Ganita, Vol. 61, 2010, 19-28

On irreducible q-representations of the Lie algebra $\mathcal{G}(0,0)$

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Abstract

In this paper, the irreducible q-representations of the Lie algebra $\mathcal{G}(0,0)$ are discussed. We construct one and two variable models of irreducible q–representations of $\mathcal{G}(0,0)$ in terms of q–dilation operators. We also give a unified set of q-commutation relations that hold for the 2-parameter collection of the 4-dimensional Lie algebras denoted by $\mathcal{G}(a,b)$. As the irreducible q–representations of the Lie algebras $\mathcal{G}(1,0)$ and $\mathcal{G}(0,1)$ already exist in the literature, this paper settles the problem of classification of the irreducible q–representations of the Lie algebra $\mathcal{G}(a,b)$.

1 Introduction

The irreducible q-representations of the special complex Lie algebra $\mathrm{sl}(2,\mathbb{C})$ were discussed by Manocha [4]. Various models of this Lie algebra in one and several variables were constructed using the techniques of fractional q-calculus. In Sahai [7], more models of this Lie algebra in terms of q-difference dilation operators were obtained using a q-integral transformation. Recently, the irreducible q-representations of the Lie algebra $\mathcal{G}(0,1)$ were discussed by Sahai and Sahai [6]. In this paper, we extend this idea to the Lie algebra $\mathcal{G}(0,0)$. Precisely, we prove a classification theorem for irreducible q-representations of the Lie algebra $\mathcal{G}(0,0)$ and give one and two variable models of this Lie algebra in terms of q-dilation operators. We also give a unified set of q-commutation relations for the 2-parameter collection of the 4-dimensional Lie algebras denoted by $\mathcal{G}(a,b)$.

For any pair of complex numbers (a,b) the 4-dimensional complex Lie algebra $\mathcal{G}(a,b)$ with basis $\mathcal{J}^+$, $\mathcal{J}^-$, $\mathcal{J}^0$ and $\mathcal{E}$ is defined by

$$\begin{aligned}
[\mathcal{J}^+,\mathcal{J}^-] &= 2a^2\mathcal{J}^0 - b\mathcal{E}, \\
[\mathcal{J}^0,\mathcal{J}^+] &= \mathcal{J}^+,\ [\mathcal{J}^0,\mathcal{J}^-] = -\mathcal{J}^-, \\
[\mathcal{J}^+,\mathcal{E}] &= [\mathcal{J}^-,\mathcal{E}] = [\mathcal{J}^0,\mathcal{E}] = 0,
\end{aligned} \tag{1}$$

where $[.,.]$ is the commutator bracket and 0 is the additive identity element.

For special choices of the parameters a and b, $\mathcal{G}(a,b)$ essentially coincides with one of the Lie algebras $\mathrm{sl}(2)$, $\mathcal{G}(0,1)$ and $\mathcal{T}_3$. Indeed, it can be shown that (see Miller [5, Lemma 2.1])

$$\mathcal{G}(a,b) \cong \begin{cases} \mathcal{G}(1,0) \cong \mathrm{sl}(2) \oplus (\mathcal{E}), & \text{if } a \neq 0 \\ \mathcal{G}(0,1), & \text{if } a = 0,\ b \neq 0 \\ \mathcal{G}(0,0) \cong \mathcal{T}_3 \oplus (\mathcal{E}), & \text{if } a = b = 0, \end{cases}$$

where $(\mathcal{E})$ is the 1-dimensional Lie algebra generated by $\mathcal{E}$. We recall here the definitions of these Lie algebras, (see Miller [5]).

The special complex Lie algebra $\mathrm{sl}(2)$ consists of all 2×2 matrices with trace zero. A basis $\{\mathcal{J}^+,\mathcal{J}^-,\mathcal{J}^0\}$ of $\mathrm{sl}(2)$ is given by

$$\mathcal{J}^+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},\ \mathcal{J}^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},\ \mathcal{J}^0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

The Lie algebra $\mathcal{G}(0,1)$ is the space of 4×4 matrices of the form

$$\alpha = \begin{pmatrix} 0 & x_2 & x_4 & x_3 \\ 0 & x_3 & x_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ x_1, x_2, x_3, x_4 \in \mathbb{C},$$

with

$$\mathcal{J}^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathcal{J}^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{J}^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathcal{E} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

as a basis.

The Lie algebra $\mathcal{T}_3$ consists of the space of matrices of the form

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & a_3 \\ 0 & -a_3 & 0 & a_2 \\ 0 & 0 & a_3 & a_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ a_1, a_2, a_3 \in \mathbb{C}.$$

A basis for $\mathcal{T}_3$ is provided by the matrices

$$\mathcal{J}^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathcal{J}^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathcal{J}^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The irreducible q-representations of sl(2) have been extensively studied by Manocha [4], Sahai [7, 8], Sahai and Srivastava [10, 11]. Further, the irreducible q-representations of the Lie algebra $\mathcal{G}(0,1)$ were discussed in Sahai and Sahai [6]. We now extend this idea to the Lie algebra $\mathcal{G}(0,0)$.

For other notable contributions in this field, see Kalnins and Miller [2], Koelink [3] and Vaksman and Korogodskiĭ [12].

2 q–representations of the Lie algebra $\mathcal{G}(0,0)$

Let V_q be a complex vector space consisting of q-special functions with a basis $\{\phi_\lambda : \lambda \in S\}$ such that the functions $\{f_\lambda = \lim_{q\to 1} \phi_\lambda : \lambda \in S\}$

form a basis for a vector space, say, V. Let $A(V_q)$ be the associative algebra of all linear operators on V_q over the complex field.

A q-representation of $\mathcal{G}\,(0,0)$ on V_q is a linear mapping $\rho_q : \mathcal{G}\,(0,0) \rightarrow A(V_q)$ satisfying the following conditions:

1. There exists a Lie algebra representation ρ of $\mathcal{G}\,(0,0)$ on V such that

$$\lim_{q \to 1} \rho_q(x)\phi_\lambda = \rho(x) f_\lambda,$$

for all $x, y \in \mathcal{G}\,(0,0)$.

2. If we define

$$J_q^+ = \rho_q\left(\mathcal{J}^+\right),\ J_q^- = \rho_q\left(\mathcal{J}^-\right),\ J_q^0 = \rho_q\left(\mathcal{J}^0\right),\ E_q = \rho_q\left(\mathcal{E}\right) \quad (2)$$

where $J_q^+, J_q^-, J_q^0, E_q \in A(V_q)$, then

$$\begin{aligned} J_q^0 J_q^+ \ - \ qJ_q^+ J_q^0 &= J_q^+, \\ qJ_q^0 J_q^- \ - \ J_q^- J_q^0 &= -J_q^-, \\ qJ_q^+ J_q^- \ - \ J_q^- J_q^+ &= 0. \end{aligned} \quad (3)$$

$$\left[E_q, J_q^+\right] = \left[E_q, J_q^-\right] = \left[E_q, J_q^0\right] = 0.$$

A q-representation ρ_q of $\mathcal{G}\,(0,0)$ is said to be *irreducible* if there is no proper subspace W_q of V_q which is invariant under ρ_q.

If we define an operator C_q on V_q by

$$C_q = qJ_q^+ J_q^-, \quad (4)$$

then it is easy to check that

$$\begin{aligned} qJ_q^+ C_q &= C_q J_q^+, \\ J_q^- C_q &= qC_q J_q^-, \\ J_q^0 C_q &= C_q J_q^0, \\ E_q C_q &= C_q E_q. \end{aligned} \quad (5)$$

Indeed, as $q \to 1$, ρ_q reduces to a Lie algebra representation ρ of $\mathcal{G}(0,0)$ on V.

Let $S_q = \left\{ [\lambda - u]_q : \lambda \in S \right\}$ be the spectrum of J_q^0, where $[t]_q$ is the q-analogue of $t \in \mathbb{C}$ given by $[t]_q = \frac{1-q^t}{1-q}$ and u is a nonzero complex constant. Let the q-representation ρ_q satisfies the conditions:

i) ρ_q is irreducible,

ii) Each eigenvalue of J_q^0 has multiplicity one. (6)

Condition (6) guarantees that S_q, and for that matter S, is countable and that there exists a basis $\{f_\lambda\}$ for V_q consisting of eigenvectors of J_q^0.

Following the analysis as in (Miller [5, p.40]), we note:

i) $\lambda \in S \Rightarrow$ either $J_q^+ f_\lambda = \xi_{\lambda+1} f_{\lambda+1}$, where $\xi_{\lambda+1}$ is a nonzero constant and $\lambda + 1 \in S$, or $J_q^+ f_\lambda = 0$.

ii) $\lambda \in S \Rightarrow$ either $J_q^- f_\lambda = \eta_\lambda f_{\lambda-1}$, where η_λ is a nonzero constant and $\lambda - 1 \in S$, or $J_q^- f_\lambda = 0$.

iii) $C_q f_\lambda = a_\lambda f_\lambda$ for some constants a_λ such that $a_{\lambda+1} = q a_\lambda$.

iv) S is connected in the sense: $S = \{\lambda + n : n \text{ is an integer such that } n_1 < n < n_2\}$, where n_1 and n_2 are integers. We do not exclude the possibility that $n_1 = -\infty$ or $n_2 = \infty$.

v) If $\lambda, \lambda+1 \in S$, then $\xi_{\lambda+1}, \eta_{\lambda+1} \neq 0$, since otherwise ρ_q would not be irreducible.

vi) The equation $C_q f_\lambda = a_\lambda f_\lambda$, that is, $\left(q J_q^+ J_q^-\right) f_\lambda = a_\lambda f_\lambda$ leads to the following relation

$$q \xi_\lambda \, \eta_\lambda = a_\lambda \,, \tag{7}$$

defined for all $\lambda \in S$, where $\eta_\lambda = 0$ if $\lambda - 1 \notin S$. Using

$$q J_q^+ J_q^- = J_q^- J_q^+ \,, \tag{8}$$

we have

$$q\xi_{\lambda+1}\ \eta_{\lambda+1} = a_{\lambda+1} \tag{9}$$

defined for all $\lambda \in S$ where $\xi_{\lambda+1} = 0$ if $\lambda + 1 \notin S$.

vii) The representation ρ_q of $\mathcal{G}(0,0)$ is uniquely determined by $a_\lambda, a_{\lambda+1} = qa_\lambda$, and the spectrum S_q of J_q^0. The nonzero constants ξ_λ and η_λ are not unique, and may be chosen arbitrarily, subject only to condition (7).

Denote by $Q_q^\mu(\alpha, u)$, the q-representation of $\mathcal{G}(0,0)$ defined for all $\alpha, u, \mu \in \mathbb{C}$ such that $u \neq 0$, $0 \leq \operatorname{Re}\alpha < 1$, α is not an integer and $S = \{\alpha + n,\ n = 0, \pm 1, \pm 2, \cdots\}$. For each q-representation $Q_q^\mu(\alpha, u)$, there is a basis of V_q consisting of vectors f_λ defined for each $\lambda \in S$ such that

$$\begin{aligned} J_q^0 f_\lambda &= [\lambda - u]_q\ f_\lambda, \\ J_q^+ f_\lambda &= uq^{\lambda+1}\ f_{\lambda+1}, \\ J_q^- f_\lambda &= u\ f_{\lambda-1}, \\ E_q f_\lambda &= \mu\, f_\lambda, \\ C_q f_\lambda &= q^{\lambda+1}u^2\ f_\lambda. \end{aligned} \tag{10}$$

We now have the main theorem.

THEOREM 2.1 *Every q-representation ρ_q of $\mathcal{G}(0,0)$ which satisfies conditions (6) and (7) and for which $J_q^+ J_q^- \not\equiv 0$ is isomorphic to a q-representation $Q_q^\mu(\alpha, u)$.*

PROOF. Since $J_q^+ J_q^- \not\equiv 0$ so $q\xi_\lambda\ \eta_\lambda = a_\lambda \neq 0$. This shows that $\lambda + n \in S$ for all $\lambda \in S$ where n is any integer. Thus S must be of the form $\{\alpha + n,\ n = 0, \pm 1, \pm 2, \cdots\}$, where α is the element of S with smallest positive real part. Set $a_\lambda = q^{\lambda+1}u^2$, then we may assume $\xi_\lambda = q^\lambda u$ and $\eta_\lambda = u$ for all $\lambda \in S$ where u is determined only up to sign. The q-representations $Q_q^\mu(\alpha, u)$ and $Q_q^\mu(\alpha, -u)$ are thus isomorphic.

A one variable model for $Q_q^\mu(\alpha,u)$ is

$$\begin{aligned} J_q^0 &= (1-q)^{-1}\left(1-q^{-u}T_t\right), \\ J_q^+ &= qutT_t, \\ J_q^- &= ut^{-1}, \\ E_q &= \mu I, \end{aligned} \tag{11}$$

$$f_\lambda(t) = t^\lambda,$$

where $\lambda \in S = \{\alpha+n : \alpha \in \mathbb{C}-\{0\},\ 0 \le \operatorname{Re}\alpha < 1,\ n = 0, \pm 1, \pm 2, \cdots\}$, and T_t is the q–dilation operator given by $T_t\left[f(t)\right] = f(qt)$. We can also construct two variable realizations of $Q_q^\mu(\alpha,u)$ as follows:

$$\begin{aligned} J_q^0 &= (1-q)^{-1}\left(1-q^{-u}T_t\right), \\ J_q^+ &= qu(1-x)\,tT_xT_t, \\ J_q^- &= uT_x^{-1}t^{-1}(1-x)^{-1}, \\ E_q &= \mu I, \end{aligned} \tag{12}$$

$$\phi_\lambda(x,t) = {}_1\phi_0\left(\begin{matrix} q^{-\lambda} \\ - \end{matrix};q,q^\lambda x\right)t^\lambda, \qquad \lambda \in S,$$

and

$$\begin{aligned} J_q^0 &= (1-q)^{-1}\left(1-q^{-u}T_t\right), \\ J_q^+ &= qu(1-x)^{-1}\,tT_xT_t, \\ J_q^- &= ut^{-1}T_x^{-1}(1-x), \\ E_q &= \mu I, \end{aligned} \tag{13}$$

$$\phi_\lambda(x,t) = {}_1\phi_0\left(\begin{matrix} q^{\lambda} \\ - \end{matrix};q,x\right)t^\lambda, \qquad \lambda \in S,$$

where ${}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix};q,x\right)$ is the q-binomial function, see Gasper and Rahman [1]. It can be verified that all above models, (11)–(13), satisfy the commutation relations (3) and (5) as well as (10).

3 Conclusion

It is interesting to compare these results with Koelink [3]. Koelink has studied the quantised universal enveloping algebra $U_q(m(2))$, the complex associative algebra with unit 1 and generators A, B, C, D satisfying relations

$$AB = qBA, \qquad AC = q^{-1}CA, \qquad AD = DA = 1, \qquad BC = CB. \tag{14}$$

Identify the operators J_q^+, J_q^-, J_q^0 with these generators as follows:

$$J_q^+ = B\ ,\ J_q^- = AC\ ,\ J_q^0 = \frac{1-A}{1-q}. \tag{15}$$

It can be easily verified that if A, B, C, D satisfy (14) then the J_q-operators defined by (15) formally satisfy (3). We remark here that these models of $\mathcal{G}(0,0)$ will lead to q–special function identities also. However, as the procedure is analogous to Manocha [4] and Sahai and Sahai [6], we avoid the details.

Finally, we unify the results in this paper for the Lie algebra $\mathcal{G}(0,0)$ with the results for the Lie algebra sl(2) obtained in Manocha [4] and the Lie algebra $\mathcal{G}(0,1)$ obtained in Sahai and Sahai [6]. Define the q-representation of the Lie algebra $\mathcal{G}(a,b)$ as follows:

A q-representation of $\mathcal{G}(a,b)$ on V_q is a linear mapping $\rho_q : \mathcal{G}(a,b) \to A(V_q)$ satisfying the following conditions:

1. There exists a Lie algebra representation ρ of $\mathcal{G}(a,b)$ on V such that
$$\lim_{q\to 1} \rho_q(x)\phi_\lambda = \rho(x) f_\lambda,$$
for all $x, y \in \mathcal{G}(a,b)$.

2. If we denote $J_q^+ = \rho_q(\mathcal{J}^+)$, $J_q^- = \rho_q(\mathcal{J}^-)$, $J_q^0 = \rho_q(\mathcal{J}^0)$, $E_q =$

$\rho_q(\mathcal{E})$, where $\{\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}^0, \mathcal{E}\}$ forms a basis of $\mathcal{G}(a,b)$, then

$$\begin{aligned} J_q^0 J_q^+ - qJ_q^+ J_q^0 &= J_q^+, \\ qJ_q^0 J_q^- - J_q^- J_q^0 &= -J_q^-, \\ qJ_q^+ J_q^- - J_q^- J_q^+ &= 2a^2 q^{2u} J_q^0 - a^2(1-q)\, q^{2u} J_q^0 J_q^0 - bq^{u-1} E_q, \end{aligned} \tag{16}$$

$$[E_q, J_q^+] = [E_q, J_q^-] = [E_q, J_q^0] = 0.$$

The operator C_q on V_q is defined by

$$C_q = qJ_q^+ J_q^- - bq^u J_q^0 E_q + a^2 q^{2u} J_q^0 J_q^0 - a^2 q^{2u} J_q^0. \tag{17}$$

It can be easily verified that C_q satisfies the following relations:

$$\begin{aligned} qJ_q^+ C_q &= C_q J_q^+, \\ J_q^- C_q &= qC_q J_q^-, \\ J_q^0 C_q &= C_q J_q^0, \\ E_q C_q &= C_q E_q. \end{aligned} \tag{18}$$

Indeed, as $q \to 1$, ρ_q reduces to a Lie algebra representation ρ of $\mathcal{G}(a,b)$ on V.

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