On nearly Kenmotsu manifolds with semi-symmetric metric connection

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Abstract

The object of the present paper is to study nearly Kenmotsu manifolds with semi-symmetric metric connection. We have obtained the relations between curvature tensors, Ricci tensors and scalar curvatures of nearly Kenmotsu manifold with semi-symmetric metric connection. Also, we obtain some results on quasi-projectively flat, $\phi-$projectively flat and Weyl conformally flat manifolds with semi-symmetric metric connection. It is shown that the manifolds satisfying $\overline{R}.\overline{S} = 0$ and $\overline{P}.\overline{S} = 0$ are $\eta-$Einstein manifolds. Finally, we prove that $\phi-$Ricci symmetric nearly Kenmotsu manifold with semi-symmetric metric connection is a $\eta-$Einstein manifold.

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1 Introduction

The notion of Kenmotsu manifolds was defined by K. Kenmotsu as a class of almost contact Riemannian manifolds [13]. After that Kenmotsu manifolds were studied by several authors including Jun, De and Pathak [12], A. De [4], De and Sarkar [6] etc. On the other hand different types of manifold have been studied widely in Differential Geometry. Some of them are: Nearly Sasakian manifolds [16], On $\phi-$recurrent Kenmotsu manifold [7], On $\phi-$Ricci Symmetric Kenmotsu manifolds [18], Nearly trans-Sasakian manifolds [17], Einstein Manifolds [2], Further, Najafi and Kashani studied some properties of nearly Kenmotsu manifolds [15].

The semi-symmetric linear connection on a differentiable manifold was introduced by Friedman and Schouten [8] and metric connection with a torsion on a Riemannian manifold was studied by Hayden [10]. Further, semi-symmetric metric connection on a Riemannian manifold was studied by Yano [19] and later studied by Amur and Pujar [14], Bagewadi et. al., [1] De [5], Sharafuddin and Hussain [11] and others. In this paper, we study nearly Kenmotsu manifold with semi-symmetric metric connection. In section 2, we collect some basic informations about nearly Kenmotsu manifolds. In next sections, we find some importent results and theorems on nearly Kenmotsu manifolds with semi-symmetric metric connection.

2 Preliminaries

Let $(M^n, \phi, \xi, \eta, g)$ be an $n-$dimensional almost contact metric manifold [3], where ϕ is a $(1,1)$ tensor field, ξ is the structure vector field, η is a $1-$form and g is a Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \phi \circ \xi = 0, \eta \circ \phi = 0, \tag{2.1}$$

$$\eta(X) = g(X, \xi), \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), g(\phi X, Y) = -g(X, \phi Y), \tag{2.3}$$

for any vector fields X, Y on M^n and rank of ϕ is $n-1$. If

$$d\eta(X, Y) = g(X, \phi Y), \tag{2.4}$$

then $(M^n, \phi, \xi, \eta, g)$ is called almost contact metric manifold. An almost contact metric manifold is called nearly Kenmotsu manifold [17] if

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = -\eta(Y)X - \eta(X)Y, \tag{2.5}$$

for any vector fields $X, Y \in \Gamma(TM)$, where ∇ is the Riemannian connection of Riemannian metric g on M^n. It is easy to see that every Kenmotsu manifold is a nearly Kenmotsu manifold but converse is not true. A nearly Kenmotsu manifold is not a $K-$contact manifold and hence is not a Sasakian manifold.

Moreover, for a nearly Kenmotsu manifold we have

$$\nabla_X \xi = (X - \eta(X)\xi), \nabla_\xi \xi = 0, \tag{2.6}$$

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.7}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$R(\xi, Y)Z = \eta(Z)Y - g(Y, Z)\xi, \tag{2.9}$$

$$R(\xi, Y)\xi = -R(Y, \xi)\xi = X - \eta(X)\xi, \tag{2.10}$$

$$\eta(R(X, Y)Z) = [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \tag{2.11}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.12}$$

$$QX = -(n-1)X, \tag{2.13}$$

$$Q\xi = -(n-1)\xi, \tag{2.14}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.15}$$

$$'R(\phi X,Y,Z,W) +' R(X,\phi Y,Z,W) +' R(X,Y,\phi Z,W) +' R(X,Y,Z,\phi W) = 0, \tag{2.16}$$

$$'R(\xi,Y,Z,W) = \eta(Z)g(Y,W) - \eta(W)g(Y,Z), \tag{2.17}$$

$$'R(\phi X,\phi Y,Z,W) =' R(X,Y,\phi Z,\phi W), \tag{2.18}$$

$$\begin{aligned} 'R(\phi X,\phi Y,\phi Z,\phi W) &= \ 'R(X,Y,Z,W) - \eta(X)'R(\xi,Y,Z,W) \\ &\quad +\eta(Y)'R(\xi,X,Z,W), \end{aligned} \tag{2.19}$$

where $g(QX,Y) = S(X,Y)$, Q is a Ricci operator, S is Ricci tensor of type $(0,2)$, R is the Riemannian curvature tensor of type $(1,3)$ and $'R$ is a Riemannian curvature tensor of type $(0,4)$ defined by $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

A nearly Kenmotsu manifold M^n is said to be $\eta-$Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are smooth functions.

A connection $\overline{\nabla}$ in M^n is called semi-symmetric connection if its torsion tensor

$$T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y], \tag{2.20}$$

satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{2.21}$$

Further, a semi-symmetric connection is called semi-symmetric metric connection if

$$(\overline{\nabla}_X g)(Y,Z) = 0. \tag{2.22}$$

Let M^n be an $n-$dimensional nearly Kenmotsu manifold and ∇ be the Levi-Civita connection on M^n, the semi$-$symmetric metric connection $\overline{\nabla}$ on M is given by

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi. \tag{2.23}$$

Throughout the chapter we denote the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature by $\overline{R}, \overline{S}, \overline{Q}$ and $\overline{r}$ respectively of nearly Kenmotsu manifolds with respect to the semi-symmetric metric connection $\overline{\nabla}$. Similarly, we denote the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature by R, S, Q and r respectively of nearly Kenmotsu manifolds with respect to the symmetric metric connection ∇ (Levi-Civita connection).

Definition 2.1. *The Weyl conformal curvature tensor* $\overline{C}$ *of type* $(1,3)$ *of an* $n-$*dimensional nearly Kenmotsu manifold with semi symmetric metric connection* $\overline{\nabla}$ *is given by*

$$\begin{aligned}\overline{C}(X,Y)Z &= \overline{R}(X,Y)Z - \frac{1}{(n-2)}[\overline{S}(Y,Z)X \\ &\quad -\overline{S}(X,Z)Y + g(Y,Z)\overline{Q}X - g(X,Z)\overline{Q}Y] \\ &\quad +\frac{\overline{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{2.24}$$

Definition 2.2. *The projective curvature tensor* $\overline{P}$ *of type* $(1,3)$ *of an* $n-$*dimensional nearly Kenmotsu manifold with semi symmetric metric connection* $\overline{\nabla}$ *is given by*

$$\overline{P}(X,Y)Z = \overline{R}(X,Y)Z - \frac{1}{n-1}[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y]. \tag{2.25}$$

3 Curvature tensor on nearly Kenmotsu manifold with semi-symmetric metric connection

Let M^n be an $n-$dimensional nearly Kenmotsu manifold. The curvature tensor $\overline{R}$ of M^n with respect to the semi-symmetric metric connection $\overline{\nabla}$ is defined by

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z. \tag{3.1}$$

By using equations $(2.2), (2.23)$ and (3.1), we get

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z + 3[g(X,Z)Y - g(Y,Z)X] + 2[g(Y,Z)\eta(X) \\ &\quad -g(X,Z)\eta(Y)]\xi + 2[\eta(Y)X - \eta(X)Y]\eta(Z),\end{aligned} \tag{3.2}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

is the Riemannian curvature tensor of metric connection ∇.

Taking inner product of equation (3.2) with W and using equation (2.2), we get

$$\begin{aligned}& g(\overline{R}(X,Y)Z,W) \\ &= g(R(X,Y)Z,W) + 3[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] + 2[g(Y,Z)\eta(X) \\ &\quad -g(X,Z)\eta(Y)]\eta(W) + 2[\eta(Y)g(X,W) - \eta(X)g(Y,W)]\eta(Z).\end{aligned} \tag{3.3}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields on nearly Kenmotsu manifold M^n. Now putting $X = W = e_i$ in the equation (3.3) and taking summation over i, $1 \leq i \leq n$, we get

$$\overline{S}(Y,Z) = S(Y,Z) + (5-3n)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z). \tag{3.4}$$

Moreover, on a nearly Kenmotsu manifold with semi-symmetric metric connection, the following equations hold:

$$\overline{S}(Y,\xi) = -2(n-1)\eta(Y), \tag{3.5}$$

$$\overline{R}(X,Y)\xi = 2[\eta(X)Y - \eta(Y)X], \tag{3.6}$$

$$\overline{R}(\xi,Y)Z = 2[\eta(Z)Y - g(Y,Z)\xi], \tag{3.7}$$

$$\overline{R}(\xi,Y)\xi = 2[Y - \eta(Y)\xi], \tag{3.8}$$

$$\overline{R}(X,Y)Z = -\overline{R}(Y,X)Z, \overline{R}(\xi,Y)\xi = -\overline{R}(Y,\xi)\xi, \tag{3.9}$$

$$\overline{Q}Y = -2(n-1)Y, \overline{Q}\xi = -2(n-1)\xi, \tag{3.10}$$

$$\overline{S}(\phi Y,\phi Z) = S(Y,Z) + (5-3n)g(Y,Z) + 2(2n-3)\eta(Y)\eta(Z), \tag{3.11}$$

$$\overline{r} = r - 3n^2 + 7n - 4, \tag{3.12}$$

$$\overline{\nabla}_X\xi = 2(X - \eta(X)\xi), \tag{3.13}$$

$$\eta(\overline{R}(X,Y)Z) = 2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)], \tag{3.14}$$

From equation (3.2), we get

$$\begin{aligned}\overline{R}(Y,Z)X &= R(Y,Z)X + 3[g(Y,X)Z - g(Z,X)Y] + 2[g(Z,X)\eta(Y) \\ &\quad -g(Y,X)\eta(Z)]\xi + 2[\eta(Z)Y - \eta(Y)Z]\eta(X),\end{aligned} \tag{3.15}$$

$$\begin{aligned}\overline{R}(Z,X)Y &= R(Z,X)Y + 3[g(Z,Y)X - g(X,Y)Z] + 2[g(X,Y)\eta(Z) \\ &\quad -g(Z,Y)\eta(X)]\xi + 2[\eta(X)Z - \eta(Z)X]\eta(Y),\end{aligned} \tag{3.16}$$

From equations $(3.2), (3.15)$ and (3.16), we get

$$\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = 0, \tag{3.17}$$

Hence, we can state the following theorem:

Theorem 3.1. *If M^n be an $n-$dimensional nearly Kenmotsu manifold with semi symmetric metric connection $\overline{\nabla}$, then curvature tensor $\overline{R}$ of it satisfies Bianchi first identity.*

Taking inner product of equation (3.2) with W, we have

$$\begin{aligned} (3.18) \quad & {}'\overline{R}(X,Y,Z,W) \\ = \ & {}'R(X,Y,Z,W) + 3[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \\ & +2[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\eta(W) + 2[\eta(Y)g(X,W) - \eta(X)g(Y,W)]\eta(Z). \end{aligned}$$

From equation (3.18), we get

$$\begin{aligned} {}'\overline{R}(X,Y,Z,W) &= -{}'\overline{R}(Y,X,Z,W), \\ {}'\overline{R}(X,Y,Z,W) &= -{}'\overline{R}(X,Y,W,Z), \\ {}'\overline{R}(X,Y,Z,W) &= {}'\overline{R}(Z,W,X,Y), \end{aligned} \tag{3.19}$$

Hence, we can state the following theorem:

Theorem 3.2. *The curvature tensor* ${}'\overline{R}$ *of a nearly Kenmotsu manifold with semi-symmetric metric connection* $\overline{\nabla}$ *satisfies the following:*

(i) It is skew symmetric in first two slots,
(ii) It is skew symmetric in last two slots,
(iii) It is symmetric in pair of slots.

Theorem 3.3. *Let* $(M^n, \phi, \xi, \eta, g)$ *be a nearly Kenmotsu manifold with semi-symmetric metric connection. Then the following curvature relations hold:*

$$'\overline{R}(\phi X,Y,Z,W) +' \overline{R}(X,\phi Y,Z,W) +' \overline{R}(X,Y,\phi Z,W) +' \overline{R}(X,Y,Z,\phi W) = 0, \tag{3.20}$$

$$'\overline{R}(\xi,Y,Z,W) = 2[\eta(Z)g(Y,W) - \eta(W)g(Y,Z)], \tag{3.21}$$

$$'\overline{R}(\phi X,\phi Y,Z,W) =' \overline{R}(X,Y,\phi Z,\phi W), \tag{3.22}$$

$$\begin{aligned} (3.23) \quad & {}'\overline{R}(\phi X,\phi Y,\phi Z,\phi W) \\ = \ & {}'R(X,Y,Z,W) - 4\eta(X)'R(\xi,Y,Z,W) + 4\eta(Y)'R(\xi,X,Z,W) \\ & +3[g(X,Z)g(Y,Z) - g(X,W)g(Y,Z)]. \end{aligned}$$

4 Quasi-Projectively flat nearly Kenmotsu manifold with semi-symmetric metric connection

A nearly Kenmotsu manifold M^n is said to be quasi-projectively flat with semi-symmetric metric connection if

$$g(\overline{P}(X,\phi Y)\phi Z,W) = 0, \tag{4.1}$$

where $\overline{P}$ is the projectively curvature tensor with respect to semi-symmetric metric connection.

Taking inner product of equation (2.25) with W and using equations (3.2), (3.4) and (4.1), we get

$$\begin{aligned} & g(R(X,\phi Y)\phi Z,W)+3[g(X,\phi Z)g(\phi Y,W)-g(\phi Y,\phi Z)g(X,W)] \\ & +2\eta(X)\eta(W)g(\phi Y,\phi Z) \\ = \; & \frac{1}{n-1}[S(\phi Y,\phi Z)g(X,W)+(5-3n)g(\phi Y,\phi Z)g(X,W) \\ & -S(X,\phi Z)g(\phi Y,W)-(5-3n)g(X,\phi Z)g(\phi Y,W)]. \end{aligned} \tag{4.2}$$

Let $\{e_1, e_2, \ldots.., e_{n-1}, \xi\}$ be local orthogonal basis of vector fields in M^n then $\{\phi e_1, \phi e_1, \ldots...., \phi e_{n-1}, \xi\}$ is also local orthonormal basis of M^n. Putting the value $Y = Z = e_i$ in the equation (4.2) and taking summation over i, $1 \leq i \leq n$, we get

$$S(X,W) = (\frac{r}{n} + \frac{2(n-2)}{n})g(X,W) - 2(n-1)\eta(X)\eta(W).$$

Hence, we have

Theorem 4.1. *An $n-$dimensional quasi projectively flat nearly Kenmotsu manifold M^n with respect to a semi-symmetric metric connection is a $\eta-$Einstein manifold.*

5 $\phi-$Projectively flat nearly Kenmotsu manifold with respect semi-symmetric metric connection

A nearly Kenmotsu manifold with respect to semi-symmetric metric connection is said to be $\phi-$projectively flat if

$$\phi^2(\overline{P}(\phi X, \phi Y)\phi Z) = 0, \tag{5.1}$$

where $\overline{P}$ is the projective curvature tensor of the manifold M^n with respect to semi-symmetric metric connection.

Let M^n be a $\phi-$projectively flat nearly Kenmotsu manifold with respect to semi-symmetric metric connection it is easy to see that

$$\phi^2(\overline{P}(\phi X, \phi Y)\phi Z) = 0,$$

holds iff

$$g(\overline{P}(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{5.2}$$

for any $X, Y, Z, W \in \Gamma(TM)$.

Taking inner product of equation (2.25) with W and using equations (3.2), (3.11) and (5.2), we get

$$(5.3) \qquad g(R(\phi X,\phi Y)\phi Z,\phi W)+3[g(\phi X,\phi Z)g(\phi Y,\phi W)-g(\phi Y,\phi Z)g(\phi X,\phi W)]$$
$$= \frac{1}{n-1}[S(\phi Y,\phi Z)g(\phi X,\phi W)+(5-3n)g(\phi Y,\phi Z)g(\phi X,\phi W)$$
$$-S(\phi X,\phi Z)g(\phi Y,\phi W)-(5-3n)g(\phi X,\phi Z)g(\phi Y,\phi W)].$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be local orthogonal basis of vector fields in M^n then $\{\phi e_1, \phi e_1, ..., \phi e_{n-1}, \xi\}$ is also local orthonormal basis of M^n. Putting the value $Y = Z = e_i$ in the equation (5.3) and taking summation over i, $1 \leq i \leq n$, we get

$$S(\phi X,\phi W)=\frac{1}{n}(r+2(n-2))g(\phi X,\phi W).$$

Using equations (2.3) and (2.15), we get

$$S(X,W)=\frac{1}{n}(r+2(n-2))g(X,W)-\frac{1}{n}[r+n^2+n-4]\eta(X)\eta(W).$$

Hence, we can state the following theorem:

Theorem 5.1. *An $n-$dimensional $\phi-$projectively flat nearly Kenmotsu manifold M^n with respect to a semi-symmetric metric connection is an $\eta-$Einstein manifold.*

6 Weyl conformal curvature tensor on nearly Kenmotsu manifold with semi-symmetric metric connection

Let us consider the nearly Kenmotsu manifold M^n with respect to the semi-symmetric metric connection is Weyl conformally flat, that is $\overline{C} = 0$. Then, from equation (2.24), we get

$$(6.1) \qquad \overline{R}(X,Y)Z$$
$$= \frac{1}{(n-2)}[\overline{S}(Y,Z)X-\overline{S}(X,Z)Y+g(Y,Z)\overline{Q}X-g(X,Z)\overline{Q}Y]$$
$$-\frac{\overline{r}}{(n-1)(n-2)}[g(Y,Z)X-g(X,Z)Y].$$

Now, taking the inner product of equation (6.1) with W and using equations (2.2), (3.2), (3.4) and (3.10), we get

$$
\begin{aligned}
(6.2) \quad & g(R(X,Y)Z,W) + 3[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] + 2[g(Y,Z)\eta(X)\eta(W) \\
& -g(X,Z)\eta(X)\eta(W)] + 2[\eta(Y)\eta(Z)g(X,W) - \eta(X)\eta(Z)g(Y,W)] \\
= \ & \frac{1}{n-2}[S(Y,Z)g(X,W) + (5-3n)g(Y,Z)g(X,W) + 2(n-2)\eta(Y)\eta(Z)g(X,W) \\
& -S(X,Z)g(Y,W) - (5-3n)g(X,Z)g(Y,W) - 2(n-2)\eta(X)\eta(Z)g(Y,W) \\
& +S(X,W)g(Y,Z) + (5-3n)g(Y,Z)g(X,W) + 2(n-2)\eta(X)\eta(W)g(X,W) \\
& -S(Y,W)g(X,Z) - (5-3n)g(X,Z)g(Y,W) - 2(n-2)\eta(Y)\eta(W)g(X,Z)] \\
& -\frac{r-3n^2+7n-4}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].
\end{aligned}
$$

Putting the value $X = W = \xi$ in equation (6.2) and using equations $(2.2), (2.9)$ and (2.12), we get

$$S(Y,Z) = (1 + \frac{r}{n})g(Y,Z) - (n + \frac{r}{n-1})\eta(Y)\eta(Z).$$

Now, we can state the following theorem:

Theorem 6.1. *Let M^n be an $n-$dimensional Weyl conformally flat nearly Kenmotsu manifold with respect to a semi-symmetric metric connection is an $\eta-$Einstein manifold.*

7 On nearly Kenmotsu manifold with semi-symmetric metric connection satisfies $\overline{R}.\overline{S} = 0$.

An $n-$dimensional nearly Kenmotsu manifold with semi-symmetric metric connection $\overline{\nabla}$ satisfying the conditions

$$\overline{R}(X,Y).\overline{S} = 0. \tag{7.1}$$

From equation (7.1), we have

$$\overline{S}(\overline{R}(X,Y)Z,W) + \overline{S}(Z,\overline{R}(X,Y)W) = 0.$$

Putting the value $X = W = \xi$ in above equation we get

$$\overline{S}(\overline{R}(\xi,Y)Z,\xi) + \overline{S}(Z,\overline{R}(\xi,Y)\xi) = 0. \tag{7.2}$$

Using the equations $(2.2), (2.9), (2.10), (2.13), (3.2)$ and (3.4), we get

$$S(Y,Z) = (n-3)g(Y,Z) - 2(n-2)\eta(Y)\eta(Z).$$

Hence, we can state the following theorem:

Theorem 7.1. *Let M^n be an $n-$dimensional nearly Kenmotsu manifold with respect to a semi-symmetric metric connection satisfies the condition $\overline{R}.\overline{S} = 0$ is an $\eta-$Einstein manifold.*

8 On nearly Kenmotsu manifold with semi-symmetric metric connection satisfies $\overline{P}.\overline{S} = 0$.

An $n-$dimensional nearly Kenmotsu manifold with semi-symmetric metric connection $\overline{\nabla}$ satisfying the conditions

$$(\overline{P}(X,Y).\overline{S})(Z,U) = 0. \tag{8.1}$$

Then, we have

$$\overline{S}(\overline{P}(X,Y)Z,U) + \overline{S}(Z,\overline{P}(X,Y)U) = 0.$$

Putting the value $X = W = \xi$ in above equation we get

$$\overline{S}(\overline{P}(\xi,Y)Z,\xi) + \overline{S}(Z,\overline{P}(\xi,Y)\xi) = 0. \tag{8.2}$$

Using equations $(2.2), (2.12), (2.25), (3.4)$ and (8.2), we get

$$S(Y,Z) = 2(n-2)g(Y,Z) - 2(n-2)\eta(Y)\eta(Z).$$

Now, we can state the following theorem:

Theorem 8.1. *If nearly Kenmotsu manifold with semi-symmetric metric connection $\overline{\nabla}$ satisfies $\overline{P}.\overline{S} = 0$, then the nearly Kenmotsu manifold is an $\eta-$Einstein manifold.*

9 $\phi-$Ricci symmetric nearly Kenmotsu manifold

A nearly Kenmotsu manifold with semi-symmetric metric connection M^n is said to be locally $\phi-$symmetric, if

$$\phi^2((\overline{\nabla}_X\overline{Q})(Y)) = 0, \tag{9.1}$$

for any vector field X, Y, Z, W orthogonal to ξ.

Let us consider that the manifold is ϕ-Ricci symmetric. Then we have

$$\phi^2((\overline{\nabla}_X\overline{Q})(Y)) = 0.$$

Using equation (2.1) and taking inner product with Z, we get

$$-g((\overline{\nabla}_X\overline{Q})(Y),Z) + \eta((\overline{\nabla}_X\overline{Q})(Y))\eta(Z) = 0.$$

By using equation (3.12) and $Y = \xi$, we have

$$-g(\overline{\nabla}_X\overline{Q}\xi,Z) + \overline{S}(\overline{\nabla}_X\xi,Z) + \eta((\overline{\nabla}_X\overline{Q})(\xi))\eta(Z) = 0. \tag{9.2}$$

Next, using equations $(2.2), (3.10)$ and (9.2), we get

$$2\overline{S}(X,Z) + 4(n-1)g(X,Z) - 4(n-1)\eta(X)\eta(Z) - 2\eta(X)\overline{S}(\xi,Z) + \eta((\overline{\nabla}_X\overline{Q})(\xi))\eta(Z) = 0.$$

Putting the value $X = \phi X, Z = \phi Z$ and using equation (2.15), we get

$$2(n-1)g(\phi X, \phi Z) + 2\overline{S}(\phi X, \phi Z) = 0.$$

From equations (2.2) and (3.11), we get

$$S(X, Z) = (n-3)g(X, Z) - 2(n-2)\eta(Y)\eta(Z).$$

Now, we can state the following theorem:

Theorem 9.1. *An $n-$dimensional $\phi-$Ricci symmetric nearly Kenmotsu manifold with semi-symmetric metric connection is an $\eta-$Einstein manifold.*

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