$(0; 0, 2)$ - INTERPOLATION ON LAGUERRE ABSCISSAS

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Abstract

In this paper, we have studied the weighted Pal type $(0; 0, 2)-$ interpolation when we have Lagrange data on the zeros of Laguerre Polynomials $L_n^{(\alpha)}(x)$ and weighted $(0, 2)$ data is prescribed on the zeros of the derivative of the Laguerre Polynomials $(L_n^{(\alpha)})'(x), \alpha > -1$ for a suitable weight function and any natural number n. Existence, uniqueness and explicit representation of the interpolatory polynomial $R_{n,\alpha}(x)$ has been obtained. A qualitative estimate for $R_{n,\alpha}(x)$ has also been dealt with.

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1 Introduction

In 1975, L.G. Pál [6] introduced the following interpolation process. Let

$$-\infty < x_{n,n} < \cdots < x_{1,n} < \infty$$

be a system of distinct real points which are zeros of $W_n(x)$, i.e.,

$$W_n(x) = \prod_{i=1}^{n}(x - x_{i,n}).$$

The roots $y_{i,n}(i = 1, 2, ..., n-1)$ of $W_n'(x)$ are interscaled between the roots of $W_n(x)$, i.e.,

$$(1.1) \qquad -\infty < x_{n,n} < y_{n-1,n} < x_{n-1,n} \cdots < y_{1,n} < x_{1,n} < +\infty.$$

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Pál proved that for given arbitrary numbers $(\alpha_{i,n})_{i=1}^{n}$ and $(\beta_{i,n})_{i=1}^{n-1}$, there exists a unique interpolatory polynomial $R_n(x)$ of degree 2n-1 satisfying the conditions:

$$R_n(x_{i,n}) = \alpha_{i,n}, \quad i = 1, 2, \cdots, n; \quad R_n'(y_{i,n}) = \beta_{i,n-1}, \quad i = 1, 2, \cdots, n-1,$$

and an initial condition $R_n(a) = 0$, where a is a given point, different from the nodal points (1.1). Szili [8] was the first to apply this method on infinite interval by taking the mixed nodes of the Hermite polynomial $H_n(x)$ and its derivative $H_n'(x)$. Later I. Joó [3] sharpened his results by improving the estimates of fundamental polynomials. Srivastava and Mathur [7] studied the problem of (0;0,1) - interpolation on the mixed zeros of $H_n(x)$ and its derivative.

In 2004, Lenard [7] studied a modified Pál type interpolation on Laguerre abscissas and showed that if $(x_i)_{i=1}^{n}$ and $(x_i^*)_{i=1}^{n}$ are the zeros of the Laguerre polynomials $L_n^k(x)$ and $L_n^{k-1}(x)$, respectively and $x_0 = 0$, then there exists a polynomial $R_m(x)$ of degree 2n+k satisfying the conditions:

$$R_m(x_i) = y_i, \quad R_m'(x_i^*) = y_i', \; i = 1, 2, \cdots, n,$$

and

$$R_m^{(j)}(x_0) = y_0^{(j)} (j = 0, 1, \cdots, k),$$

where y_i, y_i' and $y_0^{(j)}$ are arbitrary real numbers. She also obtained the explicit representation of the interpolatory polynomial and gave the corresponding quadrature formula.

In this paper, we have considered $\{x_k\}_{k=1}^{n}$ and $\{y_k\}_{k=1}^{n-1}$ to be the zeros of Laguerre Polynomial $L_n^{(\alpha)}(x)$, its derivative $(L_n^{(\alpha)})'(x)$ respectively, which are interscaled as:

$$0 < x_1 < y_1 < x_2 < ... < x_{n-1} < y_{n-1} < x_n < \infty. \tag{1.2}$$

For an arbitrarily given set of real numbers:

$$\{\alpha_k, k = 0(1)n; \beta_k, k = 1(1)n-1; \gamma_k, k = 1(1)n-1\}, \tag{1.3}$$

we seek to determine a polynomial $R_{n,\alpha}(x)$ of minimal possible degree such that:

$$\begin{cases} R_{n,\alpha}(x_k) = \alpha_k, & k = 1(1)n, \\ R_{n,\alpha}(y_k) = \beta_k, & k = 1(1)n-1, \\ (wR_{n,\alpha})''(y_k) = \gamma_k, & k = 1(1)n-1, \\ R_{n,\alpha}(0) = \alpha_0 \end{cases} \tag{1.4}$$

where

$$w(x) = e^{-\frac{x}{2}} x^{\frac{(\alpha+2)}{2}} \tag{1.5}$$

is the weight function satisfying the condition:

$$(wL_n^{(\alpha)}(L_n^{(\alpha)})')''(y_j) = 0. \tag{1.6}$$

In Section 2, we give some preliminaries. Section 3, is devoted to the existence and explicit representation of the interpolatory polynomial. In Section 4, the estimates of the fundamental polynomials have been obtained and lastly in section 5 we prove the main theorem of the paper, where the quantitative estimate of the interpolatory polynomial has been dealt with.

2 Preliminaries

The differential equation of Laguerre Polynomial $L_n^{(\alpha)}(x)$ is given by

$$x\frac{d^2y}{dx^2}+(\alpha+1-x)\frac{dy}{dx}+ny=0 \tag{2.1}$$

where n is a positive integer.The recurrence relations between Laguerre polynomial and its derivative are as follows:

$$L_{n-1}^{(\alpha+1)}(x)=-(L_n^{(\alpha)})'(x). \tag{2.2}$$

$$\binom{n+\alpha}{n}\approx\frac{n^\alpha}{\Gamma\alpha+1}. \tag{2.3}$$

Let the fundamental polynomials of Lagrange interpolation on the nodes x_k and y_k be

$$l_k(x)=\frac{L_n^{(\alpha)}(x)}{(x-x_k)(L_n^{(\alpha)})'(x_k)},\qquad k=1(1)n \tag{2.4}$$

and

$$l_k^*(x)=\frac{(L_n^{(\alpha)})'(x)}{(x-y_k))(L_n^{(\alpha)})''(y_k)},\qquad k=1(1)n-1, \tag{2.5}$$

respectively. We shall also need the following estimate:

$$L_n^{(\alpha)}(x)=O(n^a),\qquad \left(a=max(\frac{\alpha}{2}-\frac{1}{4},\alpha);0\le x\le d;d>1,\alpha>-1\right) \tag{2.6}$$

(see [9], (7.6.11)).

$$\max_{0<x<d} e^{-\frac{x}{2}}x^{\frac{\alpha}{2}+\frac{1}{4}}|L_n^{(\alpha)}(x)|\sim n^{\frac{\alpha}{2}-\frac{1}{4}} \tag{2.7}$$

(see [9], (Theorem 7.6.5)).

$$\sum_{k=1}^{n}x_k^{m-1}((L_n^{(\alpha)})'(x_k))^{-2}=\frac{\Gamma(n+1)\Gamma(m+\alpha+1)}{\Gamma(n+\alpha+1)}\sim O(n^{-\alpha}) \tag{2.8}$$

(see [9], (14.7.5)).

$$|(L_n^{(\alpha)})'(x_k)|\sim x_k^{-\frac{\alpha}{2}-\frac{3}{4}}n^{\frac{\alpha}{2}+\frac{1}{4}} \tag{2.9}$$

(see [9], (8.9.10), (8.9.11)).

$$x_k \sim \frac{k^2}{n} \tag{2.10}$$

$$y_k \sim \frac{k^2}{n} \tag{2.11}$$

$$\sum_{k=1}^{k=n} \frac{e^{x_k}}{x_k^{\alpha+1}} l_k^2(x) \leq \frac{e^x}{x^{\alpha+1}} \tag{2.12}$$

(see [2], (Inequality 11)). Further, we have

$$\sum_{k=1}^{n} k^{v-1} = \begin{cases} O(1), & v < 0 \\ O(log\ n), & v = 0 \\ O(n^v), & v > 0. \end{cases} \tag{2.13}$$

3 Existence and Explicit representation of the interpolatory polynomial $R_{n,\alpha}(x)$.

Let $(2n-1)$ points in $(0,\infty)$ be given by (1.2). Then to the prescribed numbers $\{\alpha_k\}_{k=0}^{n}$, $\{\beta_k\}_{k=1}^{n-1}$ and $\{\gamma_k\}_{k=1}^{n-1}$, there exists a unique polynomial $R_{n,\alpha}(x)$ of degree $\leq 3n-2$ satisfying the conditions (1.4). It can be explicitly represented as

$$R_{n,\alpha}(x) = \sum_{k=0}^{n} \alpha_k A_k(x) + \sum_{k=1}^{n-1} \beta_k B_k(x) + \sum_{k=1}^{n-1} \gamma_k C_k(x), \tag{3.1}$$

where $\{A_k(x)\}_{k=0}^{n}$, $\{B_k(x)\}_{k=1}^{n-1}$ and $\{C_k(x)\}_{k=1}^{n-1}$ are uniquely determined polynomials each of degree $\leq 3n-2$ and can be explicitly represented as:

$$\begin{aligned} A_k(x) = {} & \frac{(L_n^{(\alpha)})'(x)[l_k(x)]^2}{(L_n^{(\alpha)})'(x_k)} - \frac{2L_n^{(\alpha)}(x)(L_n^{(\alpha)})'(x)}{[(L_n^{(\alpha)})'(x_k)]^2}\Bigg[\frac{L_n^{(\alpha)}(0)}{2x_k^2(L_n^{(\alpha)})'(x_k)} \\ & + \int_0^x \frac{(l_k)'(t) - \frac{x_k-\alpha-1}{2x_k}(L_n^{(\alpha)})'(t)}{t-x_k}dt\Bigg], \end{aligned} \tag{3.2}$$

$$\begin{aligned} B_k(x) = {} & \frac{(L_n^{(\alpha)}(x))^2[l_k^*(x)]^2}{L_n^{(\alpha)}(y_k)} + \\ & + \frac{y_k L_n^{(\alpha)}(x)(L_n^{(\alpha)})'(x)}{n[L_n^{(\alpha)}(y_k)]^2}\Bigg[\int_0^x \frac{(l_k^*)'(t) - \frac{y_k-\alpha-2}{2y_k}l_k^*(t)}{t-y_k}dt + \\ & + \frac{y_k^2 - 2(3+2n)y_k + (2\alpha+4)}{8y_k^2}\int_0^x l_k^*(t)dt - \frac{(L_n^{(\alpha)})'(0)}{ny_k L_n^{(\alpha)}(y_k)}\Bigg] \end{aligned} \tag{3.3}$$

$$C_k(x) = -\frac{y_k L_n^{(\alpha)}(x)(L_n^{(\alpha)})'(x)\int_0^x l_k^*(t)dt}{2nw(y_k)[L_n^{(\alpha)}(y_k)]^2} \tag{3.4}$$

where $l_k(x)$ and $l_k^*(x)$ are given by (2.4) and (2.5) respectively. Lastly,

$$A_0(x) = \frac{L_n^{(\alpha)}(x)(L_n^{(\alpha)})'(x)}{L_n^{(\alpha)}(0)(L_n^{(\alpha)})'(0)}. \tag{3.5}$$

The interpolatory polynomial $R_{n,\alpha}(x)$ satisfies the quantitative estimates:

Theorem 3.1. *Let $f^{(r)} \in Lip\ \ \gamma$, $0 < \gamma \leq 1$, in $[0,\infty)$ for some $r > \alpha$, where $\alpha \geq 0$and integer. Then*

$$|f(x) - R_{n,\alpha}(f;x)| = \begin{cases} O(n^{-\nu-\frac{3}{2}})e^{\frac{x}{2}}x^{-\frac{\alpha}{2}-\frac{5}{4}}, & a < \frac{\alpha}{2} - \frac{5}{4} \\ O(n^{-\nu-\frac{3}{2}}\ log\ n)e^{\frac{x}{2}}x^{-\frac{\alpha}{2}-\frac{5}{4}}, & a = \frac{\alpha}{2} - \frac{5}{4} \\ O(n^{-\nu-\frac{1}{2}})e^{x}x^{-\alpha-1}, & \frac{\alpha}{2} - \frac{5}{4} < a < \frac{\alpha}{2} - 1 \\ O(n^{-\nu-\frac{1}{2}}\ log\ n)e^{x}x^{-\alpha-1}, & a = \frac{\alpha}{2} - 1 \\ O(n^{2a-\nu-\alpha+\frac{3}{2}})e^{x}x^{-\alpha-1}, & a > \frac{\alpha}{2} - 1, \end{cases}$$

for $0 \leq x \leq x_n$, $a = max\left(\frac{\alpha}{2} - \frac{1}{4}, \alpha\right)$ and $\nu = \frac{r+\gamma}{2}$.

The Proof of the Theorem 3.1 has been sketched in Section 5 which needs the estimation of the fundamental polynomials given in the following section.

4 Estimation of the fundamental polynomials

We shall need the following:

Lemma 4.1. *If $d > 1$ and $\alpha > -1$, we have*

$$\max_{0<x<d} e^{\frac{-x}{2}} x^{\frac{\alpha}{2}+\frac{3}{4}} |(L_n^{(\alpha)})'(x)| \sim n^{\frac{\alpha}{2}+\frac{1}{4}}. \tag{4.1}$$

Proof. On replacing α by $\alpha + 1$ in (2.7), we get

$$\max_{0<x<d} e^{\frac{-x}{2}} x^{\frac{\alpha}{2}+\frac{3}{4}} |L_n^{(\alpha+1)}(x)| \sim n^{\frac{\alpha}{2}+\frac{1}{4}}.$$

Now, on using (2.2) in above equation, we get the required result. □

Lemma 4.2. *For the zeros of $L_n^{(\alpha)}(x)$ the estimate*

$$\sum_{k=1}^{n} x_k^{\eta} e^{-\lambda x_k} = O(1)n^{\frac{1}{2}}$$

holds for $x \geq 0$, where η is constant and λ is positive constant.

Lemma 4.3. *For* $0 < x < d; d > 1$ *and* $\alpha > -1$,

$$|\sum_{k=1}^{n} l_k^2(x)| \leq 1 + Cn^{-\frac{1}{4}} \tag{4.2}$$

and

$$|\sum_{k=1}^{n} (l_k^*)^2(x)| \leq 1 + C'n^{-\frac{1}{4}}, \tag{4.3}$$

where C and C' are constants.

Lemma 4.4. *For* $k = 1, 2, \cdots, n-1$ *and* $x \in (0, \infty)$, *we have*

$$|l_k^*(x)| = O(n^{-\frac{1}{2}})e^{-\frac{y_k}{2}} y_k^{\frac{\alpha}{2}+\frac{5}{4}} e^{\frac{x}{2}} x^{-(\frac{\alpha}{2}+\frac{3}{4})}. \tag{4.4}$$

The above lemmas are due to [9], we omit the details.

Lemma 4.5. *For* $k = 1, 2, \cdots, n$, $x \in (0, \infty)$ *and* $a = max\left(\frac{\alpha}{2} - \frac{1}{4}, \alpha\right)$, *we have*

$$\left| x_k^{-\frac{\alpha}{2}+\frac{3}{4}} [(L_n^{(\alpha)})'(x_k)]^2 \int_0^x l_k(t)dt \right| = \begin{cases} O(n^{\alpha-1}), & a < \frac{\alpha}{2} - \frac{5}{4} \\ O(n^{\alpha-1} \ log \ n), & a = \frac{\alpha}{2} - \frac{5}{4} \\ O(n^{a+\frac{\alpha}{2}+\frac{1}{4}}), & a > \frac{\alpha}{2} - \frac{5}{4}. \end{cases} \tag{4.5}$$

Proof. By Christoffel Darboux formula for $L_n^{(\alpha+1)}(x)$, we have

$$\sum_{i=0}^{n-1} \frac{L_i^{(\alpha+1)}(t) L_i^{(\alpha+1)}(y)}{\binom{i+\alpha+1}{i+1}} = \frac{n}{\binom{n+\alpha}{n}} \frac{L_n^{(\alpha+1)}(y) L_{n-1}^{(\alpha+1)}(t) - L_n^{(\alpha+1)}(x) L_{n-1}^{(\alpha+1)}(y)}{(t-y)}. \tag{4.6}$$

On substituting $y = x_k$ and integrating with respect to t from 0 to x, we get

$$\int_0^x l_k(t)dt = \frac{\binom{n+\alpha}{n}}{n[(L_n^{(\alpha)})'(x_k)]^2} \sum_{i=0}^{n-1} \frac{L_i^{(\alpha+1)}(x_k)}{\binom{i+\alpha+1}{i+1}} \int_0^x L_i^{(\alpha+1)}(t)dt, \tag{4.7}$$

which, due to (2.2), gives

$$\int_0^x l_k(t)dt = \frac{\binom{n+\alpha}{n}}{n[(L_n^{(\alpha)})'(x_k)]^2} \sum_{i=0}^{n-1} \frac{(L_{i+1}^{(\alpha)})'(x_k)[L_{i+1}^{(\alpha)}(x) - L_{i+1}^{(\alpha)}(0)]}{\binom{i+\alpha+1}{i+1}}.$$

Now, using (2.3), (2.6) and (2.9) in above equation, we get

$$\left| \int_0^x l_k(t)dt \right| \leq \left| \frac{n^{\alpha-1} x_k^{\frac{\alpha}{2}-\frac{3}{4}}}{[(L_n^{(\alpha)})'(x_k)]^2} \sum_{i=0}^{n-1} [i^{a-\frac{\alpha}{2}+\frac{1}{4}} + i^{\frac{\alpha}{2}+\frac{1}{4}}] \right|.$$

from which, due to (2.13), the lemma follows. □

Lemma 4.6. *For $k = 1, 2, \cdots, n-1$, $x \in (0, \infty)$ and $a = max\left(\frac{\alpha}{2} - \frac{1}{4}, \alpha\right)$, we have*

$$(4.8) \qquad \left|[L_n^{(\alpha)}(y_k)]^2 \int_0^x l_k^*(t)dt\right| \approx \begin{cases} O(n^{\alpha-2}), & a < \frac{\alpha}{2} - 1 \\ O(n^{\alpha-2}\ log\ n), & Q = \frac{\alpha}{2} - 1 \\ O(n^{2a}), & Q > \frac{\alpha}{2} - 1 \end{cases}$$

Proof. The proof of this lemma follows on the same lines as the proof of Lemma 4.5. We omit the details. □

Lemma 4.7. *For $k = 1, 2, \cdots, n$ and $x \in (0, \infty)$, we have*

$$\int_0^x \frac{l_k'(t)}{(t-x_k)}dt - \frac{x_k - \alpha - 1}{2x_k (L_n^{(\alpha)})'(x_k)} \int_0^x \frac{(L_n^{(\alpha)})'(t)}{(t-x_k)}dt + \frac{n}{2x_k}\int_0^x l_k(t)dt$$
$$= \frac{\{(L_n^{(\alpha)})'(x_k)(l_k(x)) - (L_n^{(\alpha)})'(x)\}l_k(x)}{2L_n^{(\alpha)}(x)} + \frac{L_n^{(\alpha)}(x) - (L_n^{(\alpha)})'(x) - (1+\frac{1}{x_k})L_n^{(\alpha)}(0)}{2x_k(L_n^{(\alpha)})'(x_k)}$$

Proof. From (2.4), we have

$$(4.9) \qquad (t-x_k)l_k(t) = \frac{L_n^{(\alpha)}(t)}{(L_n^{(\alpha)})'(x_k)}.$$

On differentiating with respect to t and dividing by $(t-x_k)^2$, we get

$$(4.10) \qquad \frac{l_k'(t)}{(t-x_k)} = \frac{(L_n^{(\alpha)})'(t)}{(t-x_k)^2(L_n^{(\alpha)})'(x_k)} - \frac{l_k(t)}{(t-x_k)^2}$$

which on integration with respect to t from $t = 0$ to $t = x$ and using (2.4) gives

$$\int_0^x \frac{l_k'(t)}{(t-x_k)}dt = \frac{1}{(L_n^{(\alpha)})'(x_k)}\left[\int_0^x \frac{(L_n^{(\alpha)})'(t)}{(t-x_k)^2}dt - \int_0^x \frac{L_n^{(\alpha)}(t)}{(t-x_k)^3}dt\right].$$

By integrating first integral term on RHS, we get

$$(4.11) \qquad \int_0^x \frac{l_k'(t)}{(t-x_k)}dt = \frac{1}{2(L_n^{(\alpha)})'(x_k)}\left[\int_0^x \frac{(L_n^{(\alpha)})'(t)}{(t-x_k)^2}dt + \frac{L_n^{(\alpha)}(x)}{(x-x_k)^2} - \frac{L_n^{(\alpha)}(0)}{x_k^2}\right].$$

So,

$$\int_0^x \frac{l_k'(t)}{(t-x_k)}dt - \frac{x_k-\alpha-1}{2x_k(L_n^{(\alpha)})'(x_k)}\int_0^x \frac{(L_n^{(\alpha)})'(t)}{(t-x_k)}dt = \frac{1}{2(L_n^{(\alpha)})'(x_k)}\Big[\int_0^x \frac{(L_n^{(\alpha)})'(t)}{(t-x_k)^2}dt$$
$$-\frac{x_k-\alpha-1}{x_k}\int_0^x \frac{(L_n^{(\alpha)})'(t)}{(t-x_k)}dt + \frac{L_n^{(\alpha)}(x)}{(x-x_k)^2} - \frac{L_n^{(\alpha)}(0)}{x_k^2}\Big].$$

On integrating first integral term by parts and using (2.1) and (2.4), we get

$$\int_0^x \frac{l_k'(t)}{(t-x_k)}dt - \frac{x_k-\alpha-1}{2x_k(L_n^{(\alpha)})'(x_k)}\int_0^x \frac{(L_n^{(\alpha)})'(t)}{(t-x_k)}dt = \frac{1}{2(L_n^{(\alpha)})'(x_k)}\Big[\frac{L_n^{(\alpha)}(x)-L_n^{(\alpha)}(0)}{x_k}$$
$$-(L_n^{(\alpha)})'(x)(\frac{1}{(x-x_k)}+\frac{1}{x_k}) - \frac{n(L_n^{(\alpha)})'(x_k)}{x_k}\int_0^x l_k(t)dt + \frac{L_n^{(\alpha)}(x)}{(x-x_k)^2} - \frac{L_n^{(\alpha)}(0)}{x_k^2}\Big].$$

Now, on using (2.4), we get the required result. □

Lemma 4.8. *For $k=1,2,\cdots,n-1$ and $x\in(0,\infty)$, we have*

$$\int_0^x \frac{(l_k^*)'(t)}{(t-y_k)}dt - \frac{y_k-\alpha-2}{2y_k}\int_0^x \frac{l_k^*(t)}{(t-y_k)}dt = -\frac{nL_n^{(\alpha)}(y_k)(l_k^*(x))^2}{2y_k(L_n^{(\alpha)})'(x)} - \frac{l_k^*(x)}{2y_k}$$
$$+\frac{nL_n^{(\alpha)}(x)l_k^*(x)}{2y_k(L_n^{(\alpha)})'(x)} + \frac{(L_n^{(\alpha)})'(0)}{ny_kL_n^{(\alpha)}(y_k)} - \frac{L_n^{(\alpha)}(0)}{2y_kL_n^{(\alpha)}(y_k)} - \frac{n-1}{2y_k}\int_0^x l_k^*(t)dt \tag{4.12}$$

Proof. The result follows on same lines as previous lemma, we omit details. □

Lemma 4.9. *([1]) If $f^{(r)}$ exists and is continuous in $[0,\infty)$, $r\geq 0$, then there exists a polynomial G_n of degree $n\geq 4r+5$ at most, such that*

$$|f^{(i)}(x)-G_n^{(i)}(f;x)| = O(1)\omega\left(f^{(r)};\frac{\sqrt{x(x_n-x)}}{n}\right)\left(\frac{\sqrt{x(x_n-x)}}{n}\right)^{r-i}$$

$0\leq x\leq x_n$, $i=0,1,\cdots,r$, where $\omega(f^{(r)},.)$ denotes the modulus of continuity of $f^{(r)}$ on $[0,x_n]$.
The lemma shows that $G_n^{(i)}(f;0)=f^{(i)}(0)$, $i=0,1,2,\cdots,r$.

Lemma 4.10. *For x_k, $k=1,\cdots,n$ and $\nu=\frac{r+\gamma}{2}$, we have*

$$e^{-\frac{3x}{2}}x^{\frac{3\alpha}{2}+\frac{5}{4}}\sum_{k=1}^{n}|x_k^{\nu}A_k(x)| = \begin{cases} O(n^{-\frac{3}{2}}), & a<\frac{\alpha}{2}-\frac{5}{4} \\ O(n^{-\frac{3}{2}}\ log\ n), & a=\frac{\alpha}{2}-\frac{5}{4} \\ O(n^{a-\frac{\alpha}{2}-\frac{1}{4}}), & a>\frac{\alpha}{2}-\frac{5}{4}. \end{cases}$$

Proof. Due to lemma 4.7 and (3), we have

$$
\begin{aligned}
\sum_{k=1}^{n}|x_k^{\nu}A_k(x)| \leq& \sum_{k=1}^{n}\Big|\frac{x_k^{\nu}[(L_n^{(\alpha)})'(x)]^2 l_k(x)}{(L_n^{(\alpha)})'(x_k)]^2}\Big| + \sum_{k=1}^{n}\Big|\frac{x_k^{\nu-1}[L_n^{(\alpha)}(x)]^2(L_n^{(\alpha)})'(x)}{(L_n^{(\alpha)})'(x_k)]^3}\Big| \\
&+ \sum_{k=1}^{n}\Big|\frac{x_k^{\nu-1}|L_n^{(\alpha)}(x)[(L_n^{(\alpha)})'(x)]^2}{(L_n^{(\alpha)})'(x_k)]^3}\Big| + \sum_{k=1}^{n}\Big|\frac{x_k^{\nu-1}L_n^{(\alpha)}(x)(L_n^{(\alpha)})'(x)L_n^{(\alpha)}(0)}{(L_n^{(\alpha)})'(x_k)]^3}\Big| \\
&+ \sum_{k=1}^{n}\Big|\frac{nx_k^{\nu-1}L_n^{(\alpha)}(x)[(L_n^{(\alpha)})'(x)]^2\int_0^x l_k(t)dt}{(L_n^{(\alpha)})'(x_k)]^3}\Big| \\
=& \ I_{A1} + I_{A2} + I_{A3} + I_{A4} + I_{A5}(say).
\end{aligned}
$$

By Lemma 4.1, we have

$$
I_{A1} = \sum_{k=1}^{n}\left|e^{-x_k}x_k^{\nu+\alpha+\frac{3}{2}}l_k(x)\right|e^x x^{-\alpha-\frac{3}{2}}.
$$

On using Cauchy's inequality, Lemma 4.2 and Lemma 4.3, we get

$$
\begin{aligned}
I_{A1} &= \left(\sum_{k=1}^{n}\left|e^{-2x_k}x_k^{2\nu+2\alpha+3}\right|\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}|l_k(x)|^2\right)^{\frac{1}{2}} e^x x^{-\alpha-\frac{3}{2}} \\
&= O(n^{\frac{1}{4}})e^x x^{-\alpha-\frac{3}{2}} \leq O(n^{\frac{1}{4}})e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{5}{4}}. \qquad (4.13)
\end{aligned}
$$

Again by using (2.7) and Lemma 4.1 in I_{A2}, we have

$$
\begin{aligned}
I_{A2} &= O(\frac{1}{n})\sum_{k=1}^{n}\left|e^{-\frac{3x_k}{2}}x_k^{\nu+\frac{3\alpha}{2}+\frac{9}{4}}\right|e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{5}{4}} \\
&= O(n^{-\frac{1}{2}})e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{5}{4}} \qquad (4.14)
\end{aligned}
$$

due to Lemma 4.2. Similarly,

$$
I_{A3} = O(1)e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{7}{4}} \leq O(1)e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{5}{4}} \qquad (4.15)
$$

and

$$
I_{A4} = O(n^{\frac{\alpha}{2}-\frac{1}{4}})e^x x^{-\alpha-1} \leq O(n^{\frac{\alpha}{2}-\frac{1}{4}})e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{5}{4}}. \qquad (4.16)
$$

On multiplying and dividing I_{A5} by $x_k^{-\frac{\alpha}{2}+\frac{3}{4}}[(L_n^{(\alpha)})'(x_k)]^2$ and using (2.7) and Lemma 4.1 for x and x_k, we get

$$
I_{A5} = O(n^{-\alpha-1})\sum_{k=1}^{n}\left|e^{-\frac{5x_k}{2}}x_k^{\nu+3\alpha+2}\left(x_k^{-\frac{\alpha}{2}+\frac{3}{4}}[(L_n^{(\alpha)})'(x_k)]^2\int_0^x l_k(t)dt\right)\right|e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{7}{4}}..
$$

By Lemma 4.5 and Lemma 4.2 it follows that

- for $a < \frac{\alpha}{2} - \frac{5}{4}$,

$$I_{A5} = O(n^{-\frac{3}{2}}) e^{\frac{3x}{2}} x^{-\frac{3\alpha}{2}-\frac{7}{4}} \leq O(n^{-\frac{3}{2}}) e^{\frac{3x}{2}} x^{-\frac{3\alpha}{2}-\frac{5}{4}}. \tag{4.17}$$

- for $a = \frac{\alpha}{2} - \frac{5}{4}$

$$I_{A5} = O(n^{-\frac{3}{2}} \ log \ n) e^{\frac{3x}{2}} x^{-\frac{3\alpha}{2}-\frac{7}{4}} \leq O(n^{-\frac{3}{2}}) e^{\frac{3x}{2}} x^{-\frac{3\alpha}{2}-\frac{5}{4}} \tag{4.18}$$

- for $a > \frac{\alpha}{2} - \frac{5}{4}$

$$I_{A5} = O(n^{a-\frac{\alpha}{2}-\frac{1}{4}}) e^{\frac{3x}{2}} x^{-\frac{3\alpha}{2}-\frac{7}{4}} \leq O(n^{a-\frac{\alpha}{2}-\frac{1}{4}}) e^{\frac{3x}{2}} x^{-\frac{3\alpha}{2}-\frac{5}{4}}. \tag{4.19}$$

Thus by (4.13), (4.14), (4.15), (4.16), (4.17), (4.18) and (4.19) the lemma follows. □

Lemma 4.11. *For y_k, $k = 1, \cdots, n-1$ and $\nu = \frac{r+\gamma}{2}$, we have*

$$e^{-\frac{3x}{2}} x^{\frac{3\alpha}{2}+\frac{7}{4}} \sum_{k=1}^{n-1} |y_k^{\nu} B_k(x)| = \begin{cases} O(n^{-\frac{1}{2}}), & a < \frac{\alpha}{2} - 1 \\ O(n^{-\frac{1}{2}} \ log \ n), & a = \frac{\alpha}{2} - 1 \\ O(n^{2a-\alpha+\frac{3}{2}}), & a > \frac{\alpha}{2} - 1. \end{cases}$$

Proof. On using Lemma 4.8 in (3.4), we get

$$\begin{aligned}
\sum_{k=1}^{n-1} |y_k^{\nu} B_k(x)| &\leq \sum_{k=1}^{n-1} \left| \frac{y_k^{\nu} L_n^{(\alpha)}(x) [l_k^*(x)]^2}{2 L_n^{(\alpha)}(y_k)} \right| + \sum_{k=1}^{n-1} \left| \frac{y_k^{\nu} [L_n^{(\alpha)}(x)]^2 l_k^*(x)}{2 [L_n^{(\alpha)}(y_k)]^2} \right| \\
&+ \sum_{k=1}^{n-1} \left| \frac{y_k^{\nu} L_n^{(\alpha)}(x) (L_n^{(\alpha)})'(x) l_k^*(x)}{n [L_n^{(\alpha)}(y_k)]^2} \right| + \sum_{k=1}^{n-1} \left| \frac{y_k^{\nu-1} L_n^{(\alpha)}(x) (L_n^{(\alpha)})'(x) L_n^{(\alpha)}(0)}{2n [L_n^{(\alpha)}(y_k)]^3} \right| \\
&+ \sum_{k=1}^{n-1} \left| \frac{y_k^{\nu-1} (y_k^2 - 2(1+4n) y_k + 2\alpha + 4) L_n^{(\alpha)}(x) (L_n^{(\alpha)})'(x)}{8n [L_n^{(\alpha)}(y_k)]^2} \int_0^x l_k^*(t) dt \right| \\
&= I_{B1} + I_{B2} + I_{B3} + I_{B4} + I_{B5} (say).
\end{aligned}$$

Now, on using (2.7) for x and y_k and Lemma 4.4 and Lemma 4.2, we have

$$I_{B1} \ = \ O(\frac{1}{n}) \sum_{k=1}^{n-1} \left| e^{-\frac{3y_k}{2}} y_k^{\nu+\frac{3\alpha}{2}+\frac{11}{4}} \right| e^{\frac{3x}{2}} x^{-\frac{3\alpha}{2}-\frac{7}{4}} = O\left(n^{-1/2}\right) e^{\frac{3x}{2}} x^{-\frac{3\alpha}{2}-\frac{7}{4}}.$$

Again by using (2.7) for x and y_k, we have

$$I_{B2} = O(1) e^x x^{-\alpha-\frac{1}{2}} \sum_{k=1}^{n-1} \left| e^{-y_k} y_k^{\nu+\alpha+\frac{1}{2}} l_k^*(x) \right|.$$

On using Cauchy's inequality, Lemma 4.2 and Lemma 4.3 in above equation, we get

$$\begin{aligned} I_{B2} &= O(1)\left(\sum_{k=1}^{n-1}|e^{-2y_k}y_k^{2\nu+2\alpha+1}|\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n-1}|(l_k^*(x))^2|\right)^{\frac{1}{2}}e^x x^{-\alpha-\frac{1}{2}} \\ &= O(n^{\frac{1}{4}})e^x x^{-\alpha-\frac{1}{2}} \leq O(n^{\frac{1}{4}})e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{7}{4}}. \end{aligned} \tag{4.20}$$

Similarly,

$$I_{B3} = O(n^{-\frac{1}{4}})e^x x^{-\alpha-1} \leq O(n^{-\frac{1}{4}})e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{7}{4}}. \tag{4.21}$$

and

$$I_{B4} = O(n^{\frac{\alpha}{2}+\frac{1}{4}})e^x x^{-\alpha-1} \leq O(n^{\frac{\alpha}{2}+\frac{1}{4}})e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{7}{4}}. \tag{4.22}$$

On multiplying and dividing I_{B5} by $[L_n^{(\alpha)}(y_k)]^2$ and using (2.7) for x and y_k and Lemma 4.1, we get

$$\begin{aligned} I_{B5} &= O(n^{-\alpha})\sum_{k=1}^{n-1}\Big|e^{-2y_k}\left(y_k^{\nu+2\alpha+2}-2(1+4n)y_k^{\nu+2\alpha+1}+(2\alpha+4)y_k^{\nu+2\alpha}\right) \\ &\quad \left([L_n^{(\alpha)}(y_k)]^2\int_0^x l_k^*(t)dt\right)\Big|e^x x^{-\alpha-1}. \end{aligned}$$

Now, due to Lemma 4.6 and Lemma 4.2, we have

- for $a < \frac{\alpha}{2}-1$

$$I_{B5} \leq O(n^{-\frac{1}{2}})e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{7}{4}}. \tag{4.23}$$

- for $a = \frac{\alpha}{2}-1$, we have

$$I_{B5} = O(n^{-\frac{1}{2}}\ log\ n)e^x x^{-\alpha-1} \leq O(n^{-\frac{1}{2}}\ log\ n)e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{7}{4}} \tag{4.24}$$

- for $a > \frac{\alpha}{2}-1$, we have

$$I_{B5} = O(n^{2a-\alpha+\frac{3}{2}})e^x x^{-\alpha-1} \leq O(n^{2a-\alpha+\frac{3}{2}})e^{\frac{3x}{2}}x^{-\frac{3\alpha}{2}-\frac{7}{4}}. \tag{4.25}$$

Thus by (4.20), (4.20), (4.21), (4.22),(4.23), (4.24) and (4.25) the lemma follows. □

Lemma 4.12. *For y_k, $k = 1,\cdots,n-1$ and $\nu = \frac{r+\gamma}{2}$, we have*

$$e^{-x}x^{\alpha+1}\sum_{k=1}^{n-1}|y_k^{\nu-1}C_k(x)| = \begin{cases} O(n^{-\frac{3}{2}}), & a < \frac{\alpha}{2}-1 \\ O(n^{-\frac{3}{2}}\ log\ n), & a = \frac{\alpha}{2}-1 \\ O(n^{2a-\alpha+\frac{1}{2}}), & a > \frac{\alpha}{2}-1. \end{cases}$$

Proof. On multiplying and dividing (3.4) by $[L_n^{(\alpha)}(y_k)]^2$ and on using (1.5), (2.7) for x and y_k and Lemma 4.1, we get

$$\sum_{k=1}^{n-1}|y_k^{\nu-1}C_k(x)| = O(n^{-\alpha})\sum_{k=1}^{n-1}\left|e^{-\frac{3y_k}{2}}y_k^{\nu+\frac{3\alpha}{2}}\left([L_n^{(\alpha)}(y_k)]^2\int_0^x l_k^*(t)dt\right)\right|e^x x^{-\alpha-1}$$

which due to Lemma 4.2 and Lemma 4.6 gives

- for $a < \frac{\alpha}{2} - 1$

$$\sum_{k=1}^{n-1}|y_k^{\nu-1}C_k(x)| = O(n^{-\frac{3}{2}})e^x x^{-\alpha-1}. \tag{4.26}$$

- for $a = \frac{\alpha}{2} - 1$

$$\sum_{k=1}^{n-1}|y_k^{\nu-1}C_k(x)| = O(n^{-\frac{3}{2}}\ log\ n)e^x x^{-\alpha-1}. \tag{4.27}$$

- for $a > \frac{\alpha}{2} - 1$

$$\sum_{k=1}^{n-1}|y_k^{\nu-1}C_k(x)| = O(n^{2a-\alpha+\frac{1}{2}})e^x x^{-\alpha-1}. \tag{4.28}$$

Owing to (4.26), (4.27) and (4.28), the lemma follows. □

5 Proofs of Theorem 3.1

Proof. Let $G_{n+\alpha}(f)$ be the polynomials defined in Lemma 4.9 then

$$\begin{aligned}
&|f(x) - R_{n,\alpha}(f;x)| \le |f(x) - G_{n+\alpha}(f;x)| + |G_{n+\alpha}(f;x) - R_{n,\alpha}(f;x)| \\
&\le O(1)\omega\left(f^{(r)};\frac{\sqrt{x(x_n - x)}}{n}\right)\left(\frac{\sqrt{x(x_n-x)}}{n}\right)^r \\
&+\sum_{k=1}^{n}|f(x_k) - G_{n+\alpha}(f;x_k)|A_k(x)| + \sum_{k=1}^{n-1}|f(y_k) - G_{n+\alpha}(f;y_k)|B_k(x)| \\
&+\sum_{k=1}^{n-1}|f''(y_k) - G_{n+\alpha}(f'';y_k)|C_k(x)| + |f(0) - G_{n+\alpha}(f;0)|A_0(x)| \\
&\equiv I_1 + I_2 + I_3 + I_4 + I_4 + I_5.
\end{aligned}$$

Now, by the definition of modulus of continuity, we have

$$I_1 = O(1)\left(\frac{\sqrt{x(x_n-x)}}{n}\right)^{r+\gamma} = O(1)x^{\frac{(r+\gamma)}{2}}x_n^{\frac{(r+\gamma)}{2}}\left(1-\frac{x}{x_n}\right)^{\frac{(r+\gamma)}{2}} n^{-(r+\gamma)},$$

which due to (2.10) and taking $\frac{r+\gamma}{2}=\nu$, gives

$$I_1 = O(1)x^{\frac{(r+\gamma)}{2}}n^{-\frac{(r+\gamma)}{2}} = O(1)x^{\nu}n^{-\nu}. \tag{5.1}$$

On using Lemma 4.9, (2.10) and taking $\frac{r+\gamma}{2}=\nu$ for I_2, we get

$$\begin{aligned} I_2 &= \sum_{k=1}^{n}\left|O(1)\omega\left(f^{(r)};\frac{\sqrt{x_k(x_n-x_k)}}{n}\right)\left(\frac{\sqrt{x_k(x_n-x_k)}}{n}\right)^{r}A_k(x)\right| \\ &= O(n^{-\nu})\sum_{k=1}^{n}|x_k^{\nu}A_k(x)|, \end{aligned}$$

which, due to Lemma 4.10, gives

$$I_2 = \begin{cases} O(n^{-\nu-\frac{3}{2}})e^{\frac{x}{2}}x^{-\frac{\alpha}{2}-\frac{5}{4}}, & a < \frac{\alpha}{2}-\frac{5}{4} \\ O(n^{-\nu-\frac{3}{2}}\ log\ n)e^{\frac{x}{2}}x^{-\frac{\alpha}{2}-\frac{5}{4}}, & a = \frac{\alpha}{2}-\frac{5}{4} \\ O(n^{a-\nu-\frac{\alpha}{2}-\frac{1}{4}})e^{\frac{x}{2}}x^{-\frac{\alpha}{2}-\frac{5}{4}}, & a > \frac{\alpha}{2}-\frac{5}{4}. \end{cases} \tag{5.2}$$

Similarly, due to Lemma 4.9, (2.11) and Lemma 4.11, we have

$$I_3 = \begin{cases} O(n^{-\nu-\frac{1}{2}})e^{\frac{x}{2}}x^{-\frac{\alpha}{2}-\frac{7}{4}}, & a < \frac{\alpha}{2}-1 \\ O(n^{-\nu-\frac{1}{2}}\ log\ n)e^{\frac{x}{2}}x^{-\frac{\alpha}{2}-\frac{7}{4}}, & a = \frac{\alpha}{2}-1 \\ O(n^{2a-\nu-\alpha+\frac{3}{2}})e^{\frac{x}{2}}x^{-\frac{\alpha}{2}-\frac{7}{4}}, & a > \frac{\alpha}{2}-1. \end{cases} \tag{5.3}$$

and

$$I_4 = \begin{cases} O(n^{-\nu-\frac{1}{2}})e^{x}x^{-\alpha-1}, & a < \frac{\alpha}{2}-1 \\ O(n^{-\nu-\frac{1}{2}}\ log\ n)e^{x}x^{-\alpha-1}, & a = \frac{\alpha}{2}-1 \\ O(n^{2a-\nu-\alpha+\frac{3}{2}})e^{x}x^{-\alpha-1}, & a > \frac{\alpha}{2}-1. \end{cases} \tag{5.4}$$

Owing to Lemma 4.9,

$$I_5 = 0. \tag{5.5}$$

Thus by (5.1), (5.2), (5.3), (5.4) and (5.5) theorem follows. □

References

[1] Balázs, Katherine, Lagrange and Hermite Interpolation Processes on the Positive Real Line, Journal of Approximation Theory, 50(1987), 18-24.

[2] Joó, I., An Interpolation-Thoretical Characterization of the Classical Orthogonal polynomials, Acta Mathematica Academiae Scientiarum Hungaricae Tomus, 26 (1-2), 1975, 163-169.

[3] Joó, I., On Pál Interpolation, Annals Univ. Sci. Budapest, Sect. Math., 37(1994) 247-262.

[4] Lenard, Margit, Pál type Interpolation and Quardature formulae on Laguerre abscissas, Mathematica Pannonica, 15/2 (2004), 265-274

[5] Mathur, K.K. and Srivastava, R.; Weighted (0; 0,2)-interpolation on infinite interval, *Acta. Math. Hung., 70(1-2), 1996, 57-73.*

[6] Pál, L.G., A new Modification of Hermite- Fejér Interpolation, Analysis Math., (1975), 197-205.

[7] Srivastava R. and Mathur K.K., An Interpolation Process on the Roots of Hermite Polynomials (0;0,1) interpolation on Infinite Interval, Bull. Ins. Of Math Acad Sinica, vol 26, No. 3, Sept 1998.

[8] Szili, L.; A convergence theorem for the Pál- method of interpolation on the roots of Hermite polynomials, *Analysis Math., 11(1985), 75-84.*

[9] Szegö, G., Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ. 23, New York 1959.