{ijk}'$ satisfies the condition (Lal and Singh [1]):

$$R^i_{jkma} = \lambda_a R^i_{jkm}, \quad \ldots \quad (1.3)$$

Or,

$$R^i_{jkma} - \lambda_a R^i_{jkm} = 0,$$

where $\lambda_a$ is a non-zero recurrence vector field and the Riemannian curvature tensor, which we have denoted by $R^h_{ijk}$, is defined as

$$R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^m_{ik} \Gamma^k_{mj} - \Gamma^m_{ij} \Gamma^h_{mk}, \quad \ldots \quad (1.4)$$

where $\partial_j = \partial/\partial x^j$ and $\{x^i\}$ denotes the real local co-ordinates.

The Ricci tensor and the scalar curvature tensor are respectively given by

$$R_{ij} = R^h_{ijh} \text{ and } R = R_{ij} g^{ij}.$$

If the curvature tensor satisfies the conditions :

$$R^h_{ijk,a_1 \ldots a_r} - \lambda a_1 \ldots a_r R^h_{ijk} = 0, \quad \ldots \quad (1.5)$$

for some non-zero recurrence tensor field $\lambda a_1 \ldots a_r$, then the space is called Tachibana r-recurrent space.

We shall call such a space an $'T_n'$ – space.
The space $T_n$ is said to be Tachibana Ricci – r recurrent, if it satisfies the condition:

$$R_{ij}a_1...a_r - \lambda a_1...a_r R_{ij} = 0,$$

...(1.6)

for some non-zero recurrence tensor field $\lambda a_1...a_r$ and is denoted in brief by an $R^{-t}T_n$ - space.

Multiplying equation (1.6) by $g^{ij}$ and using the fact that $g^{ij}$, $a_1...a_r = 0$, we obtain

$$R, a_1...a_r - \lambda a_1...a_r R = 0.$$  \(\text{...(1.7)}\)

The Weyl projective curvature tensor and Weyl conformal curvature tensor in a $T_n$-space are respectively given by

$$W^h_{ijk} = \frac{1}{n-1} (R^h_{iju} + \delta_j^h (R_{i} \delta_k^h - R_{ij} \delta^h_k)) \quad \text{...(1.8)}$$

and

$$C^h_{ijk} = \frac{1}{n-2} (R^h_{ijk} + \delta^h_j (R_{i} \delta^h_k + g_{ik} R_{j}^h - g_{ij} R^h_k)) - \frac{R}{(n-1)(n-2)} (g_{ik} \delta^h_j - g_{ij} \delta^h_k). \quad \text{...(1.9)}$$

In view of (1.8) and (1.9), we have

$$C^h_{ijk} = W^h_{ijk} + \frac{1}{n-1} (R_{ik} \delta^h_j - R_{ij} \delta^h_k) + \frac{1}{n-2} (g_{ik} R^h_j - g_{ij} R^h_k) + \frac{R}{(n-1)(n-2)} (g_{ik} \delta^h_j - g_{ij} \delta^h_k). \quad \text{...(1.10)}$$

Remark 1.1. From (1.5), it follows that every $T_n$-space is an $R^{-t}T_n$-space, but the converse is not necessarily true.

2. WEYL–TACHIBANA PROJECTIVE r-RECURRENT AND WEYL-TACHIBANA CONFORMAL r-RECURRENT SPACES

Definition 2.1. A Tachibana space $T_n$ satisfying the condition:

$$W^h_{ijk,a_1...a_r} - \lambda a_1...a_r W^h_{ijk} = 0,$$

...(2.1)
for some non-zero recurrence tensor $\lambda a_1 \ldots a_r$ is called Weyl-Tachibana projective r-recurrent space and is denoted by $W{}^rT_n$-space.

**Definition 2.2.** A Tachibana space $T_n$ satisfying the condition:

$$C_{ijk}^h, a_1 \ldots a_r - \lambda a_1 \ldots a_r C_{ijk}^h = 0.$$ ... (2.2)

for some non-zero recurrence tensor $\lambda a_1 \ldots a_r$ is called a Weyl-Tachibana conformal r-recurrent space and is denoted by an $C{}^rT_n$-space.

We, now, have the following theorems:

**Theorem 2.1.** Every $^rT_n$-space is $W{}^rT_n$-space.

**Proof.** Differentiating (1.8), we have

$$W_{ijk,a_1 \ldots a_r}^h = R_{ijk,a_1 \ldots a_r}^h + \cdots (\delta_j^h R_{ik,a_1 \ldots a_r}^h - \delta_k^h R_{ij,a_1 \ldots a_r}^h).$$ ... (2.3)

Multiplying (1.8) by $\lambda a_1 \ldots a_r$ and subtracting the result thus obtained from (2.3), we get

$$W_{ijk,a_1 \ldots a_r}^h - \lambda a_1 \ldots a_r W_{ijk}^h = R_{ijk,a_1 \ldots a_r}^h + \cdots \{\delta_j^h (R_{ik,a_1 \ldots a_r}^h - \lambda a_1 \ldots a_r R_{ij})\}.$$ ... (2.4)

If the space is $^rT_n$-space, then (1.5) and (1.6) are satisfied and (2.4), in view of (1.5) and (1.6), gives

$$W_{ijk,a_1 \ldots a_r}^h - \lambda a_1 \ldots a_r W_{ijk}^h = 0,$$

which shows that the space is $W{}^rT_n$-space.

This completes the proof of the theorem.

**Theorem 2.2.** Every $^rT_n$-space is $C{}^rT_n$-space.

**Proof.** Differentiating (1.9), we get

$$C_{ijk,a_1 \ldots a_r}^h = R_{ijk,a_1 \ldots a_r}^h + \cdots (\delta_j^h R_{ik,a_1 \ldots a_r}^h - \delta_k^h R_{ij,a_1 \ldots a_r}^h + g_{ik} R_{j,a_1 \ldots a_r}^h).$$ n-2

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Multiplying (1.9) by $\lambda a_1 \ldots a_r$ and subtracting the result thus obtained from (2.5), we get

$$C_{ijk, a_1 \ldots a_r - \lambda a_1 \ldots a_r}^h = R_{ijk, a_1 \ldots a_r}^h - \lambda a_1 \ldots a_r R_{ijk}^h + \frac{1}{n-2} \{ \delta_j^h (R_{ik}, a_1 \ldots a_r - R_{ij}^h) - \delta_k^h (R_{ij}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R_{ij}^h) - g_{ij} (R_{ik}^h, a_1 \ldots a_r - \lambda a_1 \ldots a_r R_{ik}^h) \} - \frac{1}{(n-1)(n-2)} (g_{ik} \delta_j^h - g_{ij} \delta_k^h) \ldots (2.6)$$

If the space is $T_n$-space, then (1.5), (1.6) and (1.7) are satisfied and (2.6), in view of (1.5), (1.6) and (1.7), becomes

$$C_{ijk, a_1 \ldots a_r}^h = 0,$$

which shows that the space is $C\, T_n$-space.

This completes the proof of the theorem.

**Theorem 2.3.** If in a $T_n$-space any two of the following properties are satisfied:
(i) the space is $R - T_n$,
(ii) the space is $W - T_n$,
(iii) the space is $C - T_n$,
then the third is also satisfied.

**Proof.** Differentiating (1.10), we have

$$C_{ijk, a_1 \ldots a_r}^h = W_{ijk, a_1 \ldots a_r}^h + \frac{1}{(n-1)(n-2)} (\delta_j^h R_{ik}, a_1 \ldots a_r - \delta_k^h R_{ij}, a_1 \ldots a_r)$$

$$+ \frac{1}{(n-2)} (g_{ik} R_{ij}^h, a_1 \ldots a_r - g_{ij} R_{ik}^h, a_1 \ldots a_r) - \frac{1}{(n-1)(n-2)} (g_{ik} \delta_j^h - g_{ij} \delta_k^h) \ldots (2.7)$$
Multiplying (1.10) by $\lambda a_1 \ldots a_r$ and subtracting the result thus obtained from (2.7), we get,

$$
C^h_{ijk}, a_1 \ldots a_r - \lambda a_1 \ldots a_r C^h_{ijk} = W^h_{ijk}, a_1 \ldots a_r - \lambda a_1 \ldots a_r W^h_{ijk} + \frac{1}{(n-1)(n-2)} \left\{ \delta^h_j (R_{ik}, a_1 \ldots a_r) \right\}
$$

\[- \lambda a_1 \ldots a_r R_{ik} \right\} - \delta^h_k (R_{ij}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R_{ij}) \left\{ g_{ik} \left( R^h_{ij}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R^h_{ij} \right) \right\}

$$
R, a_1 \ldots a_r - \lambda a_1 \ldots a_r R
$$

\[- g_{ij} \left( R^h_{k}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R^h_{k} \right) \right\} - \frac{1}{(n-1)(n-2)} \left( g_{ik} \delta^h_j - g_{ji} \delta^h_k \right) \quad \text{(2.8)}

Making use of equations (1.6), (1.7), (2.1), (2.2) and (2.8), we obtain the proof of the above theorem.

**Theorem 2.4.** The necessary and sufficient condition for a $W - T_n$ space to be $T_n$ is that the space be $R - T_n$ one.

**Proof.** Let the $W - T_n$ space be $T_n$ space, so that equations (1.5) and (2.1) are satisfied and (2.4), in view of (1.5) and (2.1), reduces to

$$
\delta^h_j (R_{ik}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R_{ik}) - \delta^h_k (R_{ij}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R_{ij}),
$$

which after some simplification and further calculation shows that the space is $R - T_n$.

Conversely, let the $W - T_n$ space be $R - T_n$, so that (1.6) and (1.7) are satisfied. Then (2.4), in view of (1.6) and (2.1), reduces to

$$
R^h_{ijk}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R^h_{ijk} = 0,
$$

which shows that the space is $T_n$ space.

Hence the theorem is completed.

**Theorem 2.5.** The necessary and sufficient condition for a $C - T_n$ space to be $T_n$ space is that the space be $R - T_n$.

**Proof.** Let the $C - T_n$ space be $T_n$ space, so that (1.5) and (2.2) are satisfied and (2.6), in view of (1.5) and (2.2), reduces to
\[
\begin{align*}
\text{(n-2)} \quad \{\delta^h_j (R_{ik}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R_{ij}) - \delta^h_k (R_{ij}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R_{ij}) + g_{ik} (R^h_j, a_1 \ldots a_r R^h_j) - g_{ij} (R^h_k, a_1 \ldots a_r R^h_k)\} \quad &\quad \frac{(R, a_1 \ldots a_r - \lambda a_1 \ldots a_r R)}{(n-1)(n-2)} (g_{ik} \delta^h_j - g_{ij} \delta^h_k) = 0.
\end{align*}
\]

or,
\[
(n-1) \{\delta^h_j (R_{ik}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R_{ij}) - \delta^h_k (R_{ij}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R_{ij}) + g_{ik} (R^h_j, a_1 \ldots a_r R^h_j - \lambda a_1 \ldots a_r R^h_j) \}
\]

\[
- (R, a_1 \ldots a_r - \lambda a_1 \ldots a_r R) (g_{ik} \delta^h_j - g_{ij} \delta^h_k) = 0,
\]

which after some simplification shows that the space in \(R - \mathring{T}_n\).

Conversely, let the \(C - \mathring{T}_n\) space be \(R- \mathring{T}_n\), so that (1.6), (1.7) and (2.2) reduces to

\[
R^h_{ijk}, a_1 \ldots a_r - \lambda a_1 \ldots a_r R^h_{ijk} = 0,
\]

which shows that the space is \(\mathring{T}_n\) space.

This completes the proof of the theorem.

In view of equations (1.5), (1.6) and (2.4), the following theorem can be proved easily:

**Theorem 2.6.** If in a \(T_n\) space, any two of the following properties are satisfied:

(i) the space is \(\mathring{T}_n\),  
(ii) the space is \(R - \mathring{T}_n\),  
(iii) the space is \(W - \mathring{T}_n\),

then it must also satisfy the third.

Similarly, making use of (1.5), (1.6), (1.7) and (2.6), we may immediately prove the following:

**Theorem 2.7.** If in a \(T_n\) space, any two of the following properties are satisfied:
(i) the space is $\mathbb{T}_n$,
(ii) the space is $\mathbb{R} - \mathbb{T}_n$,
(iii) the space is $\mathbb{C} - \mathbb{T}_n$,

then it must satisfy the third also.

REFERENCES


