CR-SUBMANIFOLDS OF A NEARLY TRANS-HYPERBOLIC SASAKIAN MANIFOLD WITH SEMI-SYMMETRIC SEMI-METRIC CONNECTION.

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Abstract

The notion of a CR-submanifolds of a Kaehler manifold was introduced by A. Bejancu [8]. CR-submanifolds have been studied by many geometers ([1], [3], [4], [6], [9], [12], [13], [14], [16], [18], [19]). On the other hand, almost contact hyperbolic (f,g,η, ξ)-structure was defined and studied by Upadhyaya and Dube in [17]. S. Kumar and K. K. Dube studied CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold in [14]. Semi-symmetric semi-symmetric connections were studied by many geometers ([2], [5], [7], [15]). In this paper, we study CR-submanifold of a nearly trans-hyperbolic Sasakian manifold admitting semi-symmetric semi-meric connection and prove some basic lemmas on CR-submanifolds for semi symmetric semi-metric connection. Also, the parallel distributions on CR-submanifolds for semi-symmetric semi-metric connection have been discussed.

1 Preliminaries

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The Torsion tensor T of ∇ is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

The connection ∇ is symmetric if torsion tensor T vanishes, otherwise it is symmetric. The connection ∇ is metric if $\nabla g = 0$ for the Riemannian metric g, otherwise it is non-metric.
A connection ∇ is said to be semi-symmetric ([11]) if its torsion tensor is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form.
Let $\overline{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost

hyperbolic contact structure ϕ, ξ, η, g where a tensor field ϕ of type (1,1), a vector field ξ and 1-form η of ξ satisfying

$$\varphi^2 X = X - \eta(X)\xi, g(X, \xi) = \eta(X) \tag{1.1}$$

$$\eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta o \varphi = 0, \tag{1.2}$$

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y) \tag{1.3}$$

fro any vector X, Y tangent to $\overline{M}$ [18]. In case we have

$$g(\phi X.Y) = -g(X, \phi Y). \tag{1.4}$$

An almost hyperbolic contact metric structure (ϕ, ξ, η, g) on $\overline{M}$ is called trans-hyperbolic Sasakian [10] if and only if

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)\phi X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{1.5}$$

for all X, Y tangent to $\overline{M}$ where α, β are functions on $\overline{M}$. On a trans-hyperbolic Sasakian manifold $\overline{M}$ we have

$$\nabla_X \xi = -\alpha(\varphi X) + \beta(X - \eta)(X)\xi) \tag{1.6}$$

for a Riemannian metric g and the Levi-Civita connection ∇. Further, an almost hyperbolic contact metric manifold $\overline{M}$ is called nearly trans-hyperbolic Sasakian manifold if [14]

$$\begin{aligned} (\overline{\nabla}_X \varphi)Y + (\overline{\nabla}_Y \varphi)X &= \alpha(2g(X, Y)\xi - \eta(Y)\varphi X - \eta(X)\varphi Y) \\ &\quad -\beta(\eta(X)\varphi Y + \eta(Y)\varphi X) \end{aligned} \tag{1.7}$$

Let M be a submanifold of nearly trans-hyperbolic Sasakian manifold $\overline{M}$. The metric induced on M is denoted by same symbol g.Let $M = TM + TM\perp$, where TM is tangent space and $TM\perp$ is the normal space.

Definition 2.1. An m- dimensional submanifold M of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ is called a CR-submanifold if ξ is tangent
to M and $T_X(M) = D_X + D_X^\perp$ such that
(i) the distribution D_x is invariant under φ, that is $\phi D_x \subset D_x$ for each $x \in M$,
(ii) the complementary orthogonal distribution $D^\perp$ is anti-invariant under φ, that is $\varphi D_x^\perp \subset$

$T_x^{\perp}(M)$ for all $x \in M$.
If $\dim D_x^{\perp} = 0$ (resp. $\dim D_x = 0$), then CR-submanifold is called invariant (resp. anti invariant). The distribution D (resp. $D^{\perp}$) is called horizontal (resp. vertical) distribution. The pair $(D, D^{\perp})$ is called ξ-horizontal (resp. ξ-vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D^{\perp}$) for any $x \in M$.

For any $X \in TM$, we write

$$X = PX + QX, \tag{1.8}$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively. For any vector $N \in TM^{\perp}$ we can put

$$\phi N = BN + CN, \tag{1.9}$$

where BN is tangential and CN is the normal component of ϕN.
Now, we remark that owing to the existence of the 1-form η, we can define a semi-symmetric semi-metric connection $\overline{\nabla}$ in a nearly trans-hyperbolic Sasakian manifold by

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(X)Y + g(X,Y)\xi \tag{1.10}$$

such that $\quad (\overline{\nabla}_X g)(Y,Z) = 2\eta(X)g(Y,Z) - \eta(Y)g(Z,X) - \eta(Z)g(X,Y)$

Inserting (2.10) in (2.5), we get

$$\begin{aligned}(\overline{\nabla}_X \varphi)Y &= \alpha(g(X,Y)\xi - \eta(Y)\varphi X) + \beta(g(\varphi X,Y)\xi - \eta(X)\varphi(Y) \\ &\quad -2\eta(X)\varphi Y + g(X,\varphi Y)\xi.\end{aligned}$$

Interchanging X and Y, we have

$$\begin{aligned}(\overline{\nabla}_Y \varphi)X &= \alpha(g(X,Y)\xi - \eta(Y)\varphi X) + \beta(g(X,\varphi Y)\xi - \eta(Y)\varphi(X) \\ &\quad -2\eta(Y)\varphi X + g(Y,\varphi X)\xi.\end{aligned}$$

Adding above tow equation, we obtain

$$\begin{aligned}(\overline{\nabla}_X \varphi)Y + (\overline{\nabla}_Y \varphi)X &= \alpha(2g(X,Y)\xi - \eta(X)\varphi Y - \eta(Y)\varphi X) - \beta(\eta(X)\varphi Y \quad (1.11)\\ &\quad +\eta(Y)\varphi X) - 2\eta(X)\varphi Y - 2\eta(Y)\varphi X. \quad (1.12)\end{aligned}$$

From (2.6) and (2.10), we get

$$\overline{\nabla}_X \xi = -\alpha(\varphi X) + \beta(X - \eta(X)\xi). \tag{1.13}$$

The Gauss formula for a CR-Submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection is

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1.14}$$

and the Weingarten formula on M is given by

$$\overline{\nabla}_X N = -A_N X - \eta(X)N + \nabla_X^{\perp} N \tag{1.15}$$

for $X, Y \in TM$, $N \in TM^{\perp}$, where h and A are called the second fundamental tensor and shape operator respectively and $\nabla^{\perp}$ denotes the normal connection. Moreover, we also have

$$g(h(X, Y), N) = g(A_N X, Y). \tag{1.16}$$

Theorem 2.1. The connection induced on CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection is also a semi- symmetric semi-metric connection.

Proof. Let $\overline{\overline{\nabla}}$ be the induced connection with the unit normal N on CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi- symmetric semi-metric connection $\overline{\nabla}$. Then

$$\overline{\nabla}_X Y = \overline{\overline{\nabla}}_X Y + m(X, Y), \tag{1.17}$$

where m is a tensor field of type $(0, 2)$ on CR-submanifold M If ∇^* be the induced connection ∇ from the CR- Riemannian connection on CR-Submanifold.

Then we have

$$\nabla_X Y = \nabla_X^* Y + h(X, Y), \tag{1.18}$$

where h is a second fundamental tensor of type (0, 2). From (2.16), (2.17) and (2.10), we get

$$\overline{\overline{\nabla}}_X Y + m(X, Y) = \nabla_X^* Y + h(X, Y) - \eta(X)Y + g(X, Y)\xi.$$

Comparing the tangential and normal components from both sides, we find

$$\overline{\overline{\nabla}}_X Y = \nabla_X^* Y - \eta(X)Y + g(X, Y)\xi.$$

Thus $\overline{\overline{\nabla}}$ is also a semi- symmetric semi-metric connection.

2 Some Basic Lemmas on CR-submanifold for semi-symmetric semi-metric connection

Lemma 3.1. Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a semi-symmetric semi-metric connection $\tilde{\nabla}$ Then

$$P\nabla_X(\varphi PY) + P\nabla_Y(\varphi PX) - PA_{\varphi QX}Y - PA_{\varphi QY}X = \varphi P\nabla_X Y + \varphi P\nabla_Y X$$

$$\begin{aligned} &+2\alpha g(X,Y)P\xi - \alpha\eta(X)\varphi PY - \alpha\eta(Y)\varphi PX - \beta\eta(Y)\varphi PX \\ &-\beta\eta(X)\varphi PY - 2\eta(X)\varphi PY - 2\eta(Y)\varphi PX, \end{aligned} \tag{2.1}$$

$$Q(\nabla_X\varphi PY)+Q(\nabla_Y\varphi PX)-Q(A_{\varphi QX}Y)-Q(A_{\varphi QY}X) = 2Bh(X,Y)+2\alpha g(X,Y)Q\xi, \tag{2.2}$$

$$\begin{aligned} &h(X,\varphi PY) + h(Y,\varphi PX) + \nabla^{\perp}_X\varphi QY + \nabla^{\perp}_Y\varphi QX - \eta(X)\varphi QY - \\ &\eta(Y)\varphi QX + \varphi Q\nabla_Y X + \varphi Q\nabla_X Y + 2Ch(X,Y) - 2\varphi h(X,Y) = 0 \end{aligned} \tag{2.3}$$

Proof. From (2.8) , we have

$$\varphi Y = \varphi PY + \varphi QY.$$

By covariant differentiation of both sides, we have

$$\tilde{\nabla}_X\varphi Y = \tilde{\nabla}_X$$

Using (2.12), (2.13) and (2.11), we get

$$\begin{aligned} (\tilde{\nabla}_X\varphi)Y + \varphi\nabla_X Y + \varphi h(X,Y) &= \nabla_X(\varphi PY) + h(\varphi PY,X) + \nabla^{\perp}_X(\varphi QY) \\ &\quad -A_{\varphi QY}X - \eta(X)\varphi QY. \end{aligned}$$

Interchanging X and Y, we have

$$\begin{aligned} (\tilde{\nabla}_X\varphi)Y + \varphi\nabla_Y X + \varphi h(X,Y) &= \nabla_Y(\varphi PX) + h(Y,\varphi PX) + \nabla^{\perp}_X(\varphi QX) \\ &\quad -A_{\varphi QX}Y - \eta(Y)\varphi QX. \end{aligned}$$

Adding above two equations, we get

$$\begin{aligned} &(\tilde{\nabla}_X\varphi)Y + (\tilde{\nabla}_Y\varphi)X + \varphi\nabla_X Y + \varphi\nabla_Y X + 2\varphi h(X,Y) = \nabla_X(\varphi PY) + \nabla_Y(\varphi PX) \\ &+h(X,\varphi PY) + h(Y,\varphi PX) + \nabla^{\perp}_X(\varphi QY) + \nabla^{\perp}_Y(\varphi QX) - A_{\varphi QX}Y - A_{\varphi QY}X \\ &-\eta(Y)\varphi QX - \eta(X)\varphi QY \end{aligned}$$

Using (2.11) in above equation, we obtain

$$\alpha(2g(X,Y)\xi-\eta(X)\varphi Y-\eta(Y)\varphi X)-\beta(\eta(Y)\varphi X+\eta(X)\varphi Y)-2\eta(X)\varphi Y-2\eta(Y)\varphi X$$
$$+\varphi\nabla_X Y+\varphi\nabla_Y X+2\varphi h(X,Y)=h(X,\varphi PX)+\nabla_X\perp(\varphi QY)+\nabla_Y^{\perp}(\varphi QX)$$
$$+\nabla_X(\varphi PY)+\nabla_Y(\varphi PX)-A_{\varphi QX}Y-A_{\varphi QY}X$$

Equations (3.1) to (3.3) followed by comparing the horizontal, vertical and normal components.

Lemma 3.2. Let M be a ξ-horizontal CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a semi-symmetric semi-metric connection. Then

$$\begin{aligned}2(\tilde{\nabla}_X\varphi)Y &= \nabla_X\varphi Y-\nabla_Y\varphi X+h(X,\varphi Y)-h(Y,\varphi X)-\varphi[X,Y]\\ &\quad+\alpha(2g(X,Y)\xi-\eta(X)\varphi Y-\eta(Y)\varphi X)-2\eta(X)\varphi Y\\ &\quad-\beta(\eta(X)\varphi Y+\eta(Y)\varphi X)-2\eta(Y)\varphi X,\end{aligned}\tag{2.4}$$

$$\begin{aligned}2(\tilde{\nabla}_X\varphi)X &= \alpha(2g(X,Y)\xi-\eta(Y)\varphi X-\eta(X)\varphi Y)\\ &\quad-\beta(\eta(X)\varphi Y+\eta(Y)\varphi X)+\varphi[X,Y]-2\eta(X)\varphi Y\\ &\quad-2\eta(Y)\varphi X-\nabla_X\varphi Y+\nabla_Y\varphi X-h(X,\varphi Y)+h(Y,\varphi X)\end{aligned}\tag{2.5}$$

for any $X,Y\in D$.

Proof. Let $X,Y\in D$. Using Gauss formula (2.13), we have

$$\tilde{\nabla}_X\phi Y-\tilde{\nabla}_Y\phi X=\nabla_X\phi Y+h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X).\tag{2.6}$$

Also, we have

$$\tilde{\nabla}_X\phi Y-\tilde{\nabla}_Y\phi X=(\tilde{\nabla}_X\phi)Y-(\tilde{\nabla}_Y\phi)X+\phi[X,Y].\tag{2.7}$$

From (3.6) and (3.7), we get

$$(\tilde{\nabla}_X\phi)Y-(\tilde{\nabla}_Y\phi)X=\nabla_X\phi Y+h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X)-\phi[X,Y].\tag{2.8}$$

Adding (2.11) and (3.8), we have

$$2(\tilde{\nabla}_X\varphi)Y=\nabla_X\varphi Y-\nabla_Y\varphi X+h(X,\varphi Y)-h(Y,\varphi X)-\varphi[X,Y]$$
$$-2g(X,Y)\xi-2\eta(X)\eta(Y)\xi+\alpha(2g(X,Y)-\eta(Y)\phi X-\eta(X)\phi Y)$$
$$-\beta(\eta(Y)\phi X+\eta(X)\phi Y).$$

Subtracting (3.8) from (2.11), we get

$$2(\tilde{\nabla}_Y\varphi)X = \alpha(2g(X,Y)\xi - \eta(Y)\varphi X - \eta(X)\varphi Y) - \beta(\eta(X)\varphi Y \\ +\eta(Y)\varphi X) - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi - \nabla_X\varphi Y \\ -\nabla_Y\varphi X - h(X,\varphi Y) + h(Y,\varphi X) + \varphi[X,Y].$$

Here lemma is proved.

Corollary 3.1 Let M be a ξ-horizontal CR-Submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a semi-symmetric semi-metric connection. Then

$$\begin{aligned} 2(\tilde{\nabla}_Y\varphi)Z &= A_{\varphi Y}Z - A_{\varphi Z}Y + \nabla^{\perp}_Y\varphi Z - \nabla^{\perp}_Z\varphi Y - \varphi[Y,Z] - \eta(Y)\varphi Z \\ &\quad +\alpha(2g(Y,Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - 3\eta(Z)\varphi Y \\ &\quad -\beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y), \end{aligned} \tag{2.9}$$

$$\begin{aligned} 2(\tilde{\nabla}_Z\varphi)Y &= -A_{\varphi Y}Z + A_{\varphi Z}Y - \nabla^{\perp}_Y\varphi Z + \nabla^{\perp}_Z\varphi Y + \varphi[Y,Z] - \eta(Z)\varphi Y \\ &\quad +\alpha(2g(Y,Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - 3\eta(Z)\varphi Z \\ &\quad -\beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y) \end{aligned} \tag{2.10}$$

for any $Y, Z \in D^{\perp}$,

Proof. From Weingarten formula (2.14), we have

$$\tilde{\nabla}_Z\varphi Y - \tilde{\nabla}_Y\varphi Z = A_{\varphi Y}Z - A_{\varphi Z}Y + \nabla^{\perp}_Y\varphi Z - \nabla^{\perp}_Z\varphi Y + \eta(Y)\varphi Z - \eta(Z)\varphi Y. \tag{2.11}$$

Also, we have

$$\tilde{\nabla}_Z\phi Y - \nabla_Y\phi Z = (\tilde{\nabla}_Y\phi)Z - (\tilde{\nabla}_Z\phi)Y + \phi[Y,Z]. \tag{2.12}$$

From (3.11) and (3.12), we get

$$\begin{aligned} (\tilde{\nabla}_Y\varphi)Z - (\tilde{\nabla}_Z\varphi)Y &= -A_{\varphi Y}Z - A_{\varphi Z}Y + \nabla^{\perp}_Y\varphi Z - \nabla^{\perp}_Z\varphi Y + \eta(Y)\varphi Z \\ &\quad -\eta(Z)\varphi Y + \varphi[Y,Z]. \end{aligned} \tag{2.13}$$

Also from (2.11), we have

$$\begin{aligned} (\tilde{\nabla}_Y\varphi)Z + (\tilde{\nabla}_Z\varphi)Y &= -\alpha(2g(Z,Y)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y - 2\eta(Z)\varphi Y \\ &\quad -\beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y) - 2\eta(Y)\varphi Z. \end{aligned} \tag{2.14}$$

Adding (3.13) from (3.14), we obtain

$$\begin{aligned} 2(\tilde{\nabla}_Y\varphi)Z &= A_{\varphi Y}Z - A_{\varphi Z}Y - \nabla^{\perp}_Y\varphi Z - \nabla^{\perp}_Z\varphi Y - \varphi[Y,Z] + \alpha(2g(Y,Z)\xi - \eta(Y)\varphi Z \\ &\quad -\eta(Z)\varphi Y) - \beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y) - \eta(Y)\varphi Z + \eta(Z)\varphi Y. \end{aligned}$$

Subtracting (3.13) from (3.14), we get

$$\begin{aligned} 2(\tilde{\nabla}_Z\varphi)Y &= -A_{\varphi Y}Z + A_{\varphi Z}Y - \nabla^{\perp}_Y\varphi Z + \nabla^{\perp}_Z\varphi Y + \varphi[Y,Z] + \alpha(2g(Y,Z)\xi - \eta(Y)\varphi Z \\ &\quad -\eta(Z)\varphi Y) - \beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y) - 3\eta(Y)\varphi Z - \eta(Z)\varphi Y. \end{aligned}$$

Hence Lemma is proved.

Corollary 3.2. Let M be a ξ-horizontal CR-submanifold of any nearly trans-hyperbolic Sasakian manifold with a semi-metric connection. Then

$$\begin{aligned} 2(\tilde{\nabla}_Z\varphi)Y &= -A_{\varphi Y}Z + A_{\varphi Z}Y - \nabla^{\perp}_Y\varphi Z + \nabla^{\perp}_Z\varphi Y + \varphi[Y,Z] + 2\alpha g(Y,Z)\xi, \\ 2(\tilde{\nabla}_Y\varphi)Z &= A_{\varphi Y}Z - A_{\varphi Z}Y + \nabla^{\perp}_Y\varphi Z - \nabla^{\perp}_Z\varphi Y - \varphi[Y,Z] + 2\alpha g(Y,Z)\xi \end{aligned}$$

for any $Y, Z \in D^{\perp}$.

Lemma 3.4. Let M be a CR-submanifold of an nearly trans-hyperbolic Sasakian manifold with a semi-symmetric connection Then

$$\begin{aligned} &2(\tilde{\nabla}_X\varphi)Y = -A_{\varphi Y}X + \nabla^{\perp}_X\varphi Y - \nabla_Y\varphi X - h(Y,\varphi X) - \varphi[X,Y] - 2\eta(Y)\varphi X \\ &-3\eta(X)\varphi Y + \alpha(-\eta(Y)\varphi X - \eta(X)\varphi Y) - \beta(\eta(X)\varphi Y + \eta(Y)\varphi X), \end{aligned} \tag{2.15}$$

$$\begin{aligned} &2(\tilde{\nabla}_Y\varphi)X = A_{\varphi Y}X - \nabla^{\perp}_X\varphi Y + \nabla_Y\varphi X + h(Y,\varphi X) - \varphi[X,Y] - \eta(X)\varphi Y \\ &-2\eta(Y)\varphi X + \alpha(-\eta(Y)\varphi X - \eta(X)\varphi Y) - \beta(\eta(X)\varphi Y + \eta(Y)\varphi X) \end{aligned} \tag{2.16}$$

for any $X \in D$ and $Y \in D^{\perp}$.

Proof. Let $X \in D$ and $Y \in D^{\perp}$. Then from (2.12) and (2.13), we have

$$\begin{aligned} \tilde{\nabla}_X\phi Y &= -A_{\phi Y}X - \eta(X)\phi Y + \nabla^{\perp}_X\phi Y, \\ \tilde{\nabla}_Y\phi X &= \nabla_Y\phi X + h(Y,\phi X). \end{aligned}$$

Subtracting above two equation, we have

$$\tilde{\nabla}_X\phi Y - \tilde{\nabla}_Y\phi X = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \eta(X)\phi Y. \tag{2.17}$$

Also, by direct covariant differentiation, we have

(2.18) $$\tilde{\nabla}_X\varphi Y - \tilde{\nabla}_Y\varphi X = (\tilde{\nabla}_X\varphi)Y - (\tilde{\nabla}_Y\varphi)X + \varphi[X,Y].$$

From (3.17) and (3.18), we get

(2.19) $$(\tilde{\nabla}_X\varphi)Y - (\tilde{\nabla}_Y\varphi)X = -A_{\varphi Y}X + \nabla^{\perp}_X\varphi Y - \nabla_Y\varphi X - h(Y,\phi X) - \eta(X)\phi Y - \varphi[X,Y]$$

Adding (3.19) and (2.11), we get

$$\begin{aligned} 2(\tilde{\nabla}_X\varphi)Y &= -A_{\varphi Y}X + \nabla^{\perp}_X\varphi Y - \nabla_Y\varphi X - h(Y,\varphi X) - 2\eta(X)\varphi Y \\ &\quad -\varphi[X,Y] - .2\eta(Y)\varphi X + \alpha(-\eta(Y)\varphi X - \eta(X)\varphi Y) \\ &\quad -\beta(\eta(X)\varphi Y + \eta(Y)\varphi X). \end{aligned}$$

Subtracting (3.19) from (2.11), we obtain

$$\begin{aligned} 2(\tilde{\nabla}_Y\varphi)X &= A_{\varphi Y}X - \nabla^{\perp}_X\varphi Y + \nabla_Y\varphi X + h(Y,\varphi X) + \varphi[X,Y] \\ &\quad +\beta(-\eta(Y)\varphi X - \eta(X)\varphi Y) - \beta(\eta(X)\varphi Y + \eta(Y)\varphi X) \\ &\quad -.2\eta(Y)\varphi X - \eta(x)\varphi Y. \end{aligned}$$

Hence lemma is proved.

Corollary 3.3 Let M be a ξ-horizontal CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection, then

$$\begin{aligned} 2(\tilde{\nabla}_X\varphi)Y &= -A_{\varphi Y}X + \nabla^{\perp}_X\varphi Y - \nabla_Y\varphi X - h(Y,\varphi X) - \varphi[X,Y] \\ &\quad -\alpha\eta(X)\varphi Y - \beta\eta(X)\varphi Y - 3\eta(X)\varphi Y, \end{aligned}$$

$$\begin{aligned} 2(\tilde{\nabla}_Y\varphi)X &= A_{\varphi Y}X - \nabla^{\perp}_X\varphi Y - \nabla_Y\varphi X + h(Y,\varphi X) + \varphi[X,Y] \\ &\quad -\alpha\eta(X)\varphi Y - \beta\eta(X)\varphi Y - \eta(X)\varphi Y, \end{aligned}$$

for any $X \in D$ and $Y \in D^{\perp}$.

Corollary 3.4. Let M be a ξ-horizontal CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection, then

$$\begin{aligned} 2(\tilde{\nabla}_X\varphi)Y &= -A_{\varphi Y}X + \nabla^{\perp}_X\varphi Y - \nabla_Y\varphi X - h(Y,\varphi X) - \varphi[X,Y] \\ &\quad -\alpha\eta(Y)\varphi X - \beta\eta(Y)\varphi X - 2\eta(Y)\varphi X, \end{aligned}$$

$$\begin{aligned} 2(\tilde{\nabla}_Y\varphi)X &= A_{\varphi Y}X - \nabla^{\perp}_X\varphi Y + \nabla_Y\varphi X + h(Y,\varphi X) + \varphi[X,Y] \\ &\quad -\alpha\eta(Y)\varphi X - \beta\eta(Y)\varphi X - 2\eta(Y)\varphi X. \end{aligned}$$

for $X \in D$ any $Y \in D^{\perp}$ and

3 Parallel distributions on CR-submanifolds for semi-symmetric semi-metric connection

Definition 4.1. The horizontal (respectively vertical) distribution D (respectively $D^{\perp}$) is said to be parallel with respect to the semi-symmetric semi-metric connection on M if $\tilde{\nabla}_X Y \in D$(respectively $\tilde{\nabla}_Z W \in D^{\perp}$) for any vector field $X, Y \in D$ (respectively $W, Z \in D^{\perp}$).

Proposition 4.1. Let M be a ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a semi-symmetric semi-metric connection. If horizontal distribution D is parallel, then

$$h(X, \varphi Y) = h(Y, \varphi X) \qquad for\ any\ X, Y \in D \tag{3.1}$$

Proof. Let D be parallel distribution, then

$$\nabla_X \phi Y \in D, \quad \nabla_Y \phi \in D \qquad for\ any\ X, Y \in D. \tag{3.2}$$

From (3.2), we get

$$Q(\nabla_X \varphi PY) + Q(\nabla_Y \varphi PX) - Q(A_{\varphi QX} Y) - Q(A_{\varphi QY} X) = 2Bh(X, Y) + 2\alpha g(X, Y)Q\xi$$

$$2Bh(X, Y) + 2\alpha g(X, Y)Q\xi = 0$$

$$Bh(X, Y) = -\alpha g(X, Y)Q\xi \qquad for\ any\ \ X, Y \in D. \tag{3.3}$$

From (2.9), we have

$$\varphi h(X, Y) = Bh(X, Y) + Ch(X, Y). \tag{3.4}$$

From (4.3) and (4.4), we have

$$\varphi h(X, Y) = -\alpha g(X, Y)Q\xi + Ch(X, Y). \tag{3.5}$$

Now, from (3.3) we have

$$h(X, \varphi Y) + h(Y, \varphi X) = 2\varphi h(X, Y) + 2\alpha g(X, Y)Q\xi. \tag{3.6}$$

Replacing X by φX,we find

$$h(\varphi X, \varphi Y) + h(Y, X) = 2\varphi h(\varphi X, Y) + 2\alpha g(X, \varphi Y)Q\xi. \tag{3.7}$$

Similarly, replacing Y by φY in (4.6), we get

$$h(\varphi Y, \varphi X) + h(X, Y) = 2\varphi h(X, \varphi Y) + 2\alpha g(X, \varphi Y)Q\xi. \tag{3.8}$$

From (4.7) and (4.8), we obtain

$$2\varphi h(\varphi X, Y) + 2\alpha g(\varphi X, Y)Q\xi = 2\varphi h(X, \varphi Y) + 2\alpha g(X, \varphi Y)Q\xi. \tag{3.9}$$

Operating φ on both sides and using $\varphi\xi = 0$, we get

$$\varphi h(\varphi X, Y) + \alpha g(\varphi X, Y)\varphi Q\xi = \varphi h(X, \varphi Y) + \alpha g(X, \varphi Y)\varphi Q\xi = 0. \tag{3.10}$$

Thus, we have

$$h(X, \varphi Y) = h(Y, \varphi X) \qquad for\ each\ \ X, Y \in D. \tag{3.11}$$

Proposition 4.2. Let M be a ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a semi-symmetric semi-metric connection. If the distribution $D^{\perp}$ is parallel with respect to the connection on M then

$$(A_{\varphi Y} Z + A_{\varphi Z} Y) \in D^{\perp} \qquad for\ any\ \ X, Z \in D^{\perp}. \tag{3.12}$$

Proof. Let $Y, Z \in D^{\perp}$. Using (2.12), we get

$$(\tilde{\nabla}_Y \varphi)Z + \varphi(\tilde{\nabla}_Y Z) = -A_{\varphi Z} Y + \nabla^{\perp}_Y \varphi Z - \eta(Y)\varphi Z. \tag{3.13}$$

Now using (2.13) we have

$$(\tilde{\nabla}_Y \varphi)Z = -A_{\varphi Z} Y + \nabla^{\perp}_Y \varphi Z - \eta(Y)\varphi Z - \varphi\nabla_Y Z - \varphi h(Y, Z). \tag{3.14}$$

Interchanging Y and Z, we have

$$(\tilde{\nabla}_Y \varphi)Z = -A_{\varphi Z} Y + \nabla^{\perp}_Z \varphi Y - \eta(Z)\varphi Y - \varphi\nabla_Z Y - \varphi h(Y, Z).$$

Adding above two equations, we obtain

$$\begin{aligned} (\tilde{\nabla}_Y \varphi)Z + (\tilde{\nabla}_Z \varphi)Y &= -A_{\varphi Z} Y - A_{\varphi Y} Z + \nabla^{\perp}_Y \varphi Z + \nabla^{\perp}_Z \varphi Y - \eta(Y)\varphi Z \\ &\quad -\eta(Z)\varphi Y - \varphi\nabla_Y Z - \varphi\nabla_Z Y - 2\varphi h(Y, Z). \end{aligned} \tag{3.15}$$

Taking inner product with $x \in D$ in (4.13), we get

$$g(A_{\varphi Y}Z,X)+g(A_{\varphi Z}Y,X) = g(\varphi\nabla_Y Z,X) + g(\varphi\nabla_Z Y,X).$$

If $D^{\perp}$ is parallel then $\nabla_Y Z\in D^{\perp}$ and $\nabla_Z Y\in D^{\perp}$ for $Y,Z\in D^{\perp}$.
Consequently, we have

$$g(A_{\varphi Y}Z,X)+g(A_{\varphi Z}Y,X)=0$$

or,

$$g(A_{\varphi Y}Z+A_{\varphi Z}Y,X)=0, \tag{3.16}$$

which implies that $(A_{\varphi Y}Z+A_{\varphi Z}Y)\in D^{\perp}$

Definition 4.2. A CR- submanifold with a semi-symmetric semi-metric connection is said to be mixed totally geodesic if $h(X,Z)=0$ for all $X\in D$ and $Z\in D^{\perp}$.
The following lemma is an easy consequence of (2.15).
Lemma 4.1. Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a semi-symmetric semi-metric connection. Then M is mixed totally geodesic if and only if $A_N X\in D$ for all $X\in D$.

Definition 4.3. A normal vector field $N\neq 0$ with a connection $\nabla^{\perp}$ is called D-parallel normal section if $\nabla^{\perp}_X N=0$ for all $X\in D$.
Now, we have the following proposition.

Proposition 4.3. Let M be a mixed totally geodesic ξ-vertical CR-submnaifold of a nearly trans-hyperbolic Sasakian Manifold $\tilde{M}$ with a semi-symmetric semi-metric connection. Then the normal section $N\in\phi D^{\perp}$ is D-parallel if and only if $\nabla_X\phi N\in D$ for all $X\in D$.
Proof. Let $N\in\phi D^{\perp}$. From (3.2) we have

$$Q(\nabla_X\varphi PY)+Q(\nabla_Y\varphi PX)-Q(A_{\varphi QX}Y)-Q(A_{\varphi QY}X) = 2Bh(X,Y)+2\alpha g(X,Y)Q\xi. \tag{3.17}$$

Also, we have

$$g(X,Y)=0,\quad \varphi PY=0,\quad \varphi QX=0 \tag{3.18}$$

From (4.18)and (4.17), we get

$$Q(\nabla_Y \phi X) = 0. \tag{3.19}$$

In particular, we have

$$Q(\nabla_Y X) = 0.$$

Using (4.19) in (3.3), we obtain

$$\nabla^{\perp}_X(\varphi QY) = \phi Q(\nabla_X Y).$$

Consequently, we get

$$\nabla^{\perp}_X N = \phi Q \nabla_X \phi N.$$

Hence the proposition is proved.

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