

Some General Identities of Rogers-Ramanujan Type-II

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Abstract :

General expansions of $R(\lambda, \mu) = \prod_{\substack{n=0 \\ n \neq 0, \mu, \lambda - \mu \pmod{\lambda}}}^{\infty} (1 - q^n)^{-1}$ in multiple series are

obtained. Expansions of $R(\lambda, \mu)$ for $\lambda = 8s+14, 16s+10, 24s+14, 32s+14, 48s+26, 48s+34, 48s+38, 72s+34, 36s-3, 36s+3, 36s+15, 36s+21, 72s+39, 72s+51, 72s+57, 108s+51, 32s+12, 32s+20, 64s+28, 64s+52, 64s+60, 96s+52, 96s+68, 96s+76, 72s-6, 72s+6, 72s+30, 72s+42, 128s+104, 128s+120$ are obtained as $(s+1)$ -fold and $(s+2)$ -fold series for some values of μ . These expansions yield RR type of identities as double and triple series on specializing s further. The moduli of these RR type identities are 2, 3, 4, 5, 6 times the prime numbers 11, 13, 17, 23, 31, 37, 41.

Key Words: Basic hypergeometric series, bibasic hypergeometric series, transformations, Rogers-Ramanujan type of identities, modulus of Rogers-Ramanujan identities.

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§1. The expansion of triple product

$$R(\lambda, \mu) = \prod_{\substack{n=0 \\ n \neq 0, \mu, \lambda - \mu \pmod{\lambda}}}^{\infty} (1 - q^n)^{-1}$$

as an infinite or multiple series, for λ, μ positive integers, has been investigated since long and are known as Rogers-Ramanujan (RR) identities (λ is called the modulus of identity). In a recent paper [2], RR type identities for $R(12s-1, i), R(12s+1, i), R(12s+5, i), R(12s+7, i)$ as $(s+1)$ -fold series are obtained for some values of i and RR type identities for $R(24s+13, i), (24s+17, i), R(24s+19, i)$ and $R(36s+17, i)$ as $(s+2)$ -fold series

are obtained for some values of i. These identities for s=1, 2 yield full quota of identities of prime moduli 11, 13, 17, 19, 23, 29, 31, 37, 41, 43 and 53.

The main tools for obtaining the RR type identities are the known transformations [5; (3.1), (4.1)] : If x, y or z is of the form q^{-n} , n a non-negative integer.

$$\begin{aligned} {}_{10}W_9[a; b, x, -x, y, -y, z, -z; q; -a^3q^3/bx^2y^2z^2] = \\ \frac{[a^2q^2, a^2q^2/x^2y^2, a^2q^2/x^2z^2, a^2q^2/y^2z^2; q^2]_\infty}{[a^2q^2/x^2, a^2q^2/y^2, a^2q^2/z^2, a^2q^2/x^2y^2z^2; q^2]_\infty} \\ {}_5\Phi_4 \left[\begin{matrix} x^2, y^2, z^2, -aq/b, -aq^2/b; q^2; q^2 \\ x^2y^2z^2/a^2, a^2q^2/b^2, -aq, -aq^2 \end{matrix} \right] \quad (1.1) \end{aligned}$$

$$\begin{aligned} {}_{10}W_9[a; b, x, xq, y, yq, z, zq; q^2; a^3q^3/bx^2y^2z^2] = \\ \frac{[aq, aq/xy, aq/xz, aq/yz; q]_\infty}{[aq/x, aq/y, aq/z, aq/xyz; q]_\infty} {}_5\Phi_4 \left[\begin{matrix} x, y, z, -\sqrt{aq/b}, \sqrt{aq/b}; q; q \\ xyz/a, aq/b, -\sqrt{aq}, \sqrt{aq} \end{matrix} \right], \quad (1.2) \end{aligned}$$

and the Lemma A[2; (1.10), (1.11)].

Lemma A If p, k, s be integers with $p \geq 0$, $k \geq 1$, $s \geq 1$ then

$$\begin{aligned} [aq; q]_\infty \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \dots \sum_{r_1=0}^{r_2} C_{r_1} a^{L[(r_s)]} q^{S[(r_s)]-pr_s} / \prod_{n=0}^s [q; q]_{r_n-r_{n-1}} = \\ \sum_{j=0}^p [q^{-p}; q]_j (-a)^j q^{j(j+1)/2} / [q; q]_j \sum_{n=0}^{\infty} A_n a^{ksn} q^{n[k^2ns-pk+2kj]}, \quad (1.3) \end{aligned}$$

$$\text{where } S[(r_s)] = r_1^2 + r_2^2 + \dots + r_s^2, \quad L[(r_s)] = r_1 + r_2 + \dots + r_s \quad \text{and} \\ C_{r_1} = \sum_{r=0}^{\lfloor r_1/k \rfloor} A_r / [aq; q]_{r_1+kr} [q; q]_{r_1-kr}. \quad (1.4)$$

In this sequel it may be pointed out that the above lemma is a special case of the transformation [2; (2.7), (2.8)]

$$\begin{aligned} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{\infty} \dots \sum_{r_1=0}^{r_2} \frac{[x, y; q]_{r_s} a^{L[(r_{s-1})]} q^{S[(r_{s-1})]}}{\prod_{n=2}^s [q; q]_{r_n-r_{n-1}}} \left(\frac{aq^{1-p}}{xy} \right)^{r_s} \beta_{r_1}^{(1)} = \frac{[aq/x, aq/y; q]_\infty}{[aq, aq/xy; q]_\infty} \\ \sum_{j=0}^p \frac{[q^{-p}; q]_j q^j}{[q, xy/a; q]_j} \sum_{n=0}^{\infty} \alpha_n^{(1)} \frac{[x, y; q]_{kn+j} a^{ksn} q^{k^2n^2(s-1)}}{[aq/x, aq/y; q]_{kn}} \left(\frac{q^{1-p}}{xy} \right)^{kn}, \quad (1.5) \end{aligned}$$

where

$$\beta_n^{(1)} = \sum_{r=0}^{[n/k]} \frac{\alpha_r^{(1)}}{[aq;q]_{n+kr} [q;q]_{n-kr}}. \quad (1.6)$$

which reduces to lemma A on replacing $\beta_n^{(1)}$ and $\alpha_n^{(1)}$ by C_n and A_n respectively and letting $x, y \rightarrow \infty$. Moreover, multiplying both sides of

$C_r = \sum_{m=0}^{[r/k]} A_m / \{[aq;q]_{r-km} [q;q]_{r-km}\}$ by $\frac{[\alpha, \beta, q^{-n}; q]_r q^r}{[\alpha\beta q^{-n}/a; q]_r}$ and summing with respect to r from 0 to n, we get [2; (3.5)]

$$\sum_{r=0}^n \frac{[\alpha, \beta; q]_r [aq/\alpha\beta; q]_{n-r} C_r}{[q; q]_{n-r}} \left(\frac{aq}{\alpha\beta}\right)^r = \sum_{m=0}^n \frac{[\alpha, \beta; q]_{km} [aq/\alpha, aq/\beta; q]_n}{[aq; q]_{n+km} [q; q]_{n-km} [aq/\alpha, aq/\beta; q]_{km}} \left(\frac{aq}{\alpha\beta}\right)^{km} A_m \quad (1.7)$$

In this paper some general RR type identities are obtained which on specialization give RR type identities on moduli 2, 3, 4, 6, 8 times the moduli of sum of the primes 11, 13, 17, 19, 23, 29, 31, 37, 41, 43 and 53. The notations used are that of [1, 2].

§2. We begin this section by obtaining multiple series expansions of $R(8s+14, i)$, $R(16s+10, i)$, $R(24s+10, i)$, $R(24s+14, i)$, $R(32s+14, i)$ or their linear combination which for $s=1$ yield known, as well new, RR type of identities of moduli 22, 26, 34, 38, 46 and their linear combinations. The expansions of $R(24s+10, i)$ and $R(24s+14, i)$ for $s=2$ reduce to RR identities of moduli 58 and 62. Identities for $R(48s+26, i)$,

$R(48s+34, i)$, $R(48s+38, i)$, $R(72s+34, i)$, will also be obtained which for $s=1$ give RR identities on modulus 74, 82, 86 and 106 (i.e. the general identities obtained contain as special cases RR identities whose moduli are twice the prime numbers 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 53). For this we start by proving the transformations:

If $F_\lambda[r, (r_s)] = q^{r^2 + \lambda s[(r_s)]} / [q; q]_r \{ \prod_{j=2}^s [q^\lambda; q^\lambda]_{r_j - r_{j-1}} \} [q^\lambda; q^\lambda]_{r_1 - r}$, then

$$\begin{aligned} & [a^2 q; q]_\infty \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{r_1} \frac{F_1[r, (r_s)] a^{2r+2L[(r_s)]} q^{-pr_s}}{[a^2 q; q^2]_r} \\ &= \sum_{j=0}^p \frac{[q^{-p}; q]_j (-a^2)^j q^{\frac{1}{2}j(j+1)}}{[q; q]_j} \sum_{n=0}^{\infty} \frac{[a^2; q^2]_n (1-a^2 q^{4n}) (-)^n a^{6n+4sn} q^{(4s+7)n^2 - n - 2pn + 4nj}}{[q^2; q^2]_n (1-a^2)} \end{aligned} \quad (2.1)$$

$$\begin{aligned}
& [a^4 q^2; q^2]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{r_1} \frac{F_2[r, (r_s)] a^{2r+4L[(r_s)]} q^{\frac{1}{2}r(r-1)-2pr_s}}{[a^2 q; q^2]_r [-a^2 q; q]_{2r_1}} \\
& = \sum_{j=0}^p \frac{[q^{-2p}; q^2]_j (-a^4)^j q^{j(j+1)}}{[q^2; q^2]_j} \sum_{n=0}^{\infty} \frac{[a^2; q^2]_n (1-a^2 q^{4n}) (-a^{4+8s})^n q^{n((8s+5)n-4p+8j-1)}}{[q^2; q^2]_n (1-a^2)} \quad (2.2)
\end{aligned}$$

$$\begin{aligned}
& [a^4 q^2; q^2]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{r_1} \frac{F_2[r, (r_s)] a^{2r+4L[(r_s)]} q^{\frac{r}{2}(r-1)+4L[(r_s)]-2pr_s}}{[a^2 q; q^2]_r [-a^2 q; q]_{2r_1} (1-a^4 q^{4r_1+2})} \\
& = \sum_{j=0}^p \frac{[q^{-2p}; q^2]_j (-a^4)^j q^{j(j+5)}}{[q^2; q^2]_j} \sum_{n=0}^{\infty} \frac{[a^2 q^2; q^2]_n (1-a^4 q^{8n+4}) (-a^{4+8s})^n q^{n((8s+5)n+8s+1-4p+8j)}}{[q^2; q^2]_n} \quad (2.3)
\end{aligned}$$

$$\begin{aligned}
& [a^6 q^3; q^3]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{r_1} \frac{F_3[r, (r_s)] [a^2 q; q]_{3r_1-r} [a^2 q/b; q^2]_r a^{2r+6L[(r_s)]} q^{-3pr_s}}{[a^2 q; q^2]_r [a^6 q^3; q^3]_{2r_1} [a^2 q/b; q]_r} \\
& = \sum_{j=0}^p \frac{[q^{-3p}; q^3]_j (-a^6)^j q^{\frac{3}{2}j(j+1)}}{[q^3; q^3]_j} \sum_{n=0}^{\infty} \frac{[a^2, b; q^2]_n (1-a^2 q^{4n}) (-a^{6+12s})^n q^{n[(12s+6)n-6p+12j]}}{[q^2, a^2 q^2/b; q^2]_n (1-a^2) b^n} \quad (2.4)
\end{aligned}$$

$$\begin{aligned}
& [a^6 q^3; q^3]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{r_1} \frac{F_3[r, (r_s)] [a^2 q; q]_{3r_1-r} a^{2r+6L[(r_s)]} q^{\frac{r}{2}(r-1)+6L[(r_s)]-3pr_s}}{[a^2 q; q^2]_r [a^6 q^3; q^3]_{2r_1+1}} \\
& = \sum_{j=0}^p \frac{[q^{-3p}; q^3]_j (-a^6)^j q^{\frac{3}{2}j(j+5)}}{[q^3; q^3]_j} \sum_{n=0}^{\infty} \frac{[a^2 q^2; q^2]_n (1-a^6 q^{12n+6}) (-a^{4+12s})^n q^{n[(12s+5)n+12s]}}{[q^2; q^2]_n q^{n(1+6p-12j)}} \quad (2.5)
\end{aligned}$$

$$\begin{aligned}
& [a^6 q^3; q^3]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{r_1} \frac{F_3[r, (r_s)] [a^2 q; q]_{3r_1-r} a^{2r+6L[(r_s)]} q^{6L[(r_s)]-3pr_s}}{[a^2 q; q^2]_r [a^6 q^3; q^3]_{2r_1+1}} \\
& = \sum_{j=0}^p \frac{[q^{-3p}; q^3]_j (-a^6)^j q^{\frac{3}{2}j(j+5)}}{[q^3; q^3]_j} \sum_{n=0}^{\infty} \frac{[a^2 q^2; q^2]_n (1-a^6 q^{12n+6}) (-a^{6+12s})^n q^{(12s+7)n^2}}{[q^2; q^2]_n q^{n(6p-12s-12j-1)}} \quad (2.6)
\end{aligned}$$

$$\begin{aligned}
& [a^2 q; q]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{r_1} \sum_{m=0}^{[r_1/2][r_1/2]} \frac{[x^2; q^4]_m [a^2 q^2/x^2; q^4]_{r-m} [aq; q^2]_r [a; q^2]_{r_1-r} H[m, r, (r_s)] a^{2m+r+2L[(r_s)]} q^{2m-pr_s}}{[-aq^2; q^2]_{2m} [a^2 q^2; q^4]_r [a^2 q^2/x^2; q^2]_r [a; q]_{r_1} [a^2 q; q^2]_{r_1} x^{2m}} \\
& = \sum_{j=0}^p \frac{[q^{-p}; q]_j (-a^2)^j q^{\frac{1}{2}j(j+1)}}{[q; q]_j} \sum_{n=0}^{\infty} \frac{[a; q^2]_n (1-aq^{4n}) [x^2; q^4]_n a^{n(4+8s)} q^{(16s+7)n^2+n}}{[q^2; q^2]_n (1-a) \left[a^2 q^4/x^2; q^4 \right]_n x^{2n} q^{(4p-8j)n}}, \quad (2.7)
\end{aligned}$$

where $H[m, r, (r_s)] = (-)^r q^{4m^2+r^2+S[(r_s)]} / [q^4; q^4]_m [q^2; q^2]_{r-2m} [q; q]_{r_1-2r} \left\{ \prod_{j=2}^s [q; q]_{r_j-r_{j-1}} \right\}$.

Proof of (2.1): Transformation (1.2) on replacing a, z by a^2, q^{-r} and letting $b, x, y \rightarrow \infty$ becomes

$$\sum_{r=0}^{r_1} \frac{a^{2r} q^{r^2}}{[q;q]_r [q;q]_{r_1-r} [a^2 q; q^2]_r} = \sum_{r=0}^{[r_1/2]} \frac{[a^2; q^2]_r (1-a^2 q^{4r}) (-a^6)^r q^{r(7r-1)}}{[q^2; q^2]_r (1-a^2) [a^2 q; q]_{r_1+2r} [q; q]_{r_1-2r}}. \quad (2.8)$$

Now, in the transformation (1.3) replacing a by a^2 and setting, $k=2$,

$A_r = [a^2; q^2]_r (1-a^2 q^{4r}) (-a^6)^r q^{r(7r-1)} / [q^2; q^2]_r (1-a^2)$ and transforming C_{r_1} by (2.8), yields (2.1).

Proof of (2.2): In (1.2) replacing a, y, z by $a^2, -q^{-r}, q^{-r}$ and letting $x \rightarrow \infty, b \rightarrow 0$ to obtain

$$\sum_{r=0}^{r_1} \frac{a^{2r} q^{r(3r-1)/2}}{[q;q]_r [q^2; q^2]_{r_1-r} [a^2 q; q^2]_r [-a^2 q; q]_{2r}} = \sum_{r=0}^{[r_1/2]} \frac{[a^2; q^2]_r (1-a^2 q^{4r}) (-a^4)^r q^{r(5r-1)}}{[q^2; q^2]_r (1-a^2) [a^4 q^2; q^2]_{r_1+2r} [q^2; q^2]_{r_1-2r}} \quad (2.9)$$

In (1.3) replacing a, q by a^4 and q^2 respectively and setting $k=2$, $A_r = [a^2; q^2]_r (1-a^2 q^{4r}) (-a^4)^r q^{r(5r-1)} / [q^2; q^2]_r (1-a^2)$, and transforming C_{r_1} by (2.9), gives (2.2).

Proof of (2.3): In (2.9) replacing $(1-a^2 q^{4r})$ by $\{(1-a^2 q^{2r})q^{2r} + (1-q^{2r})\}$ and breaking the series into two series and adjusting the summation index in the second series appropriately and combining the two series, we have

$$\sum_{r=0}^{r_1} \frac{a^{2r} q^{r(3r-1)/2} (1-a^4 q^2) (1-a^4 q^4)}{[q;q]_r [q^2; q^2]_{r_1-r} (1-a^4 q^{4r_1+2}) [a^2 q; q^2]_r [-a^2 q; q]_{2r}} = \sum_{r=0}^{[r_1/2]} \frac{[a^2 q^2; q^2]_r (1-a^4 q^{8r+4}) (-a^4)^r q^{r(5r+1)}}{[q^2; q^2]_r [a^4 q^6; q^2]_{r_1+2r} [q^2; q^2]_{r_1-2r}}. \quad (2.10)$$

In the transformation (1.3) replacing a, q by $a^4 q^4$ and q^2 respectively and setting $k=2$, $A_r = [a^2 q^2; q^2]_r (1-a^4 q^{8r+4}) (-a^4)^r q^{r(5r+1)} / [q^2; q^2]_r$, transforming C_{r_1} by (2.10) gives (2.3).

Proof of (2.4): In (1.2) replacing a, x, y, z by $a^2, \omega q^{-m}, \omega^2 q^{-m}, q^{-m}$ respectively (where ω is cube root of unity), gives

$$\begin{aligned} & \sum_{r=0}^{[m/2]} \frac{[a^2, b; q^2]_r (1-a^2 q^{4r}) a^{6r} q^{6r^2}}{[q^2, a^2 q^2/b; q^2]_r (1-a^2) [a^6 q^3; q^3]_{m+2r} [q^3; q^3]_{m-2r} b^r} \\ &= \sum_{r=0}^m \frac{[a^2 q/b; q^2]_r [a^2 q; q]_{3m-r} a^{2r} q^{r^2}}{[q, a^2 q/b; q]_r [a^2 q; q^2]_r [q^3; q^3]_{m-r} [a^6 q^3; q^3]_{2m}} \end{aligned} \quad (2.11)$$

Next, in the transformation (1.3) replacing a, q by a^6 and q^3 respectively and setting $k = 2$, $A_r = [a^2, b; q^2]_r (1 - a^2 q^{4r}) q^{6r^2} a^{6r} / [q^2, a^2 q^2 / b; q^2]_r (1 - a^2) b^r$, transforming C_{r_1} by (2.11), yields (2.4).

Proof of (2.5): In the transformation (2.11) letting $b \rightarrow 0$ and rewriting $(1 - a^2 q^{4r})$ in the LHS as $[(1 - a^2 q^{2r}) + a^2 q^{2r} (1 - q^{2r})]$ breaking it into two series, adjusting the summation index suitably, we get the transformation

$$\sum_{r=0}^{[m/2]} \frac{[a^2 q^2; q^2]_r (1 - a^6 q^{12r+6}) (-a^4)^r q^{r(5r-1)}}{[q^2; q^2]_r [a^6 q^9; q^3]_{m+2r} [q^3; q^3]_{m-2r}} = \frac{(1 - a^6 q^3)(1 - a^6 q^6)}{[a^6 q^3; q^3]_{2m+1}} \sum_{r=0}^m \frac{[a^2 q; q]_{3m-r} a^{2r} q^{r(3r-1)/2}}{[q; q]_r [a^2 q; q^2]_{m+r} [q^3; q^3]_{m-r}} \quad (2.12)$$

Now, in the transformation (1.3) replacing a, q by $a^6 q^6$ and q^3 respectively and setting $k = 2$, $A_r = [a^2 q^2; q^2]_r (1 - a^6 q^{12r+6}) (-a^4)^r q^{r(5r-1)} / [q^2; q^2]_r$, transforming C_{r_1} by (2.12), yields (2.5).

Proof of (2.6): In the transformation (2.11) letting $b \rightarrow \infty$ and rewriting $(1 - a^2 q^{4r})$ in the LHS as $[(1 - q^{2r}) + q^{2r} (1 - a^2 q^{2r})]$, proceeding as in the proof of (2.12), we get the transformation

$$\sum_{r=0}^{[m/2]} \frac{[a^2 q^2; q^2]_r (1 - a^6 q^{12r+6}) (-a^6)^r q^{r(7r+1)}}{[q^2; q^2]_r [a^6 q^9; q^3]_{m+2r} [q^3; q^3]_{m-2r}} = \frac{(1 - a^6 q^3)(1 - a^6 q^6)}{[a^6 q^3; q^3]_{2m+1}} \sum_{r=0}^m \frac{[a^2 q; q]_{3m-r} a^{2r} q^{r^2}}{[q; q]_r [a^2 q; q^2]_r [q^3; q^3]_{m-r}}. \quad (2.13)$$

Now, in the transformation (1.3) replacing a, q by $a^6 q^6$ and q^3 respectively and then setting $k = 2$, $A_r = [a^2 q^2; q^2]_r (1 - a^6 q^{12r+3}) (-a^6)^r q^{r(7r+1)} / [q^2; q^2]_r$, transforming C_{r_1} by (2.13), we get (2.6).

Proof of (2.7): In (1.1) replacing q by q^2 , letting $b \rightarrow 0$ and then setting $z = q^{-r}$, $y = q^{1-r}$, gives the transformation

$$\begin{aligned} & \sum_{m=0}^{[r/2]} \frac{[a; q^2]_m [x^2; q^4]_m (1 - aq^{4m}) a^{2m} q^{m(3m+1)}}{[q^2; q^2]_m [a^2 q^4 / x^2; q^4]_m (1 - a)[q^2; q^2]_{r-2m} [a^2 q^2; q^2]_{r+2m} x^{2m}} \\ &= \frac{1}{[a^2 q^2; q^4]_r [a^2 q^2 / x^2; q^2]_r} \sum_{m=0}^{[r/2]} \frac{[x^2; q^4]_m [a^2 q^2 / x^2; q^4]_{r-m} a^{2m} q^{2m(2m+1)}}{[q^4; q^4]_m [-aq^2; q^2]_{2m} [q^2; q^2]_{r-2m} x^{2m}} \end{aligned} \quad (2.14)$$

Multiplying both sides by $[C, q^{-n}, q^{1-n}; q^2]_r q^{2r} / [Cq^{1-2n} / a^2; q^2]_r$ and summing both sides with respect to r from 0 to $[n/2]$, interchanging the order of summation on the LHS and summing the inner series and finally letting $c = a/q$, yields the transformation

$$\begin{aligned}
& \sum_{m=0}^{[n/4]} \frac{[a;q^2]_m [x^2;q^4]_m (1-aq^{4m}) a^{4m} q^{m(7m+1)}}{[q^2;q^2]_m [a^2q^4/x^2;q^4]_m (1-a)[q;q]_{n-4m} [a^2q;q^2]_{n+4m} x^{2m}} \\
& = \sum_{r=0}^{[n/2]} \sum_{m=0}^{[r/2]} \frac{[x^2;q^4]_m [a^2q^2/x^2;q^4]_{r-m} [aq;q^2]_r [a;q^2]_{n-r} (-)^r a^{r+2m} q^{4m^2+2m+r^2} x^{-2m}}{[q^4;q^4]_m [-aq^2;q^2]_{2m} [q^2;q^2]_{r-2m} [a^2q^2;q^4]_r [a^2q^2/x^2;q^2]_r [q;q]_{n-2r} [a^2q;q^2]_n [a;q]_n}.
\end{aligned} \tag{2.15}$$

In the transformation (1.3) replacing a by a^2 , setting $k=4$,

$A_r = [a;q^2]_r [x^2;q^4]_r (1-aq^{4r}) a^{4r} q^{r(7r+1)} / [q^2;q^2]_r [a^2q^4/x^2;q^4]_r (1-a)x^{2r}$, transforming C_r by (2.15), yields (2.7).

Identities for $R(8s+14, i)$:

Transformation (2.1) for $a=1, p=0; a=1, p=1; a=q, p=0; a=q, p=1$ and $a=q, p=2$ give identities which on some manipulations yield RR of identities for $R(8s+14, 2i)$: $i=1, 2, 3, 2s+2, 2s+3$ as $(s+1)$ fold series. These identities for $s=1$ reduce to known [4; (3.18)-(3.22)] RR type of identities for $R(2, 2, 2, i) : i \leq 5$ as double series.

Identities for $R(16s+10, i)$:

Transformation (2.2) for $a=1, p=0; a=1, p=1; a=q, p=0; a=q, p=1$ and (5.3) for $a=1, p=0; a=1, p=1$ give, on combining amongst themselves, RR type of identities as $(s+1)$ -fold series for $R(16s+10, i)$: $i=2, 4, 6, 8, 8s+4$ and an expansion for $R(16s+10, 8s) + R(16s+10, 8s+2)$. These identities for $s=1$ reduce to known [4; (3.23)-(3.28)] double series RR type of identities for $R(26, 2i) : 1 \leq i \leq 6$.

Identities for $R(24s+10, i)$:

The transformation (2.4) for $b \rightarrow 0$ and $a=1, p=0; a=1, p=1; a=q, p=0; a=q, p=1; a=q^2, p=0$ and (2.5) for $a=1, p=0; a=1, p=1; a=q, p=1$ yield $(s+1)$ -fold series expansions for $R(24s+10, 12s+4), R(24s+10, 12s-2) + R(24s+10, 12s)$,

$$R(24s+10, 2) + R(24s+10, 8) - q^2 R(24s+10, 4),$$

$$R(24s+10, 12s+4) - R(24s+10, 12s+2),$$

$R(24s+10, 6), R(24s+10, 12)$ and $R(24s+10, 12s-6) - R(24s+10, 12s-8)$ on some manipulations. These expansions reduce for $s=1$ to the known [4; equations (3.30)-(3.37)] RR type of double series identities for $R(34, 2i) : 1 \leq i \leq 8$. Furthermore (2.4) for $b \rightarrow 0$ and $a=q^3, p=0; a=1, p=2; a=1, p=3; a=q, p=2; a=q^2, p=1$ and (2.5) for $a=1, p=2$ give $(s+1)$ -fold series expansions for $R(24s+10, 4), R(24s+10, 12s-8) + R(24s+10, 12s-6)$,

$$R(24s+10, 12s-14) + R(24s+10, 12s-12),$$

$R(24s+10,14) - q^2 R(24s+10,10)$, $R(24s+10,12s-4)$,
 $R(24s+10,18)$. All the above expansions for $s=2$ yield RR type of identities for $R(58,2i)$: $1 \leq i \leq 14$ as triple series on some manipulations.

Identities for $R(24s+14,i)$:

Transformation (5.4) for $b \rightarrow \infty$ and $a=1, p=0; a=1, p=1; a=q, p=0; a=q, p=1; a=q, p=2; a=q^2, p=0$ and (5.6) for $a=1, p=0; a=1, p=1; a=q, p=1$ give expansions which on straight forward manipulation, yields $(s+1)$ -fold series expansions for

$$R(24s+14, i) : \quad i=2, \quad 6, \quad 12, 12s+4, \quad 12s+6 \quad \text{and}$$

$$R(24s+14, 12s) + R(24s+14, 12s+2),$$

$$R(24s+14,14) - q^2 R(24s+14,10), R(24s+14,8) - q^2 R(24s+14,4),$$

$R(24s+14,12s-4) - R(24s+14,12s-6)$. These expansions yield for $s=1$, known RR type of identities $R(38, i)$: $1 \leq i \leq 9$. In this sequel it may also be noted that for $s=2$ the above identities and the identities obtained from (5.4) on letting $b \rightarrow \infty$ and setting $a=q^3, p=0; a=1, p=2; a=1, p=3; a=q, p=3; a=q^2, p=1$ and (5.5) for $a=1, p=2; a=1, p=3$ that is $R(24s+14, 4)$,

$$R(24s+14, 12s-6) + R(24s+14, 12s-4),$$

$$R(24s+14, 20) - q^2 R(24s+14, 16),$$

$$R(24s+14, 12s-12) + R(24s+14, 12s-10),$$

$R(24s+14, 12s-2)$, $R(24s+14, 18)$ and $R(24s+14, 24)$ respectively also yield RR type of identities for $R(62, 2i)$: $1 \leq i \leq 15$.

Identities for $R(32s+14, i)$:

Transformation (2.7) for $x^2 = -aq^2$ and then $a=1, p=0; a=q^2, p=0; a=q^4, p=0; a=1, p=1; a=q^2, p=1; a=q^4, p=1; a=q^2, p=2$ and (2.7) for $x^2 = -aq^4$ and then $a=q^2, p=0; a=q^2, p=1; a=q^2, p=2$ and $a=q^4, p=1$ give $(s+2)$ -fold series expansions for $R(32s+14, i)$: $i=16s+6, 2, 16s+4, 16s+2, 6, 16s, 10, 4, 8, 12$ and $16s-2$, on some manipulations. These expansions for $s=1$ become RR type identities for $R(46s+14, 2i)$: $1 \leq i \leq 11$.

Next, for obtaining RR type of identities of moduli $48s+26, 48s+34, 48s+38$ and $72s+34$, we prove the following transformations:

If $M_\lambda[m, r, (r_s)] = q^{m^2 + \lambda r^2 + 6s[(r_s)]} / [q; q]_m [q^\lambda; q^\lambda]_{r-m} [q^6; q^6]_{r_1-r} \prod_{n=2}^s [q^6; q^6]_{r_n-r_{n-1}}$,

then

$$[a^{12}q^6; q^6]_\infty \sum_{r_s=0}^{\infty} \sum_{r_{s-1}}^{r_s} \dots \sum_{r_1=0}^{r_s} \sum_{r=0}^{r_1} \sum_{m=0}^r \frac{[a^4q^2; q^2]_{3r_1-r} M_2[m, r, (r_s)] a^{2m+4r+12L[(r_s)]} q^{-6pr_s+m(m-1)/2}}{[a^2q; q^2]_m [-a^2q; q]_{2r} [a^{12}q^6; q^6]_{2r}}$$

$$= \sum_{j=0}^p \frac{[q^{-6p}; q^6]_j (-a^{12})^j q^{3j(j+1)}}{[q^6; q^6]_j} \sum_{n=0}^{\infty} \frac{[a^2; q^2]_n (1 - a^2 q^{4n}) (-)^n a^{2n(6+12s)} q^{n[(24s+13)n-12p+24j-1]}}{[q^2; q^2]_n (1 - a^2)} \quad (2.16)$$

$$[a^{12}q^6; q^6]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \dots \sum_{r_1=0}^{r_s} \sum_{r=0}^{r_1} \sum_{m=0}^r \frac{[a^4q^2; q^2]_{3r_1-r+2} M_2[m, r.(r_s)] a^{2m+4r+12L[(r_s)]_s} q^{4r+12L[\{r_s\}]-6pr_s+m(m-1)/2}}{[a^2q; q^2]_m [-a^2q; q]_{2r} [a^{12}q^6; q^6]_{2r_1+2} (1 - a^4 q^{4r+2})}$$

$$= \sum_{j=0}^p \frac{[q^{-6p}; q^6]_j (-a^{12})^j q^{3j(j+5)}}{[q^6; q^6]_j} \sum_{n=0}^{\infty} \frac{[a^2q^2; q^2]_n (1 - a^4 q^{8n+4}) (-)^n a^{2n(6+12s)} q^{n[(24s+13)n-12p+24j+24s+9]}}{[q^2; q^2]_n} \quad (2.17)$$

$$[a^{12}q^6; q^6]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \dots \sum_{r_1=0}^{r_s} \sum_{r=0}^{r_1} \sum_{m=0}^r \frac{[a^4q^2; q^2]_{3r_1-r} M_2[m, r.(r_s)] a^{2m+4r+12L[(r_s)]_s} q^{12L[\{r_s\}]-6pr_s+m(m-1)/2}}{[a^2q; q^2]_m [-a^2q; q]_{2r} [a^{12}q^6; q^6]_{2r_1+1}}$$

$$= \sum_{j=0}^p \frac{[q^{-6p}; q^6]_j (-a^{12})^j q^{3j(j+5)}}{[q^6; q^6]_j} \sum_{n=0}^{\infty} \frac{[a^2q^2; q^2]_n (1 - a^{12} q^{24n+12}) (-)^n a^{2n(6+12s)} q^{n[(24s+13)n-12p+24j+24s+1]}}{[q^2; q^2]_n} \quad (2.18)$$

$$[a^6q^6; q^6]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \dots \sum_{r_1=0}^{r_s} \sum_{r=0}^{r_1} \sum_{m=0}^r \frac{[aq; q]_{3r-m} [aq/b; q^2]_m M_3[m, r.(r_s)] a^{m+3r+6L[(r_s)]} q^{-6pr_s}}{[aq; q^2]_m [aq/b; q]_m [a^3q^3; q^3]_{2r} [-a^3q^3; q^3]_{2r_1}}$$

$$= \sum_{j=0}^p \frac{[q^{-6p}; q^6]_j (-a^6)^j q^{3j(j+1)}}{[q^6; q^6]_j} \sum_{n=0}^{\infty} \frac{[a, b, q^2]_n (1 - aq^{4n}) a^{n(9+12s)} q^{n[(24s+18)n-12p+24j]}}{[q^2, aq^2/b; q^2]_n (1-a)b^n} \quad (2.19)$$

$$[a^6q^6; q^6]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \dots \sum_{r_1=0}^{r_s} \sum_{r=0}^{r_1} \sum_{m=0}^r \frac{[aq; q]_{3r-m} M_3[m, r.(r_s)] a^{m+3r+6L[(r_s)]} q^{-6pr_s+6r+12L[(r_s)]+m(m-1)/2}}{[aq; q^2]_m [a^3q^3; q^3]_{2r+1} [-a^3q^3; q^3]_{2r_1+2}}$$

$$= \sum_{j=0}^p \frac{[q^{-6p}; q^6]_j (-a^6)^j q^{3j(j+5)}}{[q^6; q^6]_j} \sum_{n=0}^{\infty} \frac{[aq^2; q^2]_n (1 - a^3 q^{12n+6}) (-)^n a^{n(8+12s)} q^{n[(24s+17)n-12p+24j+24s+11]}}{[q^2; q^2]_n} \quad (2.20)$$

$$\begin{aligned}
& [a^6 q^6; q^6]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}}^{\infty} \dots \sum_{r_1=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^r \frac{[aq; q]_{3r-m} M_3[m, r.(r_s)] a^{m+3r+6L[(r_s)]} q^{-6pr_s+6r+12L[(r_s)]}}{[aq; q^2]_m [a^3 q^3; q^3]_{2r+1} [-a^3 q^3; q^3]_{2r+2}} \\
& = \sum_{j=0}^p \frac{[q^{-6p}; q^6]_j (-a^6)^j q^{3j(j+5)}}{[q^6; q^6]_j} \sum_{n=0}^{\infty} \frac{[aq^2; q^2]_n (1 - a^3 q^{12n+6}) (-)^n a^{n(9+12s)} q^{n[(24s+19)n-12p+24j+24s+13]}}{[q^2; q^2]_n}
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
& [a^9 q^9; q^9]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}}^{\infty} \dots \sum_{r_1=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^r \frac{[aq; q]_{3r-m} [a^3 q^3; q^3]_{3r_1-r} N[m, r.(r_s)] a^{m+3r+9L[(r_s)]} q^{-9pr_s}}{[aq; q^2]_m [a^3 q^3; q^3]_{2r} [a^9 q^9; q^9]_{2r_1}} \\
& = \sum_{j=0}^p \frac{[q^{-9p}; q^9]_j (-a^9)^j q^{9j(j+1)/2}}{[q^9; q^9]_j} \sum_{n=0}^{\infty} \frac{[a; q^2]_n (1 - a q^{4n}) (-)^n a^{n(8+18s)} q^{n[(36s+17)n-18p+36j-1]}}{[q^2; q^2]_n (1 - a)}
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
& [a^9 q^9; q^9]_{\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}}^{\infty} \dots \sum_{r_1=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^r \frac{[aq; q]_{3r-m} [a^3 q^6; q^3]_{3r_1-r+1} N[m, r.(r_s)] a^{m+3r+9L[(r_s)]} q^{-9pr_s+6r+18L[(r_s)]}}{[aq; q^2]_m [a^3 q^6; q^3]_{2r} [a^9 q^9; q^9]_{2r_1+1}} \\
& = \sum_{j=0}^p \frac{[q^{-9p}; q^9]_j (-a^9)^j q^{9j(j+5)/2}}{[q^9; q^9]_j} \sum_{n=0}^{\infty} \frac{[aq^2; q^2]_n (1 - a^3 q^{12n+6}) (-)^n a^{n(8+18s)} q^{n[(36s+17)n-18p+36j+36s+11]}}{[q^2; q^2]_n}
\end{aligned} \tag{2.23}$$

where

$$N[m, r, (r_s)] = q^{3r^2 + 9S[(r_s)] + m(3m-1)/2} / [q; q]_m [q^3; q^3]_{r-m} [q^9; q^9]_{r_1-r} \prod_{n=2}^s [q^9; q^9]_{r_n-r_{n-1}} \tag{2.24}$$

Proof of (2.16): In (1.7) replacing a and q by a^4 and q^2 respectively and evaluating ζ by (2.9), we have, on setting $\alpha = \omega q^{-2n}, \beta = \omega^2 q^{-2n}$ (ω is a cube root of unity)

$$\begin{aligned}
& \sum_{r=0}^n \sum_{m=0}^r \frac{[a^4 q^2; q^2]_{3n-r} a^{2m+4r} q^{2r^2+m(3m-1)/2}}{[q; q]_m [a^2 q; q^2]_m [q^2; q^2]_{r-m} [q^6; q^6]_{n-r} [a^{12} q^6; q^6]_{2n} [-a^2 q; q]_{2r}} \\
& = \sum_{m=0}^{[n/2]} \frac{[a^2; q]_m (1 - a^2 q^{4m}) (-)^m a^{12m} q^{m(13m-1)}}{[q^2; q^2]_m (1 - a^2) [a^{12} q^6; q^6]_{n+2m} [q^6; q^6]_{n-2m}}.
\end{aligned} \tag{2.25}$$

(2.16) is obtained from (1.3) by replacing a, q by a^{12}, q^6 respectively and then setting $k = 2, A_n = [a^2; q]_n (1 - a^2 q^{4n}) (-)^n a^{12n} q^{n(13n-1)} / [q^2; q^2]_n (1 - a^2)$ and evaluating ζ by (2.25).

Proof of (2.17): In (1.7) replacing a and q by $a^4 q^4$ and q^2 respectively and evaluating ζ by (2.10), we have, on setting $\alpha = \omega q^{-2n}$, $\beta = \omega^2 q^{-2n}$ (ω is a cube root of unity)

$$\begin{aligned} & \sum_{r=0}^n \sum_{m=0}^r \frac{[a^4 q^2; q^2]_{3n-r+2} a^{2m+4r} q^{2r(r+2)+m(3m-1)/2}}{[q; q]_m [a^2 q; q^2]_m [q^2; q^2]_{r-m} [q^6; q^6]_{n-r} [a^{12} q^{18}; q^6]_{2n} [-a^2 q; q]_{2r} (1 - a^4 q^{4r+2})} \\ &= \sum_{m=0}^{[n/2]} \frac{[a^2 q^2; q^2]_m (1 - a^4 q^{8m+4}) (-)^m a^{12m} q^{m(13m+9)}}{[q^2; q^2]_m [a^{12} q^{18}; q^6]_{n+2m} [q^6; q^6]_{n-2m}}. \end{aligned} \quad (2.26)$$

(2.17) is obtained from (1.3) by replacing a, q by $a^{12} q^{12}, q^6$ respectively and then setting $k = 2$, $A_n = [a^2 q^2; q^2]_n (1 - a^4 q^{8n+4}) (-)^n a^{12n} q^{n(13n+9)} / [q^2; q^2]_n$ and evaluating ζ by (2.26).

Proof of (2.18): In the RHS of (2.25) replacing $(1 - a^2 q^{4m})$ by $\{(1 - a^2 q^{2m})q^{2m} + (1 - q^{2m})\}$ and proceeding as in the proof of (2.10), we get

$$\begin{aligned} & \sum_{r=0}^n \sum_{m=0}^r \frac{[a^4 q^2; q^2]_{3n-r} (1 - a^{12} q^6) (1 - a^{12} q^{12}) a^{2m+4r} q^{2rr+m(3m-1)/2}}{[q; q]_m [a^2 q; q^2]_m [q^2; q^2]_{r-m} [q^6; q^6]_{n-r} [a^{12} q^6; q^6]_{2n} [-a^2 q; q]_{2r} (1 - a^{12} q^{12n+6})} \\ &= \sum_{m=0}^{[n/2]} \frac{[a^2 q^2; q]_m (1 - a^{12} q^{24m+12}) (-a^6)^m q^{m(13m+1)}}{[q^2; q^2]_m [a^{12} q^{18}; q^6]_{n+2m} [q^6; q^6]_{n-2m}}. \end{aligned} \quad (2.27)$$

(2.18) is obtained from (1.3) by following the same lines as that of (2.17) except we use (2.27) instead of (2.26).

Proof of (2.19): (1.2) for $x = \omega q^{-r}, y = \omega^2 q^{-r}, z = q^{-r}$ (ω cube root of unity) becomes

$$\begin{aligned} & \sum_{m=0}^r \frac{[aq; q]_{3r-m} [aq/b; q^2]_m a^m q^{m^2}}{[q, aq/b; q]_m [aq; q^2]_m [q^3; q^3]_{r-m} [a^3 q^3; q^3]_{2r}} \\ &= \sum_{m=0}^{[r/2]} \frac{[a, b; q^2]_m (1 - aq^{4m}) a^{3m} q^{6m^2}}{[q^2, aq^2/b; q^2]_m (1 - a) [a^3 q^3; q^3]_{r+2m} [q^3; q^3]_{r-2m} b^m}. \end{aligned} \quad (2.28)$$

In (1.7) replacing a and q by a^3 and q^3 respectively, letting $\alpha \rightarrow \infty$ and setting $k = 2, \beta = -q^{-n}$ and $A_n = [a, b; q^2]_n (1 - aq^{4n}) a^{3n} q^{6n^2} / [q^2, aq^2/b; q^2]_n (1 - a)$ and evaluating ζ by (2.28), we get

$$\begin{aligned} & \sum_{r=0}^n \sum_{m=0}^r \frac{[aq;q]_{3r-m} [aq/b;q^2]_m a^{m+3r} q^{3r^2+m^2}}{[q, aq/b; q]_m [aq; q^2]_m [q^3; q^3]_{r-m} [q^6; q^6]_{n-r} [a^3 q^3; q^3]_{2r} [-a^3 q^3; q^3]_{2n}} \\ &= \sum_{m=0}^{[n/2]} \frac{[a, b; q^2]_m (1 - aq^{4m}) a^{9m} q^{18m^2}}{[q^2, aq^2/b; q^2]_m (1 - a) [a^6 q^6; q^6]_{n+2m} [q^6; q^6]_{n-2m} b^m}. \end{aligned} \quad (2.29)$$

(2.19) is obtained from (1.3) by replacing a, q in it by a^6, q^6 and setting $k = 2$, $A_n = [a, b; q^2]_n (1 - aq^{4n}) a^{9n} q^{18n^2} / [q^2, aq^2/b; q^2]_n (1 - a) b^n$ and evaluating C_r by (2.29)

Proof of (2.20): In (2.28) letting $b \rightarrow 0$ and replacing $(1 - aq^{4m})$ by $\{(1 - aq^{2m}) + aq^{2m}(1 - q^{2m})\}$ and proceeding as in the proof of (2.10), we get

$$\sum_{m=0}^r \frac{[aq;q]_{3r-m} (1 - a^3 q^6) a^m q^{m(3m-1)/2}}{[q;q]_m [aq;q^2]_m [q^3;q^3]_{r-m} [a^3 q^6; q^3]_{2r}} = \sum_{m=0}^{[r/2]} \frac{[aq^2;q^2]_m (1 - a^3 q^{12m+6}) (-a^2)^m q^{m(5m-1)}}{[q^2;q^2]_m [a^3 q^9; q^3]_{r+2m} [q^3;q^3]_{r-2m}}. \quad (2.30)$$

(2.30) is obtained from (1.3) by following the proof of (2.19), except we use (2.30) instead of (2.28).

Proof of (2.21): In (2.28) letting $b \rightarrow \infty$ and replacing $(1 - aq^{4m})$ by $\{(1 - aq^{2m})q^{2m} + (1 - q^{2m})\}$ and proceeding as in the proof of (2.10), we have

$$\sum_{m=0}^r \frac{[aq^2;q]_{3r-m} (1 - a^3 q^6) a^m q^{m^2}}{[q;q]_m [aq;q]_{2m} [q^3;q^3]_{r-m} [a^3 q^6; q^3]_{2r}} = \sum_{m=0}^{[r/2]} \frac{[aq^2;q^2]_m (1 - a^3 q^{12m+6}) (-a^3)^m q^{m(7m+1)}}{[q^2;q^2]_m [a^3 q^9; q^3]_{r+2m} [q^3;q^3]_{r-2m}}. \quad (2.31)$$

Following the proof of (2.20) and using (2.31) instead of (2.30), (2.21) is obtained.

Proof of (2.22): In (1.7) replacing a and q by a^3 and q^3 respectively and setting $k = 2$, $\alpha = \omega q^{-3n}$, $\beta = \omega^2 q^{-3n}$, $A_n = [a; q^2]_n (1 - aq^{4n}) (-a^2)^n q^{n(5n-1)} / [q^2; q^2]_n (1 - a)$, and evaluating C_r by a limiting case ($b \rightarrow 0$) of (2.28), we get

$$\begin{aligned} & \sum_{r=0}^n \sum_{m=0}^r \frac{[aq;q]_{3r-m} [a^3 q^3; q^3]_{3n-r} a^{m+3r} q^{3r^2+m(3m-1)/2}}{[q;q]_m [aq;q^2]_m [q^3;q^3]_{r-m} [q^9;q^9]_{n-r} [a^3 q^3; q^3]_{2r} [a^9 q^9; q^9]_{2n}} \\ &= \sum_{m=0}^{[n/2]} \frac{[a; q^2]_m (1 - aq^{4m}) (-a^8)^m q^{m(17m-1)}}{[q^2; q^2]_m (1 - a) [a^9 q^9; q^9]_{n+2m} [q^9; q^9]_{n-2m}}. \end{aligned} \quad (2.32)$$

(2.22) is obtained from (1.3) by replacing a, q by a^9, q^9 respectively and then setting $k = 2$, $A_n = [a; q^2]_n (1 - aq^{4n}) (-a^8)^n q^{n(17n-1)} / [q^2; q^2]_n (1 - a)$ and evaluating C_r by (2.32).

Proof of (2.23): Follows on the same lines as that of (2.22) except we use (2.30) instead of the limiting case ($b \rightarrow 0$) of (2.28).

Identities for $R(48s + 26, i)$:

Transformation (2.16) for $a=1, p=0; a=q, p=0; a=q^2, p=0; a=1, p=1; a=q, p=1; a=q^2, p=1; a=1, p=2; a=q, p=2$, (2.17) for $a=1, p=0; a=q, p=0; a=q^2, p=0; a=1, p=1; a=q, p=1; a=1, p=2$ and (5.18) for $a=1, p=0; a=1, p=1; a=1, p=2$ yield $(s+2)$ -fold expansions

$$R(48s + 26, 24s + 12), R(48s + 26, 2), \\ R(48s + 26, 24s + 10), R(48s + 26, 24s) + R(48s + 26, 24s + 2), \\ R(48s + 26, 14) - q^2 R(48s + 26, 10), \\ R(48s + 26, 24s - 2) + R(48s + 26, 24s + 4), \\ R(48s + 26, 24s - 12) + R(48s + 26, 24s - 10), \\ R(48s + 26, 26) - q^2 R(48s + 26, 22), \\ R(48s + 26, 4) R(48s + 26, 24s + 10) - R(48s + 26, 24s + 8), \\ R(48s + 26, 6) - q^2 R(48s + 26, 2), R(48s + 26, 16) - \\ q^4 R(48s + 26, 8), R(48s + 26, 24s - 4) + R(48s + 26, 24s + 6), \\ R(48s + 26, 28) - q^4 R(48s + 26, 20), R(48s + 26, 12),$$

$R(48s + 26, 24)$ and $R(48s + 26, 36)$. These identities for $s=1$ give RR type identities for $R(74, 2i)$: $i=1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17$ and 18.

Identities for $R(48s + 34, i)$:

In transformation (2.19) letting $b \rightarrow 0$ and then setting $a=1, p=0; a=q^2, p=0; a=q^4, p=0; a=q^6, p=0; a=1, p=1; a=q^2, p=1$ yield $(s+2)$ -fold expansions for

$$R(48s + 34, 24s + 16), R(48s + 34, 2), R(48s + 34, 24s + 14),$$

$$R(48s + 34, 4), R(48s + 34, 24s + 4) +$$

$$R(48s + 34, 24s + 6), R(48s + 34, 14) - q^2 R(48s + 34, 10),$$

whereas (2.20) for $a=1, p=0; a=q^2, p=0; a=1, p=1$ reduce to $(s+2)$ -fold expansions for

$$R(48s + 34, 6), R(48s + 34, 24s + 12) -$$

$R(48s + 34, 24s + 10)$ and $R(48s + 34, 18)$. These expansions for $s=1$, give RR type of identities on modulus 82.

Identities for $R(48s + 38, i)$:

In the transformation (2.19) letting $b \rightarrow \infty$ and then setting $a=1, p=0; a=q^2, p=0; a=q^4, p=0; a=q^6, p=0; a=1, p=1; a=q^2, p=1$, yield $(s+2)$ -fold expansions for

$$R(48s + 38, 24s + 18), R(48s + 38, 2), R(48s + 38, 24s + 16),$$

$$R(48s + 38, 4),$$

$$R(48s + 38, 24s + 6)$$

$+ R(48s + 38, 24s + 8)$, $R(48s + 38, 14) - q^2 R(48s + 38, 10)$,
on the other hand (2.21) for $a=1, p=0$; $a=q^2, p=0$ and $a=1, p=1$ reduce to $(s+2)$ -
expansions of $R(48s + 38, 6)$, $R(48s + 38, 24s + 14)$
 $R(48s + 38, 24s + 12)$, $R(48s + 38, 18)$. These expansions for $s=1$
give RR type identities on modulus 86.

Identities for $R(72s + 34, i)$:

The transformation (2.22) for $a=1, p=0$; $a=q^2, p=0$; $a=q^4, p=0$; $a=q^6, p=0$;
 $a=1, p=1$; $a=q^2, p=1$ and (2.23) for $a=1, p=0$; $a=q^2, p=0$; $a=1, p=1$ yield $(s+2)$ -fold
expansions for $R(72s + 34, 36s + 14)$, $R(72s + 34, 2)$,
 $R(72s + 34, 4)$, $R(72s + 34, 36s - 2)$ + $R(72s + 34, 36s)$,
 $R(72s + 34, 36s + 16)$, $R(72s + 34, 20)$ $-q^2 R(72s + 34, 16)$,
 $R(72s + 34, 6)$, $R(72s + 34, 36s + 12)$
 $R(72s + 34, 36s + 10)$ and $R(72s + 34, 24)$
 $q^6 R(72s + 34, 12)$ on some manipulations. For $s=1$ these identities reduce to RR
type identities on moduli 74, 82, 86, 106.

§3. In this section expansions of $R(36s - 3, i)$, $R(36s + 3, i)$,
 $R(36s + 15, i)$, $R(36s + 21, i)$ (as $(s+1)$ -fold series), $R(72s + 39, i)$,
 $R(72s + 51, i)$, $R(72s + 57, i)$ and $R(108s + 51, i)$, (as $(s+2)$ -
fold series will be obtained. These expansions for $s=1, 2$ yield some RR type identities
on moduli 33, 39, 51, 57 (as double series) and 69, 81, 87, 93, 111, 123, 129, 159 (as
triple series). The moduli of the RR identities are three times the prime numbers 11,
13, 17, 19, 23, 29, 31, 37, 41, 43 and 53. For obtaining the said RR type identities we
begin by proving the following transformations: Define

$$K[r, (r_s)] = q^{6r^2 + 2S[(r_s)]} / [q^6; q^6]_r [q^2; q^2]_{r_1-3r} \left\{ \prod_{n=2}^s [q^2; q^2]_{r_n-r_{n-1}} \right\},$$

$$H[m, r, (r_s)] = q^{6m^2 + r^2 + 4S[(r_s)]} / [q^6; q^6]_m [q^4; q^4]_{r_1-r} [q^2; q^2]_{r-3m} \left\{ \prod_{n=2}^s [q^4; q^4]_{r_n-r_{n-1}} \right\},$$

$$V[m, r, (r_s)] = q^{9m^2 + 2r^2 + 6S[(r_s)]} / [q^6; q^6]_m [q^6; q^6]_{r_1-r} [q^2; q^2]_{r-3m} \left\{ \prod_{n=2}^s [q^6; q^6]_{r_n-r_{n-1}} \right\}, \text{ then}$$

$$\begin{aligned}
& \frac{[a^2 q^2; q^2]_\infty}{[a^2 q^2/x; q^2]_\infty} \sum_{r_s=0}^\infty \sum_{r_{s-1}=0}^{r_s} \dots \sum_{r_1=0}^{r_2} \sum_{r=0}^{[r_1/3]} \frac{[-aq^3/b; q^3]_{2r} [a^2; q^6]_{r_1-r} [x; q^2]_{r_s} K[r, (r_s)] (-)^{r_s} a^{2r+2L[(r_s)]} q^{r_s(1-r_s-2p)}}{[-aq^3; q^3]_{2r} [a^2 q^6/b^2; q^6]_r [a^2; q^2]_{2r_1} x^{r_s}} \\
& = \sum_{j=0}^p \frac{[q^{-2p}; q^2]_j}{[q^2; q^2]_j} (a^2 q^2/x)^j \sum_{n=0}^\infty \frac{[x; q^2]_{3n+j} [a, b; q^3]_n (1-aq^{6n}) (-a^{3+6s})^n q^{18sn^2+n(6j-6p+3)}}{[q^3; q^3]_n [a^2 q^2/x; q^2]_{3n} [aq^3/b; q^3]_n (1-a)x^{3n} b^n} \\
(3.1)
\end{aligned}$$

$$\begin{aligned}
& [a^4 q^4; q^4]_\infty \sum_{r_s=0}^\infty \sum_{r_{s-1}=0}^{r_s} \dots \sum_{r=0}^{r_1} \sum_{m=0}^{[r/3]} \frac{[-aq^3/b; q^3]_{2m} [a^2; q^6]_{r-m} [\alpha; q^2]_r [-a^2 q^2/\alpha; q^2]_{2r_1-r} H[m, r, (r_s)] (-)^r a^{2m+2r+4L[(r_s)]}}{[-aq^3; q^3]_{2m} [a^2 q^6/b^2; q^6]_m [a^2; q^2]_{2r} [-a^2 q^2; q^2]_{2r_1} [\alpha^4 q^4/\alpha^2; q^4]_r \alpha^r q^{4pr_s-r}} \\
& = \sum_{j=0}^p \frac{[q^{-4p}; q^4]_j (-a^4)^j q^{2j(j+1)}}{[q^4; q^4]_j} \sum_{n=0}^\infty \frac{[a, b; q^3]_n (1-aq^{6n}) [\alpha; q^2]_{3n} (-a^{9+12s})^n q^{n[(36s+18)n+24j-12p+3]}}{[q^3, aq^3/b; q^3]_n [a^2 q^2/\alpha; q^2]_{3n} (1-a) \alpha^{3n} b^n} \\
(3.2)
\end{aligned}$$

and

$$\begin{aligned}
& [a^6 q^6; q^6]_\infty \sum_{r_s=0}^\infty \sum_{r_{s-1}=0}^{r_s} \dots \sum_{r=0}^{r_1} \sum_{m=0}^{[r/3]} \frac{[a^2; q^6]_{r-m} [a^2 q^{2r}; q^2]_{3r_1-r} V[m, r, (r_s)] (-)^m a^{2m+2r+6L[(r_s)]}}{[-aq^3; q^2]_{2m} [a^6 q^6; q^6]_{2r_1} [a^2; q^2]_{2r} q^{6pr_s}} \\
& = \sum_{j=0}^p \frac{[q^{-6p}; q^6]_j (-a^6)^j q^{3j(j+1)}}{[q^6; q^6]_j} \sum_{n=0}^\infty \frac{[a; q^3]_n (1-aq^{6n}) (-a^{8+18s})^n q^{n[(108s+51)n+72j-36p-3]/2}}{[q^3; q^3]_n (1-a)} \\
(3.3)
\end{aligned}$$

Proof of (3.1): In the transformation (1.1) first replacing q by q^3 and then setting $x^2 = q^{-2m}$, $y^2 = q^{2-2m}$, $z^2 = q^{4-2m}$ yields on some simplification

$$\begin{aligned}
& \sum_{r=0}^{[m/3]} \frac{[a; q^3]_r (1-aq^{6r}) [b; q^3]_r a^{3r} q^{9r^2}}{[q^3; q^3]_r (1-a) [aq^3/b; q^3]_r [a^2 q^{2r}; q^2]_{m+3r} [q^2; q^2]_{m-3r} b^r} \\
& = \frac{1}{[a^2; q^2]_{2m}} \sum_{r=0}^{[m/3]} \frac{[-aq^3/b; q^3]_{2r} [a^2; q^6]_{m-r} a^{2r} q^{6r^2}}{[q^6; q^6]_r [-aq^3; q^3]_{2r} [a^2 q^6/b^2; q^6]_r [q^2; q^2]_{m-3r}} \\
(3.4)
\end{aligned}$$

Next, in (1.5) letting $y \rightarrow \infty$ and then replacing q , a by q^2 , a^2 respectively and then setting $k=3$, $\alpha_r^{(1)} = [a; q^3]_r (1-aq^{6r}) [b; q^3]_r a^{3r} q^{9r^2} / [q^3; q^3]_r (1-a) [aq^3/b; q^3]_r b^r$, transforming $\beta_{r_1}^{(1)}$ by (3.4), yields (3.1).

Proof of (3.2): In the transformation (1.7) replacing a , q by a^2 , q^2 respectively and then setting $k=3$, $\beta = -q^{-n}$, $A_n = [a, b; q^3]_n (1-aq^{6n}) a^{3n} q^{9n^2} / [q^3, aq^3/b; q^3]_n (1-a) b^n$ and evaluating C by (3.4), we get

$$\begin{aligned}
& \sum_{r=0}^n \sum_{m=0}^{[r/3]} \frac{[a^2; q^6]_{r-m} [-aq^3/b; q^3]_{2m} [\alpha; q^2]_r [-a^2q^2/\alpha; q^2]_{2n-r} (-)^r a^{2m+2r} q^{6m^2+r(r+1)} \alpha^{-r}}{[q^6; q^6]_m [-aq^3; q^3]_{2m} [a^2q^6/b^2; q^6]_m [q^2; q^2]_{r-3m} [a^2; q^2]_{2r} [-a^2q^2; q^2]_{2n} [q^4; q^4]_{n-r} [a^4q^4/\alpha^2; q^4]_n} \\
& = \sum_{m=0}^{[n/3]} \frac{[a, b; q^3]_m (1 - aq^{6m}) [\alpha; q^2]_{3m} (-)^m a^{9m} q^{3m(6m+1)}}{[q^3, aq^3/b; q^3]_m (1 - a) [a^2q^2/\alpha; q^2]_{3m} [a^4q^4; q^4]_{n+3m} [q^4; q^4]_{n-3m} b^m \alpha^{3m}}. \tag{3.5}
\end{aligned}$$

(3.2) is obtained from (1.3) on replacing q, a by q^4, a^4 respectively and setting

$$k=3, A_n = [a, b; q^3]_n (1 - aq^{6n}) [\alpha; q^2]_{3n} (-a^9)^n q^{3n(6n+1)} / [q^3, aq^3/b; q^3]_n [a^2q^2/\alpha; q^2]_{3n} (1 - a) b^n \alpha^{3n}$$

transforming the C_r by using (3.5).

Proof of (3.3): In (1.7) replacing a, q by a^2, q^2 respectively and then setting $k = 3, \alpha = \omega q^{-2n}, \beta = \omega^2 q^{-2n}, A_n = [a; q^3]_n (1 - aq^{6n}) (-a^2)^n q^{3n(5n-1)/2} / [q^3; q^3]_n (1 - a)$ and valuating C_r by the limiting case $b \rightarrow 0$ of (3.4), we have

$$\begin{aligned}
& \sum_{r=0}^n \sum_{m=0}^{[r/3]} \frac{[a^2; q^6]_{r-m} [a^2q^2; q^2]_{3n-r} (-)^m a^{2m+2r} q^{9m^2+2r^2}}{[q^6; q^6]_m [-aq^3; q^3]_{2m} [q^2; q^2]_{r-3m} [a^2; q^2]_{2r} [q^6; q^6]_{n-r} [a^6q^6; q^6]_{2n}} \\
& = \sum_{m=0}^{[n/3]} \frac{[a; q^3]_m (1 - aq^{6m}) (-)^m a^{8m} q^{3m(17m-1)/2}}{[q^3; q^3]_m (1 - a) [a^6q^6; q^6]_{n+3m} [q^6; q^6]_{n-3m}}. \tag{3.6}
\end{aligned}$$

(3.3) is obtained from (1.3) on replacing q, a by q^6, a^6 respectively and setting $k=3, A_n = [a; q^3]_n (1 - aq^{6n}) (-a^8)^n q^{3n(17n-1)/2} / [q^3; q^3]_n (1 - a)$ and transforming C_r by (3.6).

Identities for $R(36s-3, i)$:

In (3.1) first letting $p=0, b \rightarrow 0$ and then setting $x = -a q = -q ; x = -a q = -q^4 ; x = -a q = -q^7 ; x = -a q = -q^5 ; x = -a q = -q^8$ give expansions for $R(36s-3, i) : i = 3, 6, 18s-6, 18s-3$ and $R(36s-3, 18s-6) - R(36s-3, 18s-9)$ as $(s+1)$ -fold series. These expansions for $s=1$ reduce to the RR type of double series identities of $R(33, 3i) : 1 \leq i \leq 5$ (see also [4]) and for $s=2$ they reduce to triple series RR type identities for $R(69, 3i) : i=1, 2, 9, 10, 11$.

Identities for $R(36s+3, i)$:

In (3.1) first letting $b \rightarrow \infty$ and then setting $p=0, x = -a q = -q ; p=0, x = -a q = -q^4 ; p=0, x = -a q = -q^7 ; p=0, x = -a q = -q^{10} ; p=1, x = -a q = -q$ and $p=1, x = -a q = -q^4$ give expansions for $R(36s+3, i) : i=3, 6, 9, 18s-6, 18s-3, 18s$ as $(s+1)$ -fold series. These expansions reduce for $s=1$ to the known [5; equations (8.31)-(8.36)] RR type of identities for $R(39, 3i) : 1 \leq i \leq 6$.

Identities for R(36s+15, i):

(3.1) on letting $x \rightarrow \infty$, $b \rightarrow 0$ and then setting $p=0, a=1; p=0, a=q^3; p=0, a=q^6; p=0, a=q^9; p=1, a=1; p=1, a=q^3; p=1, a=q^6; p=1, a=q^9$ give, on some reduction, RR type of identities for $R(36s+15, i)$: $i=3, 6, 9, 12, 18s-3, 18s, 18s+3$ and $18s+6$ as $(s+1)$ -fold series. These identities reduce for $s=1$ to the known [5; equations (8.19)-(8.29)] RR type of identities for $R(51, 3i)$: $1 \leq i \leq 8$ and for $s=2$ they reduce to triple series RR type identities for $R(87, 3i)$: $i = 1, 2, 3, 4, 11, 12, 13, 14$.

Identities for R(36s+21, i):

In (3.1) letting $x \rightarrow \infty$, $b \rightarrow \infty$, setting for $p=0, 1$ and giving a the values $1, q^3, q^6, q^9$ and for $p=2$ setting $a=1$ give RR type of identities for $R(36s+21, i)$: $i=3, 6, 9, 12, 18s, 18s+3, 18s+6, 18s+9$ and expansion for $R(36s+21, 18s-3) + q^{-2} R(36s+21, 18s+9)$. For $s=1$, the above reduce to known [5; equations (8.7)-(8.17)] RR type of identities for $R(57, 3i)$: $1 \leq i \leq 9$. For $s=2$ they give RR type of identities for $R(93, 3i)$: $i=1, 2, 3, 4, 11, 12, 13, 14, 15$.

Identities for R(72s+39, i):

In (3.2) letting $b \rightarrow \infty$ and then substituting $a = q^3, \alpha = -q^4, p = 0;$
 $a = 1, \alpha = -q, p = 0; a = q^6, \alpha = -q^7, p = 0;$
 $a = 1, \alpha = -q, p = 1; a = q^3, \alpha = -q^4, p = 1; a = q^6, \alpha = -q^7, p = 1;$
 $a = q^3, \alpha = -q^5, p = 0; a = q^3, \alpha = -q^5, p = 1; a = q^3, \alpha = -q^5, p = 2$ we get $(s+2)$ -fold expansions for $R(72s+39, 3), R(72s+39, 36s+18);$
 $R(72s+39, 36s+15);$
 $R(72s+39, 36s+6) + R(72s+39, 36s+9);$
 $R(72s+39, 15) - q^3 R(72s+39, 9);$
 $R(72s+39, 36s+3) + R(72s+39, 36s+12);$
 $R(72s+39, 6); R(72s+39, 18)$ and $R(72s+39, 30) + q^8 R(72s+39, 6)$. These expansions for $s=1$ give RR type identities for modulus 111.

Identities for R(72s+51, i):

In (3.2), letting $b \rightarrow 0, \alpha \rightarrow \infty$ and then substituting
 $a = 1, p = 0; a = q^3, p = 0; a = q^6, p = 0; a = q^9, p = 0; a = 1, p = 1;$
 $a = q^3, p = 1; a = q^6, p = 1; a = q^9, p = 1$ to obtain $(s+2)$ -fold expansions for

$$R(72s + 51, 3), R(72s + 51, 3)$$

$$R(72s + 51, 3), R(72s + 51, 6)$$

$$R(72s + 51, 3) + R(72s + 51, 3) + R(72s + 51, 15)$$

$$R(72s + 51, 15) - q^3 R(72s + 51, 9), R(72s + 51, 3) +$$

$R(72s + 51, 3) \text{ and } R(72s + 51, 18)$. The above identities for $s=1$ reduce to the RR identities on modulus 123.

Identities for $R(72s+57, i)$:

In (3.2) letting $b, \alpha \rightarrow \infty$ and then substituting $a = 1, p = 0; a = q^3, p = 0; a = q^6, p = 0; a = q^9, p = 0; a = 1, p = 1; a = q^3, p = 1; a = q^6, p = 1; a = q^9, p = 1$ to obtain $(s+2)$ -fold expansions for

$$R(72s + 57, 3), R(72s + 57, 3)$$

$$R(72s + 57, 3), R(72s + 57, 6)$$

$$R(72s + 57, 3) + R(72s + 57, 3) + R(72s + 57, 18)$$

$$R(72s + 57, 15) - q^3 R(72s + 57, 9),$$

$$R(72s + 57, 3) - R(72s + 57, 3) \text{ and}$$

$R(72s + 57, 18)$. These identities for $s=1$ yield RR type identities on modulus 129.

Identities for $R(108s+51, i)$:

In (3.3) for both $p=0, 1$ setting $a = 1, q^3, q^6, q^9$ we get $(s+2)$ -fold expansions for

$$R(108s + 51, 3), R(108s + 51, 3)$$

$$R(108s + 51, 3), R(108s + 51, 6)$$

$$R(108s + 51, 3) + R(108s + 51, 3) + R(108s + 51, 9)$$

$$R(108s + 51, 21) - q^3 R(108s + 51, 15)$$

$$R(108s + 51, 3) + R(108s + 51, 3) + R(108s + 51, 5) \text{ and}$$

$$R(108s + 51, 24) - q^6 R(108s + 51, 12).$$

These expansions for $s=1$ become RR type identities on modulus 159.

§ 4. In this section general RR type identities for $R(32s + 12, i), R(32s + 20, i), R(64s + 28, i), R(64s + 52, i), R(64s + 60, i), R(96s + 52, i), R(96s + 68, i), R(96s + 76, i)$ will be obtained for some values of i . These identities for $s=1, 2$ reduce to RR type identities on moduli 44, 52, 76, 92, 116, 124, 148, 164, 172 (i.e. they yield some RR type identities with moduli

four times the prime numbers 11, 13, 19, 23, 31, 37, 41 and 43). For that we first prove following four general transformations : Define

$$M[r, (r_s)] = q^{S[(r_s)]+r^2} / [q^2; q^2]_r [q; q]_{r_1-2r} \left\{ \prod_{n=2}^s [q; q]_{r_n-r_{n-1}} \right\}$$

$$N[m, r, (r_s)] = q^{m^2+r^2+2S[(r_s)]} / [q^2; q^2]_m [q; q]_{r-2m} \left\{ \prod_{n=2}^s [q^2; q^2]_{r_n-r_{n-1}} \right\} [q^2; q^2]_{r_1-r} ,$$

$$\text{and } K[m, r, (r_s)] = q^{2m^2+3r^2+3S[(r_s)]} / [q^2; q^2]_m [q^6; q^6]_{r-m} [q^3; q^3]_{r_1-2r} \prod_{n=2}^s [q^3; q^3]_{r_n-r_{n-1}} ,$$

then

$$\begin{aligned} & [aq; q]_\infty \sum_{r_s=0}^\infty \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{[r_1/2]} \frac{[aq^2/b; q^4]_r [x; q^2]_r [aq/x; q^2]_{r_1-r} (-)^r a^{r+L[(r_s)]} q^{r-pr_s} M[r, (r_s)]}{[aq^2; q^4]_r [aq^2/b; q^2]_r [aq; q^2]_{r_1} [aq/x; q]_{r_1} x^r} \\ &= \sum_{j=0}^p \frac{[q^{-p}; q]_j (-a)^j q^{\frac{1}{2}j(j+1)}}{[q; q]_j} \sum_{n=0}^\infty \frac{[a, b; q^4]_n (1 - aq^{8n}) [x; q^2]_{2n} a^{n(3+4s)} q^{(16s+8)n^2+n(8j-4p+2)}}{[q^4, aq^4/b; q^4]_n (1-a) [aq^2/x; q^2]_{2n} (bx^2)^n}, \end{aligned} \quad (4.1)$$

$$\begin{aligned} & \frac{[a^2q^2; q^2]_\infty}{[a^2q^2/y; q^2]_\infty} \sum_{r_s=0}^\infty \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{[r_1/2]} \sum_{m=0}^{[r/2]} \frac{[x; q^2]_m [aq/x; q^2]_{r-m} [y; q^2]_{r_s} N[m, r, (r_s)] (-)^{m+r_s} a^{m+r+2L[(r_s)]}}{[aq^2; q^4]_m [aq; q^2]_r [aq/x; q]_r [-aq; q]_{2r_1} x^m y^{r_s(r_s-1)+2pr_s-m}} \\ &= \sum_{j=0}^p \frac{[q^{-2p}; q^2]_j a^{2j} q^{2j}}{[q^2; q^2]_j y^j} \sum_{n=0}^\infty \frac{[a; q^4]_n (1 - aq^{8n}) [x; q^2]_{2n} [y; q^2]_{4n+j} (-a^{7+8s})^n q^{(32s+10)n^2+n(8j-8p+4)}}{[q^4; q^4]_n (1-a) [aq^2/x; q^2]_{2n} [a^2q^2/y; q^2]_{4n} x^{2n} y^{4n}}, \end{aligned} \quad (4.2)$$

$$\begin{aligned} & [a^6q^3; q^3]_\infty \sum_{r_s=0}^\infty \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{[r_1/2]} \sum_{m=0}^r \frac{[\alpha; q^6]_m [a^6q^3/\alpha; q^6]_{r_1-r} [a^2q^2; q^2]_{3r-m} [a^2q^2/b; q^4]_m K[m, r, (r_s)] (-)^r a^{m+6r+6L[(r_s)]}}{[a^2q^2; q^4]_m [a^6q^6; q^6]_{2r} [a^2q^2/b; q^2]_m [a^6q^3/\alpha; q^3]_{r_1} [a^6q^3; q^6]_{r_1} \alpha^r q^{3pr_s-3r}} \\ &= \sum_{j=0}^p \frac{[q^{-3p}; q^3]_j (-a^6)^j q^{3j(j+1)/2}}{[q^3; q^3]_j} \sum_{n=0}^\infty \frac{[a^2, b; q^4]_n (1 - a^2q^{8n}) [\alpha; q^6]_{2n} a^{(18+24s)n} q^{6n[(8s+4)n^2+4j-2p+1]}}{[q^4, a^2q^4/b; q^4]_n (1-a) [a^6q^6/\alpha; q^6]_{2n} b^n \alpha^{2n}}, \end{aligned} \quad (4.3)$$

$$[a^6q^{12}; q^3]_\infty \sum_{r_s=0}^\infty \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{[r_1/2]} \sum_{m=0}^r \frac{[a^2q^2; q^2]_{3r-m} K[m, r, (r_s)] a^{2m+6r+6L[(r_s)]} q^{m(m-1)+3r^2+12L[(r_s)]}}{[a^2q^2; q^4]_m [a^6q^{12}; q^6]_{2r} [a^6q^{15}; q^6]_{r_1} q^{3pr_s-12r}} \quad (4.4)$$

$$\begin{aligned} & [a^6q^{12}; q^3]_\infty \sum_{r_s=0}^\infty \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{[r_1/2]} \sum_{m=0}^r \frac{[\alpha; q^6]_r [a^6q^{15}/\alpha; q^6]_{r_1-r} [a^2q^2; q^2]_{3r-m} K[m, r, (r_s)] (-)^r a^{2m+6r+6L[(r_s)]}}{[a^2q^2; q^4]_m [a^6q^{12}; q^6]_{2r} [a^6q^{15}/\alpha; q^3]_{r_1} [a^6q^{15}; q^6]_{r_1} q^{3pr_s-15r-12L[(r_s)]} \alpha^r} \\ &= \sum_{j=0}^p \frac{[q^{-3p}; q^3]_j (-a^6)^j q^{3j(j+9)/2}}{[q^3; q^3]_j} \sum_{n=0}^\infty \frac{[a^2q^4; q^4]_n (1 - a^6q^{24n+12}) [\alpha; q^6]_{2n} (-)^n a^{(18+24s)n}}{[q^4; q^4]_n [a^6q^{18}/\alpha; q^6]_{2n} \alpha^{2n} q^{-n[(48s+26)n+24j-12p+32+48s]}} \end{aligned} \quad (4.5)$$

Proof of (4.1): In (1.2) first replacing q by q^2 and then setting $z = q^{-m}$, $y = q^{-m+1}$ yields

$$\begin{aligned} & \sum_{r=0}^{[m/4]} \frac{[a,b;q^4]_r (1-aq^{8r}) [x;q^2]_{2r} (a^3/bx^2)^r q^{8r^2+2r}}{[q^4, aq^4/b; q^4]_r (1-a) [aq^2/x; q^2]_{2r} [aq; q]_{m+4r} [q; q]_{m-4r}} \\ &= \frac{1}{[aq/x; q]_m [aq; q^2]_m} \sum_{r=0}^{[m/2]} \frac{[aq^2/b; q^4]_r [x; q^2]_r [aq/x; q^2]_{m-r} (-a/x)^r q^{r(r+1)}}{[q^2, aq^2/b; q^2]_r [aq^2; q^4]_r [q; q]_{m-2r}}. \end{aligned} \quad (4.6)$$

Next, in the transformation (1.3) setting $k = 4$,

$$A_r = [a,b;q^4]_r (1-aq^{8r}) [x;q^2]_{2r} (a^3/bx^2)^r q^{8r^2+2r} / \{ [q^4, aq^4/b; q^4]_r (1-a) [aq^2/x; q^2]_{2r} \},$$

transforming C_n by using (4.6), we get (4.1).

Proof of (4.2): In (1.7) replacing $\beta_{\beta, k}$ by $-q^{-n}, 4$ respectively and letting $\alpha \rightarrow \infty$,
 $A_r = [a; q^4]_r (1-aq^{8r}) [x; q^2]_{2r} (-a^3/x^2)^r q^{10r^2} / \{ [q^4; q^4]_r (1-a) [aq^2/x; q^2]_{2r} \},$
transforming C_n by the limiting case ($b \rightarrow \infty$) of the transformation (4.6) yields

$$\begin{aligned} & \sum_{r=0}^n \sum_{m=0}^{[r/2]} \frac{[x; q^2]_m [aq/x; q^2]_{r-m} (-)^m q^{m(m+1)+r^2} a^{m+r}}{[q^2; q^2]_m [aq^2; q^4]_m [q; q]_{r-2m} [aq; q^2]_r [aq/x; q]_r [q^2; q^2]_{n-r} [-aq; q]_{2n} x^m} \\ &= \sum_{m=0}^{[n/4]} \frac{[a; q^4]_m (1-aq^{8m}) [x; q^2]_{2m} q^{26m^2}}{[q^4; q^4]_m (1-a) [aq^2/x; q^2]_{2m} [a^2q^2; q^2]_{n+4m} [q^2; q^2]_{n-4m}} \left(\frac{-a^7}{x^2} \right)^m \end{aligned} \quad (4.7)$$

Next, in the transformation (1.5) letting $x \rightarrow \infty$ and then replacing q, a by q^2, a^2 respectively, setting $k = 4$,

$$\alpha_m^{(1)} = [a; q^4]_m (1-aq^{8m}) [x; q^2]_{2m} q^{26m^2} (-a^7/x^2)^m / [q^4; q^4]_m (1-a) [aq^2/x; q^2]_{2m},$$

transforming $\beta_n^{(1)}$ by using (4.7), gives (4.2).

Proof of (4.3): In (1.2) first replacing a, q by a^2, q^2 respectively and then setting $x = q^{-2r}$, $y = \omega q^{-2r}$, $z = \omega^2 q^{-2r} t$ (ω is a cube root of unity), we have

$$\begin{aligned} & \sum_{m=0}^r \frac{[a^2q^2; q^2]_{3r-m} [a^2q^2/b; q^4]_m a^{2m} q^{2m^2}}{[q^2, a^2q^2/b; q^2]_m [a^2q^2; q^4]_m [q^6; q^6]_{r-m} [a^6q^6; q^6]_{2r}} \\ &= \sum_{m=0}^{[r/2]} \frac{[a^2, b; q^4]_m (1-a^2q^{8m}) a^{6m} q^{12m^2}}{[q^4, a^2q^4/b; q^4]_m (1-a^2) [a^6q^6; q^6]_{r+2m} [q^6; q^6]_{r-2m} b^m} \end{aligned} \quad (4.8)$$

Let

$$C_r = \sum_{m=0}^{[r/k]} A_m / [a^2q^2; q^2]_{r+m} [q^2; q^2]_{r-m}. \quad (4.9)$$

Multiplying both sides by $[\alpha, q^{-n}, q^{-n+1}; q^2]_r q^{2r} / [\alpha q^{1-2n}/a^2; q^2]_r$ and summing w.r.t. r from 0 to $[n/2]$, we get

$$\begin{aligned}
& \sum_{r=0}^{[n/2]} \frac{[\alpha; q^2]_r [a^2 q / \alpha; q^2]_{n-r} (-a^2)^r q^{r(r+1)}}{[q; q]_{n-2r} [a^2 q / \alpha; q]_n [a^2 q; q^2]_n \alpha^r} C_r \\
&= \sum_{m=0}^{[n/4]} \frac{[\alpha; q^2]_{km} (-a^2)^{km} q^{km(km+1)}}{[a^2 q^2 / \alpha; q^2]_{km} [a^2 q; q]_{n+2km} [q; q]_{n-2km} \alpha^{km}} A_m. \tag{4.10}
\end{aligned}$$

In (4.10) replacing a, q by a^3, q^3 respectively and setting $k=2$,

$A_n = [a^2, b; q^4]_n (1 - a^2 q^{8n}) a^{6n} q^{12n^2} / [q^4, a^2 q^4 / b; q^4]_n (1 - a^2) b^n$ and evaluating C by (4.8), we get

$$\begin{aligned}
& \sum_{r=0}^{[n/2]} \sum_{m=0}^r \frac{[a^2 q^2 / b; q^4]_m [a^2 q^2; q^2]_{3r-m} [\alpha; q^6]_r [a^6 q^3 / \alpha; q^6]_{n-r} (-)^r a^{2m+6r} q^{2m^2+3r(r+1)}}{[q^2; q^2]_m [a^2 q^2 / b; q^2]_m [a^2 q^2; q^4]_m [q^6; q^6]_{r-m} [a^6 q^6; q^6]_{2r} [q^3; q^3]_{n-2r} [a^6 q^3 / \alpha; q^3]_n [a^6 q^3; q^6]_n \alpha^r} \\
&= \sum_{m=0}^{[n/4]} \frac{[a^2, b; q^4]_m (1 - a^2 q^{8m}) [\alpha; q^6]_{2m} a^{18m} q^{6m(4m+1)}}{[q^4, a^2 q^4 / b; q^4]_m (1 - a^2) [a^6 q^6 / \alpha; q^6]_{2m} [a^6 q^3; q^3]_{n+4m} [q^3; q^3]_{n-4m} \alpha^{2m} b^m} \tag{4.11}
\end{aligned}$$

Now, (4.3) is obtained from (1.3) on replacing a, q by a^6, q^3 respectively and then setting $k=4$, $A_n = \frac{[a^2, b; q^4]_n (1 - a^2 q^{8n}) [\alpha; q^6]_{2n} a^{18n} q^{6n(4n+1)}}{[q^4, a^2 q^4 / b; q^4]_n (1 - a^2) [a^6 q^6 / \alpha; q^6]_{2n} \alpha^{2n} b^n}$ and evaluating C by (4.11).

Proof of (4.4): In (4.8) letting $b \rightarrow 0$ and on right hand side replacing $(1 - a^2 q^{8m})$ by $\{(1 - a^2 q^{4m}) + a^2 q^{4m} (1 - q^{4m})\}$ and following the line of proof of (2.10), we get

$$\begin{aligned}
& \sum_{m=0}^r \frac{[a^2 q^2; q^2]_{3r-m} (1 - a^6 q^{12}) a^{2m} q^{m(3m-1)}}{[q^2; q^2]_m [a^2 q^2; q^4]_m [q^6; q^6]_{r-m} [a^6 q^{12}; q^6]_{2r}} \\
&= \sum_{m=0}^{[r/2]} \frac{[a^2 q^4; q^4]_m (1 - a^6 q^{24m+12}) (-a^4)^m q^{2m(5m-1)}}{[q^4; q^4]_m [a^6 q^{18}; q^6]_{r+2m} [q^6; q^6]_{r-2m}}. \tag{4.12}
\end{aligned}$$

In (4.10) replacing a, q by $a^3 q^6, q^3$ respectively and then setting $k = 2$, $A_n = [a^2 q^4; q^4]_n (1 - a^6 q^{24n+12}) (-a^4)^n q^{2n(5n-1)} / [q^4; q^4]_n$ and evaluating C by (4.12), we get

$$\begin{aligned}
& \sum_{r=0}^{[n/2]} \sum_{m=0}^r \frac{[a^2 q^2; q^2]_{3r-m} (1 - a^6 q^{12}) a^{2m+6r} q^{m(3m-1)+6r(r+2)}}{[a^2 q^2; q^4]_m [q^6; q^6]_{r-m} [a^6 q^{12}; q^6]_{2r} [q^3; q^3]_{n-2r} [q^2; q^2]_m [a^6 q^{15}; q^6]_n} \\
&= \sum_{m=0}^{[n/4]} \frac{[a^2 q^4; q^4]_m (1 - a^6 q^{24m+12}) (-a^{16})^m q^{2m(17m+11)}}{[q^4; q^4]_m [a^6 q^{15}; q^3]_{n+4m} [q^3; q^3]_{n-4m}}. \tag{4.13}
\end{aligned}$$

Now, (4.4) is obtained from (1.3) on replacing a, q by a^6q^{12}, q^3 respectively and then setting $k = 4$, $A_n = [a^2q^4; q^4]_n(1 - a^6q^{24n+12})(-a^{16})^n q^{2n(17n+11)} / [q^4; q^4]_n$ and evaluating C_i by (4.13).

Proof of (4.5): In (4.8) letting $b \rightarrow \infty$ and on the right hand side replacing $(1 - a^2q^{8m})$ by $\{(1 - a^2q^{4m})q^{4m} + (1 - q^{4m})\}$ and following the line of proof of (2.10), we get

$$\begin{aligned} & \sum_{m=0}^r \frac{[a^2q^2; q^2]_{3r-m}(1 - a^6q^{12})a^{2m}q^{2m^2}}{[q^2; q^2]_m[a^2q^2; q^4]_m[q^6; q^6]_{r-m}[a^6q^{12}; q^6]_{2r}} \\ &= \sum_{m=0}^{[r/2]} \frac{[a^2q^4; q^4]_m(1 - a^6q^{24m+12})(-a^6)^m q^{2m(7m+1)}}{[q^2; q^2]_m[a^6q^{18}; q^6]_{r+2m}[q^6; q^6]_{r-2m}}. \end{aligned} \quad (4.14)$$

The proof of (4.5) follows on the lines of (4.3) except we use (4.14) instead of (4.8).

Identities for R (32s+12, i):

Transformation (4.1) on letting $x \rightarrow 0, b \rightarrow \infty$ and then giving p and a the set of values $p=0, a=1; p=0, a=q^4; p=1, a=q^8; p=1, a=1; p=1, a=q^4$ yield on some reduction, RR type of identities for $R(32s+12, 4i): i=1, 2, 4s, 4s+1$ and expansion for $R(32s+12, 16s) - R(2s+12, 16s-4)$ as $(s+1)$ -fold series. For $s=1$ they reduce to RR type of identities for $R(44, 4i) : 1 \leq i \leq 5$.

Identities for R (32s+20, i):

The transformation (4.1) on letting $x \rightarrow \infty, b \rightarrow 0$ and giving p and a the values $p=0, a=1; p=0, a=q^4; p=1, a=1; p=1, a=q^4; p=1, a=q^8$ yields RR identities for $R(32s+20, 4i) : i=1, 2, 4s+1, 4s+2$ and an expansion for $R(32s+20, 16s+4) - R(32s+20, 16s)$ as $(s+1)$ -fold series. For $s=1$ they reduce to RR type of identities on modulus 52.

Identities for R (64s+28, i):

In (4.2) letting $x \rightarrow \infty, p = 0$ we get for $a = 1, y = -q; a = q^4, y = -q^5; a = q^8, y = -q^9; a = q^4, y = -q^6; a = q^8, y = -q^{10}; a = q^{12}, y = -q^{14}$ expansions for $R(64s+28, 2i) : i = 1, 6s+6, 2, 1, 6s+4, 4, 1, 6s+2, 3$

as $(s+2)$ -fold series. For $s = 1$ they reduce to RR type identities for $R(92, 2i) : i = 2, 3, 4, 18, 20, 22$.

Identities for $R(64s+52, i)$:

In (4.2) letting $y \rightarrow \infty$, we get for $x = -1, a = 1, p = 0$;
 $x = -1, a = 1, p = 1; x = -q^2, a = q^4, p = 0$;
 $x = -q, a = 1, p = 0; x = -q^3, a = q^4, p = 0$;
 $x = -q, a = 1, p = 1; x = -q^3, a = q^4, p = 1$ expansions for
 $R(64s + 52, 2i) : i = 16s + 13, 16s + 9, 1, 16s + 12, 2, 6$
and $R(64s + 52, 32s + 16) + R(64s + 52, 32s + 20)$ as
 $(s+2)$ -fold series. For $s=1$ they reduce to RR type identities
for $R(116, 2i) : i = 1, 2, 6, 25, 28, 29$ and a triple series expansion for the
linear combination $R(116, 48) + R(116, 52)$.

Identities for $R(64s+60, i)$:

Transformation (4.2) on letting $x, y \rightarrow \infty$ and then setting $a = 1, p = 0$;
 $a = q^4, p = 0; a = q^8, p = 0; a = 1, p = 1; a = q^4, p = 1; a = q^8, p = 1; a = q^{12}, p = 0$;
give $(s+2)$ -fold expansions for $R(64s + 60, 2i) : i = 16s + 14, 2, 6, 4$;
 $\{R(64s + 60, 32s + 28) - R(64s + 60, 32s + 24)\}$,
 $\{R(64s + 60, 32s + 20) + R(64s + 60, 32s + 24)\}$ and
 $\{R(64s + 60, 32s + 20) - R(64s + 60, 32s + 16)\}$. For $s=1$
they give RR identities for $R(124, 4i) : i = 1, 2, 3, 12, 13, 14, 15$ as triple
series.

Identities for $R(96s+52, i)$:

In (4.3) letting $b \rightarrow \infty$ and then for $a = 1, p = 0, \alpha = -q^3$;
 $a = q^2, p = 0, \alpha = -q^9; a = q^4, p = 0, \alpha = -q^{15}; a = 1, p = 1, \alpha = -q^3; a = q^2, p = 1$,
 $\alpha = -q^9$ and (4.5) for $a = 1, p = 0, \alpha = -q^9; a = 1, p = 0, \alpha = -q^6; a = q^2, p = 0$,
 $\alpha = -q^{12}; a = 1, p = 1, \alpha = -q^9; a = q^2, p = 1, \alpha = -q^{15}; a = 1, p = 1, \alpha = -q^6$;
 $a = 1, p = 2, \alpha = -q^9; a = q^2, p = 1, \alpha = -q^{12}$ give expansions of

$R(96s + 52, 2i) : i = 24s + 12, 24s + 10, 6, 3, 12, 9, 18, 2$
and $R(96s + 52, 48s + 12) + R(96s + 52, 48s + 16)$

$R(96s+52,16) - q^4 R(96s+52,8)$, $R(96s+52,48s+22) - R(96s+52,48s+18)$,
 $R(96s+52,48s+4) - R(96s+52,48s)$,
 $R(96s+52,48s+10) - R(96s+52,48s+6)$. These reduce for
 $s = 1$ to RR identities for
 $R(148,2i) : i = 2, 3, 6, 9, 12, 18, 34, 36$ and triple series
expansions for the linear combinations $R(148,60) + R(148,64)$,
 $R(148,16) - q^4 R(148,8)$, $R(148,70) - R(148,66)$,
 $R(148,52) - R(148,48)$ and $R(148,58) - R(148,54)$.

Identities for $R(96s+68, i)$:

In (4.3) letting $b \rightarrow 0$, $\alpha \rightarrow \infty$ then for $a = 1, p = 0$; $a = q^2, p = 0$;
 $a = q^4, p = 0$; $a = q^6, p = 0$; $a = 1, p = 1$, $a = q^2, p = 1$; $a = q^4, p = 1$;
 $a = q^6, p = 1$ and (4.4) for $a = 1, p = 0$; $a = 1, p = 1$;
 $a = 1, p = 2$ yield $(s+2)$ -fold expansions for
 $R(96s+68, 2i) : i = 24s + 16, 2, 24s + 14, 4, 8, 24s + 8, 10, 6, 12, 18$
and $R(96s+68, 48s+24) + R(96s+68, 48s+20)$. For
 $s = 1$ they become RR type identities for
 $R(164, 4i) : i = 1, 2, 3, 4, 5, 6, 9, 16, 19, 20$ and triple series
expansion for the linear combination $R(164, 68) + R(164, 72)$.

Identities for $R(96s+76, i)$:

In (4.3) first letting $b, \alpha \rightarrow \infty$ and then setting $a = 1, p = 0$;
 $a = q^2, p = 0$; $a = q^4, p = 0$; $a = q^6, p = 0$; $a = 1, p = 1$, $a = q^2, p = 1$; $a = q^4, p = 1$
and in (4.5) first letting $\alpha \rightarrow \infty$ then setting $a = 1, p = 0$; $a = 1, p = 1$;
 $a = 1, p = 2$ give expansions of
 $R(96s+76, 2i) : i = 24s + 18, 2, 24s + 16, 4, 8, 24s + 10, 10, 6,$
12, 18 and $R(96s+76, 48s+24) + R(96s+76, 48s+28)$ as
 $(s+2)$ -fold series for $s = 1$ they reduce to RR type identities for
 $R(172, 4i) : i = 1, 2, 3, 4, 5, 6, 9, 17, 20, 21$ and a triple series expansion
for $R(172, 72) + R(172, 76)$.

§5. In this section expansions for $R(72s - 6, i)$, $R(72s + 6, i)$, $R(72s + 30, i)$ and $R(72s + 42, i)$ (for certain values of i) as $(s+1)$ -fold series will be obtained. These expansions for $s=1,2$ reduce to RR type identities on moduli 66, 78, 102, 114, 138, 174 and 186 (which are six times the prime numbers 11, 13, 17, 19, 29, 31). For obtaining the aforesaid expansions we begin by proving the transformation

$$\begin{aligned} & \frac{[aq;q]_\infty}{[aq/x;q]_\infty} \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \dots \sum_{r_1=0}^{r_2} \sum_{r=0}^{[r_1/3]} \frac{[x;q]_{r_s} [a;q^3]_{r_1-r} [aq^3/b;q^6]_r a^{r+L[(r_s)]} (-)^{r_s} q^{-\frac{1}{2}r_s(r_s-1)-pr_s} P[r, (r_s)]}{[aq^3;q^6]_r [aq^3/b;q^3]_r [a;q]_{2r_1} x^{r_s}} \\ & = \sum_{j=0}^p \frac{[q^{-p};q]_j}{[q;q]_j} \left(\frac{aq}{x}\right)^j \sum_{n=0}^{\infty} \frac{[x;q]_{6n+j} [a,b;q^6]_n (1-aq^{12n})}{[aq/x;q]_{6n} [q^6, aq^6/b;q^6]_n (1-a)} \left(\frac{a^{3+6s}}{bx^6}\right)^n q^{36sn^2+n(6j-6p+3)} \quad (5.1) \end{aligned}$$

where

$$P[r, (r_s)] = q^{3r^2+s[(r_s)]} / [q^3; q^3]_r [q; q]_{r_1-3r} \left\{ \prod_{n=2}^s [q; q]_{r_n-r_{n-1}} \right\}.$$

Proof of (5.1): In (1.2) first replacing q by q^3 and then setting $x = q^{-m}$, $y = q^{1-m}$, $z = q^{2-m}$ yield on some reduction

$$\begin{aligned} & \sum_{r=0}^{[m/6]} \frac{[a,b;q^6]_r (1-aq^{12r}) a^{3r} q^{18r^2}}{[q^6, aq^6/b;q^6]_r (1-a) [aq;q]_{m+6r} [q;q]_{m-6r} b^r} \\ & = \sum_{r=0}^{[m/3]} \frac{[a;q^3]_{m-r} [aq^3/b;q^6]_r a^r q^{3r^2}}{[q^3, aq^3/b;q^3]_r [aq^3;q^6]_r [q;q]_{m-3r} [a;q]_{2m}}. \quad (5.2) \end{aligned}$$

Next, in (1.5) setting $k = 6$, $y \rightarrow \infty$,

$$\alpha_r^{(1)} = [a,b;q^6]_r (1-aq^{12r}) a^{3r} q^{18r^2} / [q^6, aq^6/b;q^6]_r (1-a) b^r,$$

transforming $\beta_{r_1}^{(1)}$ by (5.2), yields (5.1).

Identities for $R(72s-6, i)$:

In (5.1) letting $b \rightarrow 0$ and then setting $a=1, p=0, x=-\sqrt{q}$;
 $a=q^6, p=0, x=-q^{7/2}$; $a=1, p=0, x=-1$; $a=q^6, p=0, x=-q^3$;
 $a=1, p=0, x=-q$; $a=q^6, p=0, x=-q^4$ and $a=q^{12}, p=0, x=-q^{13/2}$
give $(s+1)$ -fold series expansions
for $R(72s-6, 36s-i)$: $i = 1, 2, s-2, 2, 1, 2, s-1, 1, 1, 2, s-3, 3$
& $R(72s-6, 36s-6) - R(72s-6, 36s-12)$ which for $s=1$ reduce to RR type

identities as double series for $R(66, 3i) : i = 1, 2, 3, 8, 9, 10, 11$ and for $s=2$ they are RR identities for $R(138, 3i) : i = 1, 2, 3, 20, 21, 22, 23$.

Identities for R(72s+6,i):

In (5.1) letting $b \rightarrow \infty$ and then setting $a=1, p=0, x=-\sqrt{q}$;

$a = 1, p = 0, x = -1 ; a = q^6, p = 0, x = -q^{7/2} ; a = q^6, p = 0, x = -q^3 ;$
 $a = 1, p = 0, x = -q ; a = q^6, p = 0, x = -q^4 ; a = q^{12}, p = 0, x = -q^{13/2}$ yield series expansions of

$$R(72s + 6, i) :$$

$i = 36s, 36s + 3, 6, 3, 36s - 3, 9, 36s - 6$ which for $s = 1$ give RR identities for $R(78, 3i) : i = 1, 2, 3, 10, 11, 12, 13$.

Identities for R(72s+30,i):

In the transformation (5.1) first letting $x \rightarrow \infty, b \rightarrow 0$ and then substituting $a = 1, p = 0 ; a = 1, p = 1 ; a = q^6, p = 0 ; a = q^6, p = 1 ; a = q^6, p = 2 ; a = 1, p = 2 ; a = q^6, p = 3$ give $(s+1)$ -fold series expansions of $R(72s + 30, 6i) : i = 6s + 2, 6s + 1, 1, 2, 3, 6s, 4$ which for $s = 1$ reduce to RR identities for $R(102, 6i) : i = 1, 2, 3, 4, 6, 7, 8$ and for $s=2$ they give RR type of identities for $R(174, 6i) : i = 1, 2, 3, 4, 12, 13, 14$.

Identities for R(72s+42,i):

In (5.1) first letting $x, b \rightarrow \infty$ and then substituting $a = 1, p = 0 ; a = 1, p = 1 ; a = 1, p = 2 ; a = q^6, p = 0 ; a = q^6, p = 1 ; a = q^6, p = 2 ; a = q^6, p = 3$ give $(s+1)$ -fold series expansions for

$R(72s + 42, 6i) : i = 6s + 3, 6s + 2, 6s + 1, 1, 2, 3, 4$. These expansions for $s=1$ are RR identities for

$R(114, 6i) : i = 1, 2, 3, 4, 7, 8, 9$ and for $s=2$ they are RR type of identities for $R(186, 6i) : i = 1, 2, 3, 4, 13, 14, 15$.

§ 6. In this section expansions for $R(128s + 104, i)$ and $R(128s + 120, i)$ (for some values of i) as $(s+2)$ -foldseries will be obtained. For $s=1$ they yield RR identities on moduli 232 and 248 (which are eight times the prime

numbers 29 and 31). To obtain the aforesaid expansions we begin by proving the transformation:

$$\begin{aligned}
 & [a^2 q; q]_\infty \sum_{r_s=0}^{\infty} \sum_{r_{s-1}=0}^{r_s} \cdots \sum_{r_1=0}^{r_2} \sum_{r=0}^{[r_1/2]} \sum_{m=0}^{[r/2]} \frac{[x; q^4]_m [a^2 q^2 / x; q^4]_{r-m} (-)^m a^{2m+2r+2L[(r_s)]} q^{2m-pr_s} Q[r, m, (r_s)]}{[a^2 q^4; q^8]_m [a^2 q^2; q^4]_r [a^2 q^2 / x; q^2]_r [a^2 q; q^2]_r x^m} \\
 & = \sum_{j=0}^p \frac{[q^{-p}; q]_j (-a^2)^j q^{\frac{j}{2}(j+1)}}{[q; q]_j} \sum_{n=0}^{\infty} \frac{[a^2; q^8]_n (1 - a^2 q^{16n}) [x; q^4]_{2n} (-a^{14+16s})^n q^{(64s+52)n^2 + 8n(2j-p)}}{[q^8; q^8]_n (1 - a^2) [a^2 q^4 / x; q^4]_{2n} x^{2n}}
 \end{aligned} \tag{6.1}$$

where

$$Q[r, m, (r_s)] = a^{2m^2 + 2r^2 + s[(r_s)]} / [q^4; q^4]_m [q^2; q^2]_{r-2m} [q; q]_{r_1-2r} \left\{ \prod_{n=2}^s [q; q]_{r_n - r_{n-1}} \right\}.$$

Proof of (6.1): In the transformation (1.2) replacing first q, a by q^2, a^2 respectively and then setting $z = q^{-r}$, $y = q^{1-r}$ and letting $b \rightarrow \infty$, yields

$$\begin{aligned}
 & \sum_{m=0}^{[r/4]} \frac{[a^2; q^4]_m (1 - a^2 q^{8m}) [x; q^2]_{2m} (-a^6)^m q^{10m^2}}{[q^4; q^4]_m (1 - a^2) [a^2 q^2 / x; q^2]_{2m} [a^2 q; q]_{r+4m} [q; q]_{r-4m} x^{2m}} \\
 & = \frac{1}{[a^2 q; q^2]_r [a^2 q / x; q]_r} \sum_{m=0}^{[r/2]} \frac{[x; q^2]_m [a^2 q / x; q^2]_{r-m} q^{m(m+1)}}{[q^2; q^2]_m [a^2 q^2; q^4]_m [q; q]_{r-2m}} \left(\frac{-a^2}{x} \right)^m.
 \end{aligned}$$

Now replacing in the above transformation q by q^2 , multiplying both sides by $[\alpha, q^{-n}, q^{-n+1}; q^2]_r q^{2r} / [\alpha q^{1-2n} / a^2; q^2]_r$ and summing with respect to r from 0 to $[n/2]$, interchanging the order of summation on the left hand side and summing the inner series and finally letting $\alpha \rightarrow \infty$, yields

$$\begin{aligned}
 & \sum_{r=0}^{[n/2]} \sum_{m=0}^{[r/2]} \frac{[x; q^4]_m [a^2 q^2 / x; q^4]_{r-m} a^{2m+2r} (-)^m q^{2m(m+1)+2r^2}}{[q^4; q^4]_m [a^2 q^4; q^8]_m [q^2; q^2]_{r-2m} [a^2 q^2; q^4]_r [a^2 q^2 / x; q^2]_r [q; q]_{n-2r} [a^2 q; q^2]_n x^m} \\
 & = \sum_{m=0}^{[n/8]} \frac{[a^2; q^8]_m (1 - a^2 q^{16m}) [x; q^4]_{2m} (-a^{14})^m q^{52m^2}}{[q^8; q^8]_m (1 - a^2) [a^2 q^4 / x; q^4]_{2m} [a^2 q; q]_{n+8m} [q; q]_{n-8m} x^{2m}}.
 \end{aligned} \tag{6.2}$$

Next, in the transformation (1.10) replacing a by a^2 , setting $k = 8$,

$$A_r = [a^2; q^8]_r (1 - a^2 q^{16r}) [x; q^4]_{2r} (-a^{14})^r q^{52r^2} / [q^8; q^8]_m (1 - a^2) [a^2 q^4 / x; q^4]_{2r} x^{2r}$$

transforming C_r by using (6.2), yields (6.1).

Identities for R(128s+104,i):

Transformation (6.1) for $x = aq^2 = q^2, p = 0$; $x = aq^2 = q^2, p = 1$;
 $x = aq^2 = q^2, p = 2$; $x = aq^2 = q^6, p = 0$; $x = aq^2 = q^6, p = 1$; $x = aq^2 = q^6, p = 2$;
 $x = aq^2 = q^6, p = 3$; $a = -x = 1, p = 0$; $a = -x = 1, p = 1$;
 $a = -x = 1, p = 2$; $x = -a = -q^4, p = 0$; $x = -a = -q^4, p = 1$ gives expansions for
 $R(128s + 104, 4i)$: $i = 1, 6, s + 1, 2, 1, 6, s + 1, 0, 2, 4, 6, 1, 6, s + 1, 3, 1, 6, s + 1, 1, 1, 3$,
 $R(128s + 104, 64s + 32) + q^{-1}R(128s + 104, 64s + 48)$.

$$R(128s + 104, 32) + q^7R(128s + 104, 16),$$

$$R(128s + 104, 64s + 36) + \frac{1}{2q}R(128s + 104, 64s + 52). \text{ For } s=1$$

they reduce RR identities for
 $R(2, 3, 2, 4i)$: $i = 1, 2, 3, 4, 6, 8, 2, 4, 2, 5, 2, 6, 2, 7, 2, 8, 2, 9$ as triple series.

Identities for R(128s+120,i):

In (6.2) letting $x \rightarrow \infty$ and then substituting $a = 1, p = 0$;
 $a = 1, p = 1$; $a = 1, p = 2$; $a = q^4, p = 0$; $a = q^4, p = 1$; $a = q^4, p = 2$;
 $a = q^4, p = 3$; yields $(s + 2)$ -fold series expansions
of $R(128s + 120, 4i)$: $i = 1, 6, s + 1, 4, 1, 6, s + 1, 2, 2, 4, 6$,
 $R(128s + 120, 64s + 40) + q^{-1}R(128s + 120, 64s + 56)$, $R(128s + 120, 32)$ +
 $q^7R(128s + 120, 16)$. For $s=1$ they become RR identities
of $R(2, 4, 8, 8i)$: $i = 1, 2, 3, 4, 1, 3$, 14, 15.

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