CR-SUBMANIFOLDS OF A NEARLY TRANS-HYPERBOLIC SASAKIAN MANIFOLD WITH SEMI-SYMMETRIC SEMI-METRIC CONNECTION.

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Abstract

The notion of a CR-submanifolds of a Kaehler manifold was introduced by A. Bejancu [8]. CR-submanifolds have been studied by many geometers ([1], [3], [4], [6], [9], [12], [13], [14], [16], [18], [19]). On the other hand, almost contact hyperbolic (f,g,η,ξ)-structure was defined and studied by Upadhyaya and Dube in [17]. S. Kumar and K. K. Dube studied CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold in [14]. Semi-symmetric semi-symmetric connections were studied by many geometers ([2], [5], [7], [15]). In this paper, we study CR-submanifold of a nearly trans-hyperbolic Sasakian manifold admitting semi-symmetric semi-metric connection and prove some basic lemmas on CR-submanifolds for semi symmetric semi-metric connection. Also, the parallel distributions on CR-submanifolds for semi-symmetric semi-metric connection have been discussed.

1 Preliminaries

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The Torsion tensor T of ∇ is given by

\[ T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \]

The connection ∇ is symmetric if torsion tensor T vanishes, otherwise it is symmetric. The connection ∇ is metric if ∇g = 0 for the Riemannian metric g, otherwise it is non-metric.

A connection ∇ is said to be semi-symmetric ([11]) if its torsion tensor is of the form

\[ T(X,Y) = \eta(Y)X - \eta(X)Y, \]

where η is a 1-form.

Let M be an n-dimensional almost hyperbolic contact metric manifold with the almost
hyperbolic contact structure $\phi, \xi, \eta, g$ where a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$ and 1-form $\eta$ of $\xi$ satisfying

\begin{align}
\phi^2 X &= X - \eta(X)\xi, \ g(X, \xi) = \eta(X) \\
\eta(\xi) &= -1, \ \phi(\xi) = 0, \ \eta \circ \phi &= 0, \\
g(\phi X, \phi Y) &= -g(X, Y) - \eta(X)\eta(Y)
\end{align}

for any vector $X, Y$ tangent to $\overline{M}$ [18]. In case we have

\begin{align}
g(\phi X, Y) &= -g(X, \phi Y).
\end{align}

An almost hyperbolic contact metric structure $(\phi, \xi, \eta, g)$ on $\overline{M}$ is called trans-hyperbolic Sasakian [10] if and only if

\begin{align}
(\nabla_X \phi)Y &= \alpha(g(X, Y)\xi - \eta(Y)\phi X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)
\end{align}

for all $X, Y$ tangent to $\overline{M}$ where $\alpha, \beta$ are functions on $\overline{M}$. On a trans-hyperbolic Sasakian manifold $\overline{M}$ we have

\begin{align}
\nabla_X \xi &= -\alpha(\phi X) + \beta(X - \eta)(X)\xi
\end{align}

for a Riemannian metric $g$ and the Levi-Civita connection $\nabla$. Further, an almost hyperbolic contact metric manifold $\overline{M}$ is called nearly trans-hyperbolic Sasakian manifold if [14]

\begin{align}
(\nabla_X \varphi)Y + (\nabla_Y \varphi)X &= \alpha(2g(X, Y)\xi - \eta(Y)\varphi X - \eta(X)\varphi Y) \\
&\quad - \beta(\eta(X)\varphi Y + \eta(Y)\varphi X)
\end{align}

Let $M$ be a submanifold of nearly trans-hyperbolic Sasakian manifold $\overline{M}$. The metric induced on $M$ is denoted by same symbol $g$. Let $M = TM + TM \perp$, where $TM$ is tangent space and $TM \perp$ is the normal space.

**Definition 2.1.** An $m$-dimensional submanifold $M$ of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ is called a $CR$-submanifold if $\xi$ is tangent to $M$ and $T_X(M) = D_X + D_X^\perp$ such that

(i) the distribution $D_x$ is invariant under $\varphi$, that is $\phi D_x \subset D_x$ for each $x \in M$,

(ii) the complementary orthogonal distribution $D^\perp$ is anti-invariant under $\varphi$, that is $\varphi D_x^\perp \subset$
$T^\perp_x(M)$ for all $x \in M$.
If $\dim D^\perp_x = 0$ (resp. $\dim D_x = 0$), then CR-submanifold is called invariant (resp. anti invariant). The distribution $D$ (resp. $D^\perp$) is called horizontal (resp. vertical) distribution. The pair $(D, D^\perp)$ is called $\xi$-horizontal (resp. $\xi$-vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D^\perp$) for any $x \in M$.

For any $X \in TM$, we write

\begin{equation}
X = PX + QX,
\end{equation}

where $PX$ and $QX$ belong to the distribution $D$ and $D^\perp$ respectively. For any vector $N \in TM^\perp$ we can put

\begin{equation}
\phi N = BN + CN,
\end{equation}

where $BN$ is tangential and $CN$ is the normal component of $\phi N$.

Now, we remark that owing to the existence of the 1-form $\eta$, we can define a semi-symmetric semi-metric connection $\nabla$ in a nearly trans-hyperbolic Sasakian manifold by

\begin{equation}
\nabla_X Y = \nabla_X Y - \eta(X)Y + g(X,Y)\xi
\end{equation}

such that

\begin{equation}
(\nabla_X g)(Y, Z) = 2\eta(X)g(Y, Z) - \eta(Y)g(Z, X) - \eta(Z)g(X, Y)
\end{equation}

Inserting (2.10) in (2.5), we get

\begin{equation}
(\nabla_X \varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)\varphi X) + \beta(g(\varphi X, Y)\xi - \eta(X)\varphi(Y) - 2\eta(X)\varphi Y + g(X, \varphi Y)\xi.
\end{equation}

Interchanging $X$ and $Y$, we have

\begin{equation}
(\nabla_Y \varphi)X = \alpha(g(X,Y)\xi - \eta(Y)\varphi X) + \beta(g(X, \varphi Y)\xi - \eta(Y)\varphi(X) - 2\eta(Y)\varphi X + g(Y, \varphi X)\xi.
\end{equation}

Adding above tow equation, we obtain

\begin{equation}
(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = \alpha(2g(X,Y)\xi - \eta(X)\varphi Y - \eta(Y)\varphi X) - \beta(\eta(X)\varphi X + \eta(Y)\varphi X - 2\eta(X)\varphi Y - 2\eta(Y)\varphi X.
\end{equation}

From (2.6) and (2.10), we get
\( (1.13) \) \[
\nabla_X \xi = -\alpha(\varphi X) + \beta(X - \eta(X))\xi.
\]

The Gauss formula for a CR-Submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection is
\( (1.14) \) \[
\nabla_X Y = \nabla_X Y + h(X, Y)
\]
and the Weingarten formula on \( M \) is given by
\( (1.15) \) \[
\nabla_X N = -A_N X - \eta(X)N + \nabla^\perp_X N
\]
for \( X, Y \in TM, \ N \in TM^\perp \), where \( h \) and \( A \) are called the second fundamental tensor and shape operator respectively and \( \nabla^\perp \) denotes the normal connection. Moreover, we also have
\( (1.16) \) \[
\quad g(h(X,Y), N) = g(A_N X, Y).
\]

**Theorem 2.1.** The connection induced on CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection is also a semi- symmetric semi-metric connection.

**Proof.** Let \( \nabla \) be the induced connection with the unit normal \( N \) on CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi- symmetric semi-metric connection \( \nabla^\perp \). Then
\( (1.17) \) \[
\nabla_X Y = \nabla^\perp_X Y + m(X, Y),
\]
where \( m \) is a tensor field of type \((0, 2)\) on CR-submanifold \( M \). If \( \nabla^* \) be the induced connection \( \nabla \) from the CR- Riemannian connection on CR-Submanifold.

Then we have
\( (1.18) \) \[
\nabla_X Y = \nabla^*_X Y + h(X, Y),
\]
where \( h \) is a second fundamental tensor of type \((0, 2)\). From \((2.16), (2.17)\) and \((2.10)\), we get
\[
\nabla_X Y + m(X,Y) = \nabla^* X Y + h(X,Y) - \eta(X) Y + g(X,Y)\xi.
\]
Comparing the tangential and normal components from both sides, we find
\[
\nabla_X Y = \nabla^* X Y - \eta(X) Y + g(X,Y)\xi.
\]
Thus \( \nabla \) is also a semi- symmetric semi-metric connection.
2 Some Basic Lemmas on CR-submanifold for semi-symmetric semi-metric connection

Lemma 3.1. Let $M$ be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a semi-symmetric semi-metric connection $\tilde{\nabla}$ Then

$$P\nabla_{X}(\varphi PY) + P\nabla_{Y}(\varphi PX) - PA_{\varphi QX}Y - PA_{\varphi QY}X = \varphi P\nabla_{X}Y + \varphi P\nabla_{Y}X
+ 2\alpha g(X,Y)P\xi - \alpha \eta(X)\varphi PY - \alpha \eta(Y)\varphi PX - \beta \eta(Y)\varphi PX$$
(2.1)

$$Q(\nabla_{X}\varphi PY) + Q(\nabla_{Y}\varphi PX) - Q(A_{\varphi QX}Y) - Q(A_{\varphi QY}X) = 2Bh(X,Y) + 2\alpha g(X,Y)Q\xi,
(2.2)$$

$$h(X, \varphi PY) + h(Y, \varphi PX) + \nabla_{X}^{\perp}\varphi QY + \nabla_{Y}^{\perp}\varphi QX - \eta(X)\varphi QY - \eta(Y)\varphi QX$$
(2.3)

**Proof.** From (2.8), we have

$$\varphi Y = \varphi PY + \varphi QY.$$ By covariant differentiation of both sides, we have

$$\tilde{\nabla}_{X}\varphi Y = \tilde{\nabla}_{X}$$ Using (2.12), (2.13) and (2.11), we get

$$(\tilde{\nabla}_{X}\varphi)Y + \varphi \nabla_{X}Y + \varphi h(X,Y) = \nabla_{X}(\varphi PY) + h(\varphi PX, X) + \nabla_{X}^{\perp}(\varphi QY) - A_{\varphi QY}X - \eta(X)\varphi QY.$$ Interchanging $X$ and $Y$, we have

$$(\tilde{\nabla}_{X}\varphi)Y + \varphi \nabla_{Y}X + \varphi h(X,Y) = \nabla_{Y}(\varphi PX) + h(Y, \varphi PX) + \nabla_{Y}^{\perp}(\varphi QX) - A_{\varphi QX}Y - \eta(Y)\varphi QX.$$ Adding above two equations, we get

$$(\tilde{\nabla}_{X}\varphi)Y + (\tilde{\nabla}_{Y}\varphi)X + \varphi \nabla_{X}Y + \varphi \nabla_{Y}X + 2\varphi h(X,Y) = \nabla_{X}(\varphi PY) + \nabla_{Y}(\varphi PX) + h(X, \varphi PY) + h(Y, \varphi PX) + \nabla_{X}^{\perp}(\varphi QY) + \nabla_{Y}^{\perp}(\varphi QX) - A_{\varphi QX}Y - A_{\varphi QY}X - \eta(Y)\varphi QX - \eta(X)\varphi QY.$$
Using (2.11) in above equation, we obtain

\[
\begin{align*}
\alpha(2g(X,Y)\xi - \eta(X)\varphi Y - \eta(Y)\varphi X) - \beta(\eta(Y)\varphi X + \eta(X)\varphi Y) - 2\eta(X)\varphi Y - 2\eta(Y)\varphi X \\
+ \varphi \nabla_X Y + \varphi \nabla_Y X + 2\varphi h(X,Y) = h(X,\varphi PX) + \nabla_X \perp (\varphi QY) + \nabla_Y \perp (\varphi QX) \\
+ \nabla_X (\varphi PY) + \nabla_Y (\varphi PX) - A\varphi QX Y - A\varphi QY X
\end{align*}
\]

Equations (3.1) to (3.3) followed by comparing the horizontal, vertical and normal components.

**Lemma 3.2.** Let $M$ be a $\xi$-horizontal CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a semi-symmetric semi-metric connection. Then

\[
2(\tilde{\nabla}_X \varphi) Y = \nabla_X \varphi Y - \nabla_Y \varphi X + h(X,\varphi Y) - h(Y,\varphi X) - \varphi [X,Y] \\
+ \alpha(2g(X,Y)\xi - \eta(X)\varphi Y - \eta(Y)\varphi X) - 2\eta(X)\varphi Y \\
- \beta(\eta(X)\varphi Y + \eta(Y)\varphi X) + \varphi [X,Y] - 2\eta(X)\varphi Y
\]

(2.4)

\[
2(\tilde{\nabla}_X \varphi) X = \alpha(2g(X,Y)\xi - \eta(Y)\varphi X - \eta(X)\varphi Y) \\
- \beta(\eta(X)\varphi Y + \eta(Y)\varphi X) + \varphi [X,Y] - 2\eta(X)\varphi Y
\]

(2.5)

for any $X, Y \in D$.

**Proof.** Let $X, Y \in D$. Using Gauss formula (2.13), we have

\[
(2.6) \quad \tilde{\nabla}_X \phi Y - \tilde{\nabla}_Y \phi X = \nabla_X \phi Y + h(X,\phi Y) - \nabla_Y \phi X - h(Y,\phi X).
\]

Also, we have

\[
(2.7) \quad \tilde{\nabla}_X \phi Y - \tilde{\nabla}_Y \phi X = (\tilde{\nabla}_X \phi) Y - (\tilde{\nabla}_Y \phi) X + \phi [X,Y].
\]

From (3.6) and (3.7), we get

\[
(2.8) \quad (\tilde{\nabla}_X \phi) Y - (\tilde{\nabla}_Y \phi) X = \nabla_X \phi Y + h(X,\phi Y) - \nabla_Y \phi X - h(Y,\phi X) - \phi [X,Y].
\]

Adding (2.11) and (3.8), we have

\[
2(\tilde{\nabla}_X \varphi) Y = \nabla_X \varphi Y - \nabla_Y \varphi X + h(X,\varphi Y) - h(Y,\varphi X) - \varphi [X,Y] \\
- 2g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \alpha(2g(X,Y) - \eta(Y)\phi X - \eta(X)\phi Y) \\
- \beta(\eta(Y)\phi X + \eta(X)\phi Y).
\]
Subtracting (3.8) from (2.11), we get
\[2(\tilde{\nabla}_Y \varphi)X = \alpha(2g(X,Y)\xi - \eta(Y)\varphi X - \eta(X)\varphi Y) - \beta(\eta(X)\varphi Y + \eta(Y)\varphi X + \varphi[X,Y].\]

Here lemma is proved.

**Corollary 3.1** Let \( M \) be a \( \xi \)-horizontal \( CR \)-Submanifold of a nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a semi-symmetric semi-metric connection. Then
\[2(\tilde{\nabla}_Y \varphi)Z = A_{\varphi Y}Z - A_{\varphi Z}Y + \nabla^\perp_Y \varphi Z - \nabla^\perp_Z \varphi Y - \varphi[Y,Z] - \eta(Y)\varphi Z + \alpha(2g(Y,Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - 3\eta(Z)\varphi Y - \beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y),\] (2.9)
\[2(\tilde{\nabla}_Z \varphi)Y = -A_{\varphi Y}Z + A_{\varphi Z}Y - \nabla^\perp_Y \varphi Z + \nabla^\perp_Z \varphi Y + \varphi[Y,Z] - \eta(Z)\varphi Y + \alpha(2g(Y,Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - 3\eta(Z)\varphi Y - \beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y),\] (2.10)
for any \( Y, Z \in D^\perp \).

**Proof.** From Weingarten formula (2.14), we have
\[(\tilde{\nabla}_Y \varphi)[Y,Z] - (\tilde{\nabla}_Z \varphi)[Y,Z] = (\tilde{\nabla}_Y \varphi)Z - (\tilde{\nabla}_Z \varphi)Y + \varphi[Y,Z].\] (2.11)
Also, we have
\[(\tilde{\nabla}_Z \phi)Y - \nabla^\perp_Y \phi Z = (\tilde{\nabla}_Y \phi)[Y,Z].\] (2.12)
From (3.11) and (3.12), we get
\[2(\tilde{\nabla}_Y \varphi)Z - (\tilde{\nabla}_Z \varphi)Y = -A_{\varphi Y}Z - A_{\varphi Z}Y + \nabla^\perp_Y \varphi Z - \nabla^\perp_Z \varphi Y + \eta(Y)\varphi Z - \eta(Z)\varphi Y - \alpha(2g(Z,Y)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y - 2\eta(Z)\varphi Y - \beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y) - 2\eta(Y)\varphi Z.\] (2.13)
Adding (3.13) from (3.14), we obtain
\[2(\tilde{\nabla}_Y \varphi)Z = A_{\varphi Y} Z - A_{\varphi Z} Y - \nabla^Y_1 \varphi Z - \nabla^Z_1 \varphi Y - \varphi[Y, Z] + \alpha(2g(Y, Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - \eta(Y)\varphi Z + \eta(Z)\varphi Y.\]

Subtracting (3.13) from (3.14), we get
\[2(\tilde{\nabla}_Z \varphi)Y = -A_{\varphi Y} Z + A_{\varphi Z} Y - \nabla^Y_1 \varphi Z + \nabla^Z_1 \varphi Y + \varphi[Y, Z] + \alpha(2g(Y, Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - \eta(Y)\varphi Z + \eta(Z)\varphi Y.\]

Hence Lemma is proved.

**Corollary 3.2.** Let \(M\) be a \(\xi\)-horizontal CR-submanifold of any nearly trans-hyperbolic Sasakian manifold with a semi-metric connection. Then
\[2(\tilde{\nabla}_Z \varphi)Y = -A_{\varphi Y} Z + A_{\varphi Z} Y - \nabla^Y_1 \varphi Z + \nabla^Z_1 \varphi Y + \varphi[Y, Z] + 2\alpha g(Y, Z)\xi,\]
\[2(\tilde{\nabla}_Y \varphi)Z = A_{\varphi Y} Z - A_{\varphi Z} Y + \nabla^Z_1 \varphi Z - \nabla^Y_1 \varphi Y - \varphi[Y, Z] + 2\alpha g(Y, Z)\xi,\]
for any \(Y, Z \in D^\perp\).

**Lemma 3.4.** Let \(M\) be a CR-submanifold of an nearly trans-hyperbolic Sasakian manifold with a semi-symmetric connection Then
\[2(\tilde{\nabla}_X \varphi)Y = -A_{\varphi Y} X + \nabla^Y_X \varphi Y - \nabla^X_1 \varphi Y - \varphi[X, Y] - 2\eta(Y)\varphi X - 3\eta(Y)\varphi X + \alpha(-\eta(Y)\varphi X - \eta(X)\varphi Y) - \beta(\eta(X)\varphi Y + \eta(Y)\varphi X),\]
\[2(\tilde{\nabla}_Y \varphi)X = A_{\varphi Y} X - \nabla^Y_X \varphi Y + \nabla^X_1 \varphi Y + \varphi[X, Y] - \eta(X)\varphi Y - 2\eta(Y)\varphi X + \alpha(-\eta(Y)\varphi X - \eta(X)\varphi Y) - \beta(\eta(X)\varphi Y + \eta(Y)\varphi X),\]
for any \(X \in D\) and \(Y \in D^\perp\).

**Proof.** Let \(X \in D\) and \(Y \in D^\perp\). Then from (2.12) and (2.13), we have
\[\tilde{\nabla}_X \phi Y = -A_{\phi Y} X - \eta(X)\phi Y + \nabla^X_1 \phi Y,\]
\[\tilde{\nabla}_Y \phi X = \nabla_Y \phi X + h(Y, \phi X).\]

Subtracting above two equation, we have
\[\tilde{\nabla}_X \phi Y - \tilde{\nabla}_Y \phi X = -A_{\phi Y} X + \nabla^X_1 \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \eta(X)\phi Y.\]
Also, by direct covariant differentiation, we have

\begin{equation}
\tilde{\nabla}_X \phi Y - \tilde{\nabla}_Y \phi X = (\tilde{\nabla}_X \phi) Y - (\tilde{\nabla}_Y \phi) X + \phi [X, Y].
\end{equation}

From (3.17) and (3.18), we get

\begin{equation}
(\tilde{\nabla}_X \phi) Y - (\tilde{\nabla}_Y \phi) X = -A \phi Y + \nabla_X \phi X - h(Y, \phi X) - \eta(X) \phi Y - \phi [X, Y].
\end{equation}

Adding (3.19) and (2.11), we get

\begin{align*}
2(\tilde{\nabla}_X \phi) Y &= -A \phi Y + \nabla_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - 2\eta(X) \phi Y \\
&\quad - \varphi [X, Y] - 2\eta(Y) \phi X + \alpha(-\eta(Y) \phi X - \eta(X) \phi Y) \\
&\quad - \beta(\eta(X) \phi Y + \eta(Y) \phi X).
\end{align*}

Subtracting (3.19) from (2.11), we obtain

\begin{align*}
2(\tilde{\nabla}_Y \phi) X &= A \phi X - \nabla_X \phi X + \nabla_Y \phi X + h(Y, \phi X) + \varphi [X, Y] \\
&\quad + \beta(-\eta(Y) \phi X - \eta(X) \phi Y) - \beta(\eta(X) \phi Y + \eta(Y) \phi X) \\
&\quad - 2\eta(Y) \phi X - \eta(x) \phi Y.
\end{align*}

Hence lemma is proved.

**Corollary 3.3** Let $M$ be a $\xi$-horizontal $CR$-submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection, then

\begin{align*}
2(\tilde{\nabla}_X \phi) Y &= -A \phi Y + \nabla_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \varphi [X, Y] \\
&\quad - \alpha \eta(X) \phi Y - \beta \eta(X) \phi Y - 3\eta(X) \phi Y,
\end{align*}

for any $X \in D$ and $Y \in D^\perp$.

**Corollary 3.4.** Let $M$ be a $\xi$-horizontal $CR$-submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection, then

\begin{align*}
2(\tilde{\nabla}_Y \phi) X &= A \phi X - \nabla_X \phi X + \nabla_Y \phi X + h(Y, \phi X) + \varphi [X, Y] \\
&\quad - \alpha \eta(X) \phi Y - \beta \eta(X) \phi Y - \eta(X) \phi Y,
\end{align*}

for $X \in D$ any $Y \in D^\perp$ and
3 Parallel distributions on CR-submanifolds for semi-symmetric semi-metric connection

**Definition 4.1.** The horizontal (respectively vertical) distribution $D$ (respectively $D^\perp$) is said to be parallel with respect to the semi-symmetric semi-metric connection on $M$ if $\tilde{\nabla}_X Y \in D$ (respectively $\tilde{\nabla}_Z W \in D^\perp$) for any vector field $X, Y \in D$ (respectively $W, Z \in D^\perp$).

**Proposition 4.1.** Let $M$ be a $\xi$-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a semi-symmetric semi-metric connection. If horizontal distribution $D$ is parallel, then

\[(3.1) \quad h(X, \varphi Y) = h(Y, \varphi X) \quad \text{for any } X, Y \in D\]

**Proof.** Let $D$ be parallel distribution, then

\[(3.2) \quad \nabla_X \varphi Y \in D, \quad \nabla_Y \varphi \in D \quad \text{for any } X, Y \in D.\]

From (3.2), we get

\[
Q(\nabla_X \varphi P Y) + Q(\nabla_Y \varphi P X) - Q(A_{\varphi Q X Y}) - Q(A_{\varphi Q Y X}) = 2Bh(X, Y) + 2\alpha g(X, Y)Q\xi
\]
\[
2Bh(X, Y) + 2\alpha g(X, Y)Q\xi = 0
\]

\[(3.3) \quad Bh(X, Y) = -\alpha g(X, Y)Q\xi \quad \text{for any } X, Y \in D.\]

From (2.9), we have

\[(3.4) \quad \varphi h(X, Y) = Bh(X, Y) + Ch(X, Y).\]

From (4.3) and (4.4), we have

\[(3.5) \quad \varphi h(X, Y) = -\alpha g(X, Y)Q\xi + Ch(X, Y).\]

Now, from (3.3) we have

\[(3.6) \quad h(X, \varphi Y) + h(Y, \varphi X) = 2\varphi h(X, Y) + 2\alpha g(X, Y)Q\xi.\]

Replacing $X$ by $\varphi X$, we find

\[(3.7) \quad h(\varphi X, \varphi Y) + h(Y, X) = 2\varphi h(\varphi X, Y) + 2\alpha g(X, \varphi Y)Q\xi.\]
Similarly, replacing $Y$ by $\varphi Y$ in (4.6), we get

$$(3.8) \quad h(\varphi Y, \varphi X) + h(X, Y) = 2\varphi h(X, \varphi Y) + 2\alpha g(X, \varphi Y)Q\xi.$$ 

From (4.7) and (4.8), we obtain

$$(3.9) \quad \varphi h(\varphi X, \varphi Y) + \alpha g(\varphi X, \varphi Y)Q\xi = \varphi h(X, \varphi Y) + \alpha g(X, \varphi Y)Q\xi.$$ 

Operating $\varphi$ on both sides and using $\varphi \xi = 0$, we get

$$(3.10) \quad \varphi h(\varphi X, \varphi Y) + \alpha g(\varphi X, \varphi Y)Q\xi = \varphi h(X, \varphi Y) + \alpha g(X, \varphi Y)Q\xi = 0.$$ 

Thus, we have

$$(3.11) \quad h(X, \varphi Y) = h(Y, \varphi X) \quad \text{for each} \quad X, Y \in D.$$ 

**Proposition 4.2.** Let $M$ be a $\xi$-vertical $CR$-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a semi-symmetric semi-metric connection. If the distribution $D^\perp$ is parallel with respect to the connection on $M$ then

$$(3.12) \quad (A_{\varphi Y}Z + A_{\varphi Z}Y) \in D^\perp \quad \text{for any} \quad X, Z \in D^\perp.$$ 

**Proof.** Let $Y, Z \in D^\perp$. Using (2.12), we get

$$(3.13) \quad (\tilde{\nabla}_Y \varphi)Z + \varphi(\tilde{\nabla}_Y Z) = -A_{\varphi Z}Y + \nabla^\perp_Y \varphi Z - \eta(Y)\varphi Z.$$ 

Now using (2.13) we have

$$(3.14) \quad (\tilde{\nabla}_Y \varphi)Z = -A_{\varphi Z}Y + \nabla^\perp_Y \varphi Z - \eta(Y)\varphi Z - \varphi \nabla_Y Z - \varphi h(Y, Z).$$

Interchanging $Y$ and $Z$, we have

$$(\tilde{\nabla}_Y \varphi)Z = -A_{\varphi Z}Y + \nabla^\perp_Y \varphi Z - \eta(Z)\varphi Z - \varphi \nabla_Z Y - \varphi h(Y, Z).$$

Adding above two equations, we obtain

$$(\tilde{\nabla}_Y \varphi)Z + (\tilde{\nabla}_Z \varphi)Y = -A_{\varphi Z}Y - A_{\varphi Y}Z + \nabla^\perp_Y \varphi Z + \nabla^\perp_Z \varphi Y - \eta(Y)\varphi Z - \eta(Z)\varphi Y - \varphi \nabla_Y Z - \varphi \nabla_Z Y - 2\varphi h(Y, Z).$$

Taking inner product with $x \in D$ in (4.13), we get
\[ g(A\varphi YZ, X) + g(A\varphi ZY, X) = g(\varphi \nabla_Y Z, X) + g(\varphi \nabla_Z Y, X). \]

If \( D^\perp \) is parallel then \( \nabla_Y Z \in D^\perp \) and \( \nabla_Z Y \in D^\perp \) for \( Y, Z \in D^\perp \). Consequently, we have

\[ g(A\varphi YZ, X) + g(A\varphi ZY, X) = 0 \]

or,

\[ g(A\varphi YZ + A\varphi ZY, X) = 0, \]

which implies that \( (A\varphi YZ + A\varphi ZY) \in D^\perp \)

**Definition 4.2.** A CR-submanifold with a semi-symmetric semi-metric connection is said to be mixed totally geodesic if \( h(X, Z) = 0 \) for all \( X \in D \) and \( Z \in D^\perp \).

The following lemma is an easy consequence of (2.15).

**Lemma 4.1.** Let \( M \) be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a semi-symmetric semi-metric connection. Then \( M \) is mixed totally geodesic if and only if \( A_N X \in D \) for all \( X \in D \).

**Definition 4.3.** A normal vector field \( N \neq 0 \) with a connection \( \nabla^\perp \) is called \( D \)-parallel normal section if \( \nabla^\perp_X N = 0 \) for all \( X \in D \).

Now, we have the following proposition.

**Proposition 4.3.** Let \( M \) be a mixed totally geodesic \( \xi \)-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold \( \tilde{M} \) with a semi-symmetric semi-metric connection. Then the normal section \( N \in \phi D^\perp \) is \( D \)-parallel if and only if \( \nabla^\perp_X \phi N \in D \) for all \( X \in D \).

**Proof.** Let \( N \in \phi D^\perp \). From (3.2) we have

\[ Q(\nabla_X \varphi PY) + Q(\nabla_Y \varphi PX) - Q(A\varphi Q_X Y) - Q(A\varphi Q_Y X) \]

\[ = 2Bh(X, Y) + 2\alpha g(X, Y)Q\xi. \]

(3.17)

Also, we have

\[ g(X, Y) = 0, \quad \varphi PY = 0, \quad \varphi QX = 0 \]

(3.18)

From (4.18) and (4.17), we get
(3.19) \[ Q(\nabla_Y \phi X) = 0. \]

In particular, we have
\[ Q(\nabla_Y X) = 0. \]

Using (4.19) in (3.3), we obtain
\[ \nabla^\perp_X (\varphi QY) = \phi Q(\nabla_X Y). \]

Consequently, we get
\[ \nabla^\perp_X N = \phi Q \nabla_X \phi N. \]

Hence the proposition is proved.

References


