CR-Submanifolds of HGF- Structure Metric Manifolds

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Abstract

The study of the differential geometry of CR-submanifolds, as a generalization of invariant and anti-invariant submanifolds ,of a Kaehler manifold was initiated by Bejancu[1] and was followed by several geometers[2,3]. In the present paper we define and study CR submanifolds of HGF structure metric manifolds[4].

1 Introduction

1.1 Definition:-

HGF metric structure manifolds: Let V_n be a differentiable manifold. Let there exists a tensor J of type (1,1) such that

(1.1) $$J^2X = -a^2X$$

For arbitrary vector field X and 'a' is a complex number. Then J is said to be a Hyperbolic differentiable structure, briefly known as HGF$-$structure and then V_n is called HGF manifold.

Let the HGF$-$structure be endowed with the Hermitian metric g, such that

(1.2) $$g(JX, JY) = a^2g(X, Y)$$

Then (J, g) is said to be a Hyperbolic differentiable metric structure, briefly known as HGF$-$ metric structure and V_n is called Hyperbolic differentiable metric structure manifold or HGF$-$ metric structure manifold.

The HGF$-$structure gives different structures for different values of a.

(i) If $a \neq 0$,it is Hyperbolic $\pi-$structure.

(ii) If $a = \pm 1$,it is an almost complex or a hyperbolic almost product structure.

(iii) If $a = \pm i$,it is an almost product or a hyperbolic almost complex structure.

(iv) If $a = 0$,it is an almost tangent or a hyperbolic almost tangent structure.

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2 CR submanifolds of HGF structure metric manifolds

2.1 Definition:

CR submanifolds of HGF structure metric manifolds:-

Let M be a submanifold of V_n then M is said to be CR submanifold of V_n if there exists a differentiable distribution.

$$D : x \to D_x \subset T_x(M)$$

On M satisfying the following conditions:-

(i). D is holomorphic, i.e. $J(D_x) = D_x$ for each $x \in M$.

(ii). The complementary orthogonal distribution , $D^{\perp} : x \to D_x^{\perp} \subset T_x M$ is anti-invariant, i.e. $J(D_x^{\perp}) \subset T_x M^{\perp}$, for each $x \in M$.

For each vector field X tangent to M, we put

$$JX = RX + SX \tag{2.1}$$

where RX and SX are respectively the tangent part and normal part of JX.

For each vector field N Normal to M, we put

$$JN = BN + CN \tag{2.2}$$

where BN and CN are respectively the tangent part and normal part of JN.

From the definition of CR-submanifolds, we have

$$TM = D \oplus D^{\perp}$$

For each $X \in T(M)$,we put

$$X = PX + QX \tag{2.3}$$

where $PX \in D$ and $QX \in D^{\perp}$

Theorem 2.1:- The submanifold M of V_n is a CR- submanifolds if and only if, we have

$$rankR = constant \tag{2.4}$$

and

$$SoR = 0 \tag{2.5}$$

Proof:- Operating (2.3) by J and using (2.1) then equating tangential and normal parts, we get

$$JX = JPX + JQX$$

$$RX + SX = JPX + JQX$$

(2.6) $$RX = JPX$$

And

(2.7) $$SX = JQX$$

for any X tangent to M.
From (2.6),we have $rankR = rankJP = 2dimP = constant$
From (2.6) and (2.7),we get (2.5).
Conversely, suppose (2.4) and (2.5) are satisfied.
Now, we define distribution D by
$D_x = ImR_x$ for each $x \in M$.
Clearly, D is an invariant distribution, since for each $X = RY \in \Gamma D$, we have

$$JX = JRY = R(RY) + S(RY) = R^2Y \in \Gamma D$$

Let $D^{\perp}$denote the complementary orthogonal distribution to D in TM. Then $D^{\perp}$is an anti invariant distribution.
For any $X \in D^{\perp}$,and
$Y = U + W$, where $U \in \Gamma D$ and $W \in \Gamma D^{\perp}$, we have

$$\begin{aligned} g(JX, Y) &= -g(X, JY) = -g(X, JU + JW) = -g(X, JW) = -g(X, RW + SW) = \\ -g(X, RW) &= 0, \end{aligned}$$

since $RW \in D$. Thus. M is a CR- Submanifold of V_n. This complete the proof.
Theorem 2.2:- The submanifold M of V_n is a CR- submanifolds if and only if, we have

(2.8) $$rankB = constant$$

and

(2.9) $$RoB = 0$$

Proof:- Suppose M is CR- submanifold. First we show that
$ImB_x \subset Dx^{\perp}$ for each $x \in M$.

$g(BN,Y) = g(JN,Y) = -g(N,JY) = 0$,for any $Y \in \Gamma D^{\perp}$ and $N \in \Gamma TM^{\perp}$.
Now to show that $D_x^{\perp} \subset ImB_x$.
If we take $U \in D_x^{\perp}$ then $JU \in T_xM^{\perp}$ and we obtain

$$-a^2U = J^2U = J(JU) = B(JU) + C(JU)$$

Equating tangential part
$a^2U = -B(JU) \in ImB_x$.Thus $D^{\perp} = ImB$, which implies $rankB = constant$.
Hence (2.8) is proven.
For each $N \in \Gamma TM^{\perp}$ we have

$$(JoB)N = (RoB)N + (SoB)N$$

Equating tangential and normal part, we have

$$(RoB)N = 0, (JoB)N = (SoB)N$$

Hence(2.9) is proven.
Conversely, Suppose (2.8) and (2.9) are satisfied. Then we define the distribution $D^{\perp}$ by $D_x^{\perp} = ImB_x$. First we see that $D^{\perp}$ is an anti-invariant distribution.
For each $X \in D^{\perp}$ and $Y \in TM$, We have
$g(JX,Y) = 0$
$g((JoB)N,Y) = g((SoB)N,Y) = 0$.
Next, the complimentary orthogonal distribution D to $D^{\perp}$ in TM is an invariant distribution. Indeed, for each $X \in \Gamma D$, $Y \in \Gamma D^{\perp}$ and $N \in \Gamma TM^{\perp}$, we have
$g(JX,Y) = -g(X,JY) = 0$,since $JY \in \Gamma TM^{\perp}$
and
$g(JX,N) = -g(X,JN) = -g(X,BN) = 0$, since $BN \in D^{\perp}$
Thus M is CR- submanifolds.
Theorem 2.3:- On each CR- submanifold M, the vector bundle morphisms R and Cdefine (f,a^2) structures on TM and $TM^{\perp}$ respectively.
Proof:- Operating (2.6) by J
$JRX = J^2PX$
Using (1.1) and (2.1) in above equation, we get
$R^2X + SRX = -a^2PX$

(2.10) $$R^2X = -a^2PX$$

Again operating above equation by J
$JR^2X = -a^2JPX$, Now using (2.1) and (2.6) ,we have
$R^3X = -a^2RX$

(2.11) $$R^3 + a^2R = 0.$$

Operating (2.2) by J

$J^2N = JBN + JCN$
$-a^2N = RBN + SBN + BCN + C^2N$
$-a^2N = SBN + C^2N$
Again operating above equation by J and using (2.2), then we get
$-a^2(BN + CN) = JSBN + BC^2N + C^3N$
Equating tangential and normal part, we get

$$C^3 + a^2C = 0. \tag{2.12}$$

This completes the proof of above theorem.

3 Integrability of distribution on a CR-submanifolds

In this section of this paper we study the integrability of both the distribution D and $D^{\perp}$on M.

The Nijenhuis tensor field of R is given by

$$[R, R](X, Y) = [RX, RY] + R^2[X, Y] - R([RX, Y] + [X, RY]) \tag{3.1}$$

the Nijenhuis Tensor field of J is given by
$[J, J](X, Y) = [JX, JY] + J^2[X, Y] - J([JX, Y] + [X, JY])$

using (2.1) and (2.3) we get ,

$$[J, J](X, Y) = [R, R](X, Y) - a^2Q[X, Y] - S([X, RY] + [RX, Y]) \tag{3.2}$$

for any $X, Y \in \Gamma D$

Theorem 3.1 :- Let M be a CR-submanifold of a HGF structure metric manifold V_n then the distribution D is integrable if and only if

$$[J, J](X, Y)^{\top} = [R, R](X, Y) \tag{3.3}$$

for any $X, Y \in \Gamma D$
where superscript T denote the tangential part of Nijenhuis tensor J
Proof :- Proof of the theorem follows easily if
Taking normal part in (3.2) , we have

$$[J, J](X, Y)^{\perp} = -S([X, RY] + [RX, Y]) \tag{3.4}$$

for any $X, Y \in \Gamma D$

Theorem 3.2 :- Let M be a CR –submanifold of a HGF structure metric manifold V_n. Then the distribution D is integrable if and only if

$$[J,J]\,(X,Y)^{\perp}=0 \tag{3.5}$$

and

$$Q\,[R,R]\,(X,Y)=0 \tag{3.6}$$

for any $X,Y\in\Gamma D$

Proof:- Suppose D is integrable then (3.5) follows from (3.4)

In view of (3.1) and (2.10), we have

$[R,R]\,(X,Y)=[RX,RY]-a^2P\,[X,Y]-R\,([RX,Y]+[X,RY])$
for any $X,Y\in\Gamma D$

Taking into account that $D=ImR$, we have
$[R,R]\,(X,Y)\in\Gamma D$
which is equivalent to (3.6)
Conversely, suppose (3.5) and (3.6) are satisfied, then from (3.4) and (3.5) ,we have
$Q\,([JX,Y]+[X,JY])=0$
Which implies (By Replacing Y by JY)

$Q\left([JX,JY]+\left[X,J^2Y\right]\right)=0$
$Q\left([JX,JY]-a^2\,[X,Y]\right)=0$
Hence $Q\,[J,J]\,(X,Y)^{\perp}=0$
From (3.2) , we have
$[J,J]\,(X,Y)^{\perp}=[R,R]\,(X,Y)-a^2Q\,[X,Y]$
$Q\,[J,J]\,(X,Y)^{\perp}=Q\,[R,R]\,(X,Y)-a^2Q^2\,[X,Y]$
$0=0-a^2Q^2\,[X,Y]$
$a^2Q^2\,[X,Y]=0$
$Q\,[X,Y]=0\Rightarrow D$ is integrable.

Theorem 3.3 :- Let M be a CR-submanifold of an integrable HGF metric manifold V_n .The distribution D is integrable if and only if the Nijenhuis tensor of R vanishes identically on D.

Proof :- Let M be a CR– submanifold of an integrable HGF metric manifold V_n . Therefore
$[J,J]\,(X,Y)=0$
The distribution D is integrable implies
$[R,R]\,(X,Y)=0$
For Converse part, we have from (3.3)

$[J, J](X, Y)^{\perp} = R[X, Y]$
this implies $[R, R](X, Y) = 0$
This completes the proof of above theorem.

Theorem 3.4 :- Let M be a CR- submanifold of a HGF structure metric manifold V_n. The distribution $D^{\perp}$is integrable iff the Nijenhuis tensor of R vanishes identically on $D^{\perp}$.

Proof :- If ,we take $X, Y \in \Gamma D$ and from(3.1)
$[R, R](X, Y) = R^2[X, Y] = -a^2 P[X, Y]$
$D^{\perp}$ is intergable iff $P[X, Y] = 0$ for any $X, Y \in \Gamma D^{\perp}$
This Completes the proof of above theorem.

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