Some Applications of Generalized Extended Fractional Derivative Operator

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Abstract

We have derived some extended fractional derivatives of certain elementary functions, extended Generalized Gauss hypergeometric functions, extended Appell's hypergeometric functions and Generalized Lauricella hypergeometric functions in one, two and more variables containing extra parameters.For the sake of clarity and easy readability, we may first study the properties of extended Appell's hypergeometric functions and then we can view the extended Lauricella's hupergeometric functions as a further generalization of Appell functions.

Keywords: Extended gamma and extended beta functions; extended gauss hypergeometric functions; extended Appell's hypergeometric functions and Generalized Lauricella hypergeometric functions; Fractional derivative operator.

1 Introduction, Definitions and Preliminaries

In 1997, M.A. Chaudhry, A. Qadir, M. Rafique and S.M. Zubair [3] presented the following extension of Euler's beta function

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(-\frac{p}{t(1-t)}\right) dt, \quad \Re(p) > 0\,. \tag{1.1}$$

Afterwards, M.A. Chaudhry and S.M. Zubair [2] used $Bp(x,y)$ to extended Gauss hypergeometric function and Kummer confluent hypergeometric function as follows

$$F_p(a,b;c;z) = \sum_0^\infty (a)_m \frac{B_p(b+m,c-b)}{B(b,c-b)} \frac{z^m}{m!}, \tag{1.2}$$
$$(p \geq 0, |z| < 1; \Re(c) > \Re(b) > 0)$$

$$\Phi_p(b;c;z) = \sum_0^\infty \frac{B_p(b+m,c-b)}{B(b,c-b)} \frac{z^m}{m!}, \tag{1.3}$$
$$(p \geq 0, |z| < 1; \Re(c) > \Re(b) > 0)$$

In 2011, E. Ozergin, M. Ali Ozarslan and A. Altin [6] introduced the following generalizations

$$\Gamma_p^{(\alpha,\beta)}(x) = \int_0^\infty t^{x-1}{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)dt, \tag{1.4}$$
$$(\Re(p)\geq 0, \Re(x)>0, \Re(\alpha)>\Re(\beta)>0)$$

$$B_p^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt, \tag{1.5}$$
$$(\Re(p)\geq 0, \Re(x)>0, \Re(y)>0, \Re(\alpha)>\Re(\beta)>0)$$

In this paper, our extensions mainly based on the following generalization of gamma and beta functions:

Definition 1.1 ([7, p.243]). *Let a function $\Theta(\{k_l\}_{l\in N_0};z)$ be analytic within the disk $|z|<R$ $(0<R<\infty)$ and let its Taylor-Maclaurin coefficients be explicitly denoted by the sequence $\{k_l\}_{l\in N_0}$. Suppose also that the function $\Theta(\{k_l\}_{l\in N_0};z)$ can be continued analytically in the right half-plane $\Re(z)>0$ with the asymptotic property given as follows:*

$$\Theta(k_l;z) \equiv \Theta(\{k_l\}_{l\in N_0};z) = \begin{cases} \sum\limits_{l=0}^{\infty} k_l \frac{z^l}{l!} & (|z|<R; 0<R<\infty; k_0=1) \\ M_0 z^\omega \exp(z)[1+O\left(\frac{1}{z}\right)] & (\Re(z)\to\infty; M_0>0; \omega\in C) \end{cases} \tag{1.6}$$

for some suitable constants M_0 and ω depending essentially on the sequence $\{k_l\}_{l\in N_0}$. We can define extended Gamma function $\Gamma_p^{(k_l)}(z)$ and the extended Beta function

$$\Gamma_p^{(k_l)}(z) = \int_0^\infty t^{z-1}\Theta\left(\{k_l\};-t-\frac{p}{t}\right)dt, \tag{1.7}$$
$$(\Re(p)\geq 0, \Re(z)>0,)$$

$$B_p^{(k_l)}(\alpha,\beta;p) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Theta\left(\{k_l\};-\frac{p}{t(1-t)}\right)dt, \tag{1.8}$$
$$(\Re(p)\geq 0, \min\{\Re(\alpha),\Re(\beta)\}>0)$$

By introducing one additional parameter q with $\Re(q)\geq 0$, we have

$$B_{p,q}^{\{k_l\}}(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Theta\left(\{k_l\};-\frac{p}{t}-\frac{q}{(1-t)}\right)dt, \tag{1.9}$$
$$(\min\{\Re(p),\Re(q)\}>0; \min\{\Re(\alpha),\Re(\beta)\}>0)$$

Definition 1.2 ([7]). *The Extended Gauss hypergeometric function ${}_2F_1{}^{(k_l)}$ is defined by*

$${}_2F_1^{\{k_l\}}\left[\begin{array}{c} a,b \\ c \end{array};z;p,q\right] = \sum_{n=0}^{\infty}(a)_n \frac{B_{p,q}^{\{k_l\}}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \tag{1.10}$$
$$(|z|<1; \min\{\Re(p),\Re(q)\}\geq 0; \Re(c),\Re(b)>0)$$

If $\Theta(k_l; z) = \exp z,$ *we write*

$$\text{(1.11)} \qquad {}_2F_1\left[\begin{array}{c} a, b \\ c \end{array}; z; p, q\right] = \sum_{n=0}^{\infty} (a)_n \frac{B_{p,q}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}$$

For the extended Gauss hypergeometric function ${}_2F_1^{(k_l)}\left[\begin{array}{c} a, b \\ c \end{array}; z; p, q\right]$, *we have the following integral representation:*

$$\text{(1.12)} \qquad {}_2F_1^{\{k_l\}}\left[\begin{array}{c} a, b \\ c \end{array}; z; p, q\right] = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \Theta\left(\{k_l\}; -\frac{p}{t} - \frac{q}{(1-t)}\right) dt,$$

$$(\Re(c) > \Re(b) > 0, \Re(p), \Re(q) \geq 0; p = q = 0\,|\arg(1-z) < \pi|)$$

Definition 1.3 ([7])**.** *The extended Appell's hypergeometric functions of two variables and extended Lauricella's hypergeometric function of three variables are as follows:*

$$\text{(1.13)} \qquad F_1(a, b, c; d; x, y; p) = \sum_{n,m=0}^{\infty} \frac{B_p(a+m+n, d-a)}{B(a, d-a)} (b)_n (c)_m \frac{x^n}{n!} \frac{y^m}{m!};$$

$$(\max\{|x|, |y|\} < 1)$$

$$\text{(1.14)} \qquad F_2(a, b, c; d, e; x, y; p) = \sum_{n,m=0}^{\infty} \frac{(a)_{m+n} B_p(b+n, d-b) B_p(c+m, e-c)}{B(b, d-b)B(c, e-c)} \frac{x^n}{n!} \frac{y^m}{m!};$$

$$(\{|x| + |y|\} < 1)$$

and

$$\text{(1.15)} \qquad F_{D,p}^3(a, b, c, d; e; x, y, z) = \sum_{n,m,r=0}^{\infty} \frac{B_p(a+m+n+r, e-a)(b)_m (c)_n (d)_r}{B(a, e-a)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^r}{r!};$$

$$(\{\sqrt{|x|} + \sqrt{|y|} + \sqrt{|z|}\} < 1)$$

Respectively, notice that the case $p = 0$ *gives the original functions.*

The extended Appell's hypergeometric functions of two variables for additional parameter q with $F_1^{\{k_l\}}(a,b,c;d;x,y;p,q)$ and $F_2^{\{k_l\}}(a,b,c;d,e;x,y;p,q)$

$$\begin{aligned}(1.16)\qquad & F_1^{\{k_l\}}(a,b,c;d;x,y;p,q)\\ &= \sum_{n,m=0}^{\infty} \frac{B_{p,q}^{\{k_l\}}(a+m+n,d-a)}{B(a,d-a)}(b)_n(c)_m \frac{x^n}{n!}\frac{y^m}{m!};\\ &(\max\{|x|,|y|\}<1, \min(\Re(p),\Re(q))\geq 0)\end{aligned}$$

$$\begin{aligned}(1.17)\qquad & F_2^{\{k_l\}}(a,b,c;d,e;x,y;p,q)\\ &= \sum_{n,m=0}^{\infty} \frac{(a)_{m+n}B_{p,q}^{\{k_l\}}(b+n,d-b)B_{p,q}^{\{k_l\}}(c+m,e-c)}{B(b,d-b)B(c,e-c)}\frac{x^n}{n!}\frac{y^m}{m!};\\ &(\{|x|+|y|\}<1, \min(\Re(p),\Re(q))\geq 0)\end{aligned}$$

Extended Lauricella's hypergeometric function,

$$\begin{aligned}(1.18)\qquad & F_{D,\{k_l\}}^{3}(a,b,c,d;e;x,y,z;p)\\ &= \sum_{n,m,r=0}^{\infty} \frac{B_{p}^{\{k_l\}}(a+m+n+r,e-a)(b)_m(c)_n(d)_r}{B(a,e-a)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^r}{r!};\\ &(\{\sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}\}<1, \Re(p)>0)\end{aligned}$$

For additional parameter q with $R(q)\geq 0$, we have

$$\begin{aligned}(1.19)\qquad & F_{D,\{k_l\}}^{r}(\alpha_1,\beta_1\ldots\beta_r;x_1\ldots x_r;p,q)\\ &= \sum_{m_1\ldots m_r=0}^{\infty} (\beta_1)_{m_1}\ldots(\beta_r)_{m_r}\frac{B_{p,q}^{\{k_l\}}(\alpha+m_1+\ldots+m_r,\gamma-\alpha)}{B(\alpha,\gamma-\alpha)}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!}\\ &(\max\{|x_1|\ldots|x_r|\}<1, \min(\Re(p),\Re(q))\geq 0)\end{aligned}$$

[7]

And integral representations of extended Appell's hypergeometric functions

(1.20)

$$F_1^{\{k_l\}}(a,b,c;d;x,y;p,q) = \frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\Theta\left(\{k_l\};-\frac{p}{t}-\frac{q}{1-t}\right)dt$$

$$(\min(\Re(p),\Re(q))\geq 0;\max\{|\arg(1-x)|,|\arg(1-y)|\}<\pi,\Re(a)<0)$$

(1.21)

$$F_2^{\{k_l\}}(a,b,c;d,e;x,y;p,q) = \frac{1}{B(b,d-b)}\frac{1}{B(c,e-c)}\int_0^1\int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-t)^{e-c-1}}{(1-xt-ys)^a} \Theta\left(\{k_l\};-\frac{p}{t}-\frac{q}{1-t}\right)\Theta\left(\{k_l\};-\frac{p}{s}-\frac{q}{1-s}\right)dtds$$

$$(\min(\Re(p),\Re(q))\geq 0;|x|+|y)|<1,\Re(d)>\Re(b)>0,\Re(e)>\Re(c)>0)$$

and integral representations of extended Lauricella hypergeometric functions

(1.22)

$$F^r_{D,\{k_l\}}(\alpha_1,\beta_1\dots\beta_r;\gamma;x_1\dots x_r;p,q) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{a-1}(1-t)^{\gamma-a-1}\prod_{j=1}^{r}(1-x_jt)^{-\beta_j}\Theta\left(\{k_l\};-\frac{p}{t}-\frac{q}{1-t}\right)dt$$

$$(\Re(p),\Re(q))\geq 0;\max\{|\arg(1-x_1)|,\dots|\arg(1-x_r)|\}<\pi,\Re(\gamma)>\Re(a)<0)$$

2 Application of Generalized Extended Riemann-Liouville Fractional Derivative Operator

H.M. Srivastava, R.K. Parmar and P. Chopra [7] introduce the following generalizations of the extended Riemann-Liouville fractional derivative operator $D_z^{\mu,p}$

$$D^{\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{f(z)\} = \begin{cases} \frac{1}{\Gamma(-\mu)}\int_0^z (z-t)^{-\mu-1}\Theta(\{k_l\}_{l\in N_0};\frac{-pz^2}{(z-t)t})f(t)dt \\ \quad (\Re(\mu)<0,\Re(p)>0) \\ \frac{d^m}{dz^m}D^{\mu-m,p}_{z,(\{k_l\}_{l\in N_0})}\{f(z)\} \\ \quad (\Re(p)>0, m-1\leq\Re(\mu)<m); m\in\aleph \end{cases} \tag{2.1}$$

Moreover, the fractional derivative operator $D^{\mu,p}_{z,(\{k_l\}_{l\in N_0})}$ defined by (2.1) can be further extended as follows:

$$D^{\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{f(z)\} = \begin{cases} \frac{1}{\Gamma(-\mu)}\int_0^z (z-t)^{-\mu-1}\Theta\left(\{k_l\}_{l\in N_0}; \frac{-pz}{t} - \frac{qz}{z-t}\right) f(t)dt \\ \qquad (\Re(\mu) < 0) \\ \frac{d^m}{dz^m} D^{\mu-m,p,q}_{z,(\{k_l\}_{l\in N_0})}\{f(z)\} \\ \qquad (m-1 \le \Re(\mu) < m); m \in \aleph \end{cases} \tag{2.2}$$

Where $(\min\{\Re(p), (\Re(q)\} > 0)$ and the path of integration in the Definition 2.1 and 2.2 is a line in the complex t-plane from $t = 0$ to $t = z$.

Theorem 2.1. *Let $Re(\lambda) > -1$, $Re(\mu) < 0$ then*

$$D^{\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^\lambda\} = \frac{B^{\{k_l\}}_p(\lambda+1, -\mu; p)}{\Gamma(-\mu)} z^{\lambda-\mu}.$$

Proof. Using (2.1) and (1.8) and let $Re(\lambda) > -1$, $Re(\mu) < 0$, we get

$$\begin{aligned} D^{\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^\lambda\} &= \frac{1}{\Gamma(-\mu)}\int_0^z t^\lambda (z-t)^{-\mu-1}\Theta\left(\{k_l\}_{l\in N_0}; \frac{-pz^2}{(z-t)t}\right) dt \\ &= \frac{1}{\Gamma(-\mu)}\int_0^1 (uz)^\lambda (z)^{-\mu-1}(1-u)^{-\mu-1}\Theta\left(\{k_l\}_{l\in N_0}; \frac{-p}{(1-u)u}\right) zdu \\ &= \frac{(z)^{\lambda-\mu}}{\Gamma(-\mu)}\int_0^1 (u)^\lambda (1-u)^{-\mu-1}\Theta\left(\{k_l\}_{l\in N_0}; \frac{-p}{(1-u)u}\right) du \\ &= \frac{B^{\{k_l\}}_p(\lambda+1, -\mu; p)}{\Gamma(-\mu)} z^{\lambda-\mu}. \end{aligned}$$

Hence the proof is completed.

Theorem 2.2. *Let $Re(\lambda) > -1$, $Re(\alpha) > 0$, $Re(\mu) < 0$ and $|z| < 1$. Then*

$$D^{\lambda-\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-z)^{-\alpha}\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F^{\{k_l\}}_p(\alpha, \lambda; \mu; z; p)$$

Proof. We can write as

$$\begin{aligned} & D^{\lambda-\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-z)^{-\alpha}\} \\ &= \frac{1}{\Gamma(\mu-\lambda)}\int_0^z t^{\lambda-1}(1-t)^{-\alpha}(z-t)^{\mu-\lambda-1}\Theta\left(\{k_l\}_{l\in N_0}; \frac{-pz^2}{(z-t)t}\right) dt \\ &= \frac{(z)^{\mu-\lambda-1}z^{\lambda-1}}{\Gamma(\mu-\lambda)}\int_0^1 u^{\lambda-1}(1-uz)^{-\alpha}(1-u)^{\mu-\lambda-1}\Theta\left(\{k_l\}_{l\in N_0}; \frac{-p}{(z-u)u}\right) zdu \\ &= \frac{(z)^{\mu-1}}{\Gamma(\mu-\lambda)} B(\lambda, \mu-\lambda) F^{\{k_l\}}_p(\alpha, \lambda; \mu; z; p) \\ &= \frac{\Gamma(\lambda)}{\Gamma(\mu)}(z)^{\mu-1} F^{\{k_l\}}_p(\alpha, \lambda; \mu; z; p) \end{aligned}$$

Whence the result.

Theorem 2.3. *For Riemann-Liouville fractional derivative operator for two parameters, let $Re(\lambda) > -1$, $Re(\mu) < 0$ then*
$D^{\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda}\} = \dfrac{B^{\{k_l\}}_{p,q}(\lambda+1,-\mu;p,q)}{\Gamma(-\mu)} z^{\lambda-\mu}.$

Proof. Using (2.2) and (1.9) and let $Re(\lambda) > -1$, $Re(\mu) < 0$. then

$$\begin{aligned} &D^{\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda}\} \\ &\quad = \frac{1}{\Gamma(-\mu)}\int_0^z t^{\lambda}(z-t)^{-\mu-1}\Theta\left(\{k_l\}_{l\in N_0};\frac{-pz}{t}-\frac{qz}{z-t}\right)dt \\ &\quad = \frac{1}{\Gamma(-\mu)}\int_0^1 (uz)^{\lambda}(z)^{-\mu-1}(1-u)^{-\mu-1}\Theta\left(\{k_l\}_{l\in N_0};\frac{-p}{u}-\frac{q}{1-u}\right)zdu \\ &\quad = \frac{z^{(\lambda-\mu)}}{\Gamma(-\mu)}\int_0^1 (u)^{\lambda}(1-u)^{-\mu-1}\Theta\left(\{k_l\}_{l\in N_0};\frac{-p}{u}-\frac{q}{1-u}\right)du \\ &\quad = \frac{B^{\{k_l\}}_{p,q}(\lambda+1,-\mu;p,q)}{\Gamma(-\mu)}z^{\lambda-\mu} \end{aligned}$$

Proof is completed.

Theorem 2.4. *For Riemann-Liouville fractional derivative operator for two parameters, let $Re(\lambda) > -1$, $Re(\alpha) > 0$, $Re(\mu) < 0$ and $|z| < 1$. Then*

$$D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-z)^{-\alpha}\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}{}_2F_1^{\{k_l\}}(\alpha,\lambda;\mu;z;p,q).$$

Proof. We can write as

$$\begin{aligned} &D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-z)^{-\alpha}\} \\ &= \frac{1}{\Gamma(\mu-\lambda)}\int_0^z t^{\lambda-1}(1-t)^{-\alpha}(z-t)^{\mu-\lambda-1}\Theta\left(\{k_l\}_{l\in N_0};\frac{-pz}{t}-\frac{qz}{z-t}\right)dt \\ &= \frac{(z)^{\mu-\lambda-1}z^{\lambda-1}}{\Gamma(\mu-\lambda)}\int_0^1 u^{\lambda-1}(1-uz)^{-\alpha}(1-u)^{\mu-\lambda-1}\Theta\left(\{k_l\}_{l\in N_0};\frac{-p}{u}-\frac{q}{1-u}\right)zdu \\ &= \frac{(z)^{\mu-1}}{\Gamma(\mu-\lambda)}B(\lambda,\mu-\lambda){}_2F_1^{\{k_l\}}(\alpha,\lambda;\mu;z;p,q) \\ &= \frac{\Gamma(\lambda)}{\Gamma(\mu)}(z)^{\mu-1}{}_2F_1^{\{k_l\}}(\alpha,\lambda;\mu;z;p,q) \end{aligned}$$

Whence the result

3 The extended fractional integral of an analytic function

The following theorems determine the extended fractional integral of an analytic function for single and double parameters.

Theorem 3.1. *Let $f(z)$ be an analytic function in the disc $|z| < \rho$ and has the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then*

$$D^{\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}f(z)\} = \sum_{n=0}^{\infty} a_n D^{\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda+n-1}\}$$
$$= \frac{z^{(\lambda-\mu-1)}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n B_p^{\{k_l\}}(\lambda+n, -\mu; p) z^n$$

Provided $Re(\lambda) > 0$, $Re(\mu) < 0$ and $|z| < \rho$.

Proof. We have

$$D^{\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}f(z)\}$$
$$= D^{\mu,p}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}\sum_{n=0}^{\infty} a_n z^n\right\}$$
$$= \frac{1}{\Gamma(-\mu)}\int_0^z t^{\lambda-1}\sum_{n=0}^{\infty} a_n t^n (z-t)^{-\mu-1}\Theta\left(\{k_l\}_{l\in N_0}; \frac{-pz^2}{(z-t)t}\right)dt$$
$$= \frac{1}{\Gamma(-\mu)}\int_0^1 (uz)^{\lambda-1}(z)^{-\mu-1}(1-u)^{-\mu-1}\Theta\left(\{k_l\}_{l\in N_0}; \frac{-p}{(1-u)u}\right)\sum_{n=0}^{\infty} a_n (uz)^n z du$$
$$= \frac{(z)^{\lambda-\mu-1}}{\Gamma(-\mu)}\int_0^1 (u)^{\lambda-1}(1-u)^{-\mu-1}\Theta\left(\{k_l\}_{l\in N_0}; \frac{-p}{(1-u)u}\right)\sum_{n=0}^{\infty} a_n (uz)^n du$$

Since the series $\sum_{n=0}^{\infty} a_n z^n u^n$ is uniformly convergent in the disc $|z| < \rho$ for $0 \le u \le 1$ and the integral $\int_0^1 \left|(u)^{\lambda-1}(1-u)^{-\mu-1}\Theta(\{k_l\}_{l\in N_0}; \frac{-p}{(1-u)u})\right| du$ is convergent provided that $Re(\lambda) > 0$, $Re(\mu) < 0$ and $|z| < \rho$, we can change the order of integration and summation and obtain

$$D^{\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}f(z)\} = \frac{(z)^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty} a_n (z)^n \int_0^1 (u)^{\lambda+n-1}(1-u)^{-\mu-1}$$
$$\Theta(\{k_l\}_{l\in N_0}; \frac{-p}{(1-u)u})du$$
$$= \frac{(z)^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty} a_n B_p^{\{k_l\}}(\lambda+n, -\mu; p) z^n$$

Hence the result.

Theorem 3.2. *Let $f(z)$ be an analytic function in the disc $|z| < \rho$ and has the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then*

$$\begin{aligned} D^{\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}f(z)\} &= \sum_{n=0}^{\infty} a_n D^{\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda+n-1}\} \\ &= \frac{z^{(\lambda-\mu-1)}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n B^{\{k_l\}}_{p,q}(\lambda+n,-\mu;p,q)z^n \end{aligned}$$

Provided that $Re(\lambda) > 0$, $Re(\mu) < 0$ and $|z| < \rho$.

Proof. We have

$$\begin{aligned} &D^{\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}f(z)\} \\ &= D^{\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}\sum_{n=0}^{\infty} a_n z^n\right\} \\ &= \frac{1}{\Gamma(-\mu)}\int_0^z t^{\lambda-1}\sum_{n=0}^{\infty} a_n t^n (z-t)^{-\mu-1}\Theta\left(\{k_l\}_{l\in N_0}; \frac{-pz}{t} - \frac{qz}{z-t}\right)dt \\ &= \frac{(z)^{\lambda-\mu-1}}{\Gamma(-\mu)}\int_0^1 (u)^{\lambda-1}(1-u)^{-\mu-1}\Theta\left(\{k_l\}_{l\in N_0}; \frac{-p}{u} - \frac{q}{1-u}\right)\sum_{n=0}^{\infty} a_n (uz)^n du \end{aligned}$$

Since the series $\sum_{n=0}^{\infty} a_n z^n u^n$ is uniformly convergent in the disc $|z| < \rho$ for $0 \le u \le 1$ and the integral $\int_0^1 \left|(u)^{\lambda-1}(1-u)^{-\mu-1}\Theta(\{k_l\}_{l\in N_0}; \frac{-p}{u} - \frac{q}{1-u})\right| du$ is convergent provided that $Re(\lambda) > 0$, $Re(\mu) < 0$ and $|z| < \rho$, we can change the order of integration and summation and obtain

$$\begin{aligned} D^{\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}f(z)\} &= \frac{(z)^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty} a_n (z)^n \int_0^1 (u)^{\lambda+n-1}(1-u)^{-\mu-1} \\ &\qquad \Theta\left(\{k_l\}_{l\in N_0}; \frac{-p}{u} - \frac{q}{1-u}\right)du \\ &= \frac{(z)^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty} a_n B^{\{k_l\}}_{p,q}(\lambda+n,-\mu;p,q)z^n \end{aligned}$$

Hence the result

4 Further Generalization of some Results

Now we will deal with the further generalizations of Theorem 2.2 and Theorem 2.4 in terms the extended Appell's hypergeometric functions in two variables, $F_1^{\{k_l\}}(a,b,c;d;x,y;p,q)$ and $F_2^{\{k_l\}}(a,b,c;d,e;x,y;p,q)$ and the extended Lauricella's hypergeometric function of r variables $x_1;\ldots;x_r$, which are defined by $F^r_{D,\{k_l\}}(\alpha_1,\beta_1\ldots\beta_r;\gamma;x_1\ldots x_r;p,q)$.

Theorem 4.1. *Let $Re(\mu)>Re(\lambda)>0$, $Re(\alpha)>Re(\beta)>0$; $|az|<1$ and $|bz|<1$. Then*

$$D^{\lambda-\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\}=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_1^{\{k_l\}}(\lambda,\alpha,\beta;\mu;az,bz;p)$$

Proof. Considering the Theorem 2.1 and using (1.16) we can write as

$$\begin{aligned}
&D^{\lambda-\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\}\\
&=\frac{1}{\Gamma(\mu-\lambda)}\int_0^z t^{\lambda-1}(1-at)^{-\alpha}(1-bt)^{-\beta}\Theta\left(\{k_l\}_{l\in N_0};\frac{-pz^2}{(z-t)t}\right)(z-t)^{\mu-\lambda-1}dt\\
&=\frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}\int_0^1 u^{\lambda-1}(1-auz)^{-\alpha}(1-buz)^{-\beta}(1-u)^{\mu-\lambda-1}\Theta\left(\{k_l\}_{l\in N_0};\frac{-p}{(1-u)u}\right)du\\
&=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_1^{\{k_l\}}(\lambda,\alpha,\beta;\mu;az,bz;p)
\end{aligned}$$

Hence proved.

Corollary 4.1. *Let $Re(\mu)>Re(\lambda)>0$, $Re(\alpha)>Re(\beta)>0$; $|az|<1$ and $|bz|<1$. Then*

$$\begin{aligned}
&D^{\lambda-\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\}\\
&\quad=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}(1-az)^{-\alpha}(1-bz)^{-\beta}F_1^{\{k_l\}}\left(\mu-\lambda,\alpha,\beta;\mu;\frac{az}{az-1},\frac{bz}{bz-1};p\right)
\end{aligned}$$

Proof. Now using, [4, p. 15, Theorem 3.2] we can easily obtain the following transformation

$$\begin{aligned}
&F_1^{\{k_l\}}(\lambda,\alpha,\beta;\mu;az,bz;p)\\
&\quad=(1-az)^{-\alpha}(1-bz)^{-\beta}F_1^{\{k_l\}}\left(\mu-\lambda,\alpha,\beta;\mu;\frac{az}{az-1},\frac{bz}{bz-1};p\right).
\end{aligned}$$

Hence the Result

Theorem 4.2. *Here we prove the Theorem 4.1 for two parameters.*
Let $Re(\mu)>Re(\lambda)>0$, $Re(\alpha)>Re(\beta)>0$; $|az|<1$ and $|bz|<1$. Then

$$D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\}=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_1^{\{k_l\}}(\lambda,\alpha,\beta;\mu;az,bz;p,q).$$

Proof. Considering the Theorem 2.2 and using (1.20) we can write as

$$
\begin{aligned}
&D^{\lambda-\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\}\\
&=\frac{1}{\Gamma(\mu-\lambda)}\int_0^z t^{\lambda-1}(1-at)^{-\alpha}(1-bt)^{-\beta}\Theta\left(\{k_l\}_{l\in N_0};\frac{-pz}{t}-\frac{qz}{z-t}\right)(z-t)^{\mu-\lambda-1}dt\\
&=\frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}\int_0^1 u^{\lambda-1}(1-auz)^{-\alpha}(1-buz)^{-\beta}(1-u)^{\mu-\lambda-1}\Theta\left(\{k_l\}_{l\in N_0};\frac{-pz}{t}-\frac{qz}{z-t}\right)du\\
&=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_1^{\{k_l\}}(\lambda,\alpha,\beta;\mu;az,bz;p,q)
\end{aligned}
$$

Hence proved the theorem for two parameters.

Corollary 4.2. *Let* $Re(\mu)>Re(\lambda)>0$*,* $Re(\alpha)>Re(\beta)>0$*;* $|az|<1$ *and* $|bz|<1$*. Then*

$$
\begin{aligned}
&D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\}\\
&\quad=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}(1-az)^{-\alpha}(1-bz)^{-\beta}F_1^{\{k_l\}}\left(\mu-\lambda,\alpha,\beta;\mu;\frac{az}{az-1},\frac{bz}{bz-1};p,q\right).
\end{aligned}
$$

Proof. We can easily obtain the following transformation using [4, p. 15, Theorem 3.2]:

$$
\begin{aligned}
&F_1^{\{k_l\}}(\lambda,\alpha,\beta;\mu;az,bz;p,q)\\
&\quad=(1-az)^{-\alpha}(1-bz)^{-\beta}F_1^{\{k_l\}}\left(\mu-\lambda,\alpha,\beta;\mu;\frac{az}{az-1},\frac{bz}{bz-1};p,q\right).
\end{aligned}
$$

Hence the result.

Generalization for r variables

Here we prove the Theorem 4.1 for (two parameters) r variables.

Theorem 4.3. *Let* $Re(\mu)>Re(\lambda)>0$*,* $Re(\beta_1)>0$*,* $Re(\beta_2)>0\ldots Re(\beta_r)>0$*; and* $|a_1z|<1$*,* $|a_2z|<1\ldots|a_rz|<1$*. Then we have*

$$
\begin{aligned}
&D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-a_1z)^{-\beta_1}(1-a_2z)^{-\beta_2}\ldots(1-a_rz)^{-\beta_r}\}\\
&\quad=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F^r_{D,\{k_l\}}(\lambda,\beta_1\ldots\beta_r;\mu;a_1z\ldots a_rz;p,q).
\end{aligned}
$$

Proof. Considering the Theorem 4.2 and using (1.19) we can write as

$$D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-a_1z)^{-\beta_1}(1-a_2z)^{-\beta_2}\dots(1-a_rz)^{-\beta_r}\}$$
$$=\frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}\sum_{m_1\dots m_r=0}^{\infty}\frac{(\beta_1)_{m_1}\dots(\beta_r)_{m_r}}{m_1!\dots m_r!}a_1^{m_1}\dots a_r^{m_r}$$
$$B^{\{k_l\}}_{p,q}(\lambda+m_1+\dots+m_r,\mu-\lambda)z^{m_1+m_2+\dots+m_r}$$
$$=\frac{B(\lambda,\mu-\lambda)}{\Gamma(\mu-\lambda)}z^{\mu-1}\sum_{m_1\dots m_r=0}^{\infty}\frac{(\beta_1)_{m_1}\dots(\beta_r)_{m_r}}{m_1!\dots m_r!}\frac{B^{\{k_l\}}_{p,q}(\lambda+m_1+\dots+m_r,\mu-\lambda)}{B(\lambda,\mu-\lambda)}$$
$$(a_1z)^{m_1}(a_1z)^{m_2}\dots(a_1z)^{m_r}$$
$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F^r_{D,\{k_l\}}(\lambda,\beta_1\dots\beta_r;\mu;a_1z\dots a_rz;p,q)$$

Hence the result.

Corollary 4.3. *If we take $p=q$ then we get the generalization of Theorem 4.1 for (single parameter) r variables*

$$D^{\lambda-\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-a_1z)^{-\beta_1}(1-a_2z)^{-\beta_2}\dots(1-a_rz)^{-\beta_r}\}$$
$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F^r_{D,\{k_l\}}(\lambda,\beta_1\dots\beta_r;\mu;a_1z\dots a_rz;p)$$

Corollary 4.4. *Putting $r=3$ in Theorem 4.3 we get*

$$D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\}$$
$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F^3_{D,\{k_l\}}(\lambda,\alpha,\beta,\gamma;\mu;az,bz,cz;p,q)$$

Corollary 4.5. *In Corollary 4.3, after putting $r=3$ we get*

$$D^{\lambda-\mu,p}_{z,(\{k_l\}_{l\in N_0})}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\}$$
$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F^3_{D,\{k_l\}}(\lambda,\alpha,\beta,\gamma;\mu;az,bz,cz;p)$$

5 More applications of Generalized Extended Riemann-Liouville Fractional Derivative Operator

Theorem 5.1. *Let $Re(\mu) > Re(\lambda) > 0$, $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\gamma) > 0$; $\left|\frac{x}{1-z}\right| < 1$, $|x| + |z| < 1$. Then we have*

$$D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}{}_2F_1^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p,q\right\}$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_2^{\{k_l\}}(\alpha,\beta,\lambda;\gamma,\mu;x,z;p,q)$$

Proof. Using Theorem 2.1 and (1.17), we get

$$D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}{}_2F_1^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p,q\right\}$$
$$= D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}\frac{1}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}(\alpha)_n\frac{B^{\{k_l\}}_{p,q}(\beta+n,\gamma-\beta)}{n!}\left(\frac{x}{1-z}\right)^n\right\}$$
$$= \frac{1}{B(\beta,\gamma-\beta)}D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}(\alpha)_n\frac{B^{\{k_l\}}_{p,q}(\beta+n,\gamma-\beta)}{n!}(x)^n(1-z)^{-\alpha-n}\right\}$$
$$= \frac{1}{B(\beta,\gamma-\beta)}\sum_{m,n=0}^{\infty}B^{\{k_l\}}_{p,q}(\beta+n,\gamma-\beta)\frac{(x)^n}{n!}\frac{(\alpha)_n(\alpha+n)_m}{m!}D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}z^{\lambda-1+m}$$
$$= \frac{1}{B(\beta,\gamma-\beta)}\sum_{m,n=0}^{\infty}B^{\{k_l\}}_{p,q}(\beta+n,\gamma-\beta)\frac{(x)^n}{n!}\frac{(\alpha)_{m+n}}{m!}\frac{B^{\{k_l\}}_{p,q}(\lambda+m,\mu-\lambda)}{\Gamma(\mu-\lambda)}z^{\mu+m-1}$$
$$= \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}\sum_{m,n=0}^{\infty}(\alpha)_{m+n}\frac{B^{\{k_l\}}_{p,q}(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)}\frac{B^{\{k_l\}}_{p,q}(\lambda+m,\mu-\lambda)}{B(\lambda,\mu-\lambda)}\frac{(x)^n}{n!}\frac{z^m}{m!}B(\lambda,\mu-\lambda)$$
$$= \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}F_2^{\{k_l\}}(\alpha,\beta,\lambda;\gamma,\mu;x,z;p,q)B(\lambda,\mu-\lambda)$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_2^{\{k_l\}}(\alpha,\beta,\lambda;\gamma,\mu;x,z;p,q)$$

Whence the result.

Corollary 5.1. *Let $Re(\mu) > Re(\lambda) > 0$, $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\gamma) > 0$; $\left|\frac{x}{1-z}\right| < 1$, $|x| + |z| < 1$. Then we have*

$$D^{\lambda-\mu,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}{}_2F_1^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p,q\right\}$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}(1-x-z)^{-\alpha}F_2^{\{k_l\}}\left(\alpha,\gamma-\beta,\mu-\lambda;\gamma,\mu;\frac{-x}{1-x-z},\frac{-z}{1-x-z};p,q\right)$$

Proof. Transformation we can easily obtain the following transformation using [4, p. 15, Theorem 3.2]:

$$
\begin{aligned}
&F_2^{\{k_l\}}(\alpha,\beta,\lambda;\gamma,\mu;x,z;p,q)\\
&\quad=(1-x-z)^{-\alpha}F_2^{\{k_l\}}\left(\alpha,\gamma-\beta,\mu-\lambda;\gamma,\mu;\frac{-x}{1-x-z},\frac{-z}{1-x-z};p,q\right)
\end{aligned}
$$

Hence the proof.

Remark 5.1. *Let $Re(\mu)>Re(\lambda)>0$, $Re(\alpha)>0$, $Re(\beta)>0$, $Re(\gamma)>0$; $\left|\frac{x}{1-z}\right|<1$, $|x|+|z|<1$. Then we get the Riemann-Liouville fractional derivative for single parameter*

$$
\begin{aligned}
&D^{\lambda-\mu,p}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_p^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p\right\}\\
&\quad=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_2^{\{k_l\}}(\alpha,\beta,\lambda;\gamma,\mu;x,z;p)
\end{aligned}
$$

Proof. If we set $p=q$ in Theorem 5.1, we get function $B_p^{\{k_l\}}(\beta+n,\gamma-\beta)$ in place of $B_{p,q}^{\{k_l\}}(\beta+n,\gamma-\beta)$ and $B_p^{\{k_l\}}(\lambda+m,\mu-\lambda)$ in place of $B_{p,q}^{\{k_l\}}(\lambda+m,\mu-\lambda)$ and using, [7, p. 243, Remark 1, Equation (2.11)], thus we obtain the above result.

Corollary 5.2. *Let $Re(\mu)>Re(\lambda)>0$, $Re(\alpha)>0$, $Re(\beta)>0$, $Re(\gamma)>0$; $|\frac{x}{1-z}|<1$, $|x|+|z|<1$. Then we have*

$$
\begin{aligned}
&D^{\lambda-\mu,p}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_p^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p\right\}\\
&\quad=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}(1-x)^{-\alpha}F_2^{\{k_l\}}\left(\alpha,\gamma-\beta,\lambda;\gamma,\mu;\frac{x}{x-1},\frac{-z}{x-1};p\right)\\
&D^{\lambda-\mu,p}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_p^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p\right\}\\
&\quad=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}(1-z)^{-\alpha}F_2^{\{k_l\}}\left(\alpha,\beta,\mu-\lambda;\gamma,\mu;\frac{-x}{z-1},\frac{z}{z-1};p\right)\\
&D^{\lambda-\mu,p}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_p^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p\right\}\\
&\quad=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}(1-x-z)^{-\alpha}F_2^{\{k_l\}}\left(\alpha,\gamma-\beta,\mu-\lambda;\gamma,\mu;\frac{-x}{1-x-z},\frac{-z}{1-x-z};p\right)
\end{aligned}
$$

Proof. Using, [7, p. 243, Remark 1, Equation (2.11)], we get the following transformations

$$F_2^{\{k_l\}}(\alpha,\beta,\lambda;\gamma,\mu;x,z;p) = (1-x)^{-\alpha} F_2^{\{k_l\}}\left(\alpha,\gamma-\beta,\lambda;\gamma,\mu;\frac{x}{x-1},\frac{-z}{x-1};p\right)$$
$$F_2^{\{k_l\}}(\alpha,\beta,\lambda;\gamma,\mu;x,z;p) = (1-z)^{-\alpha} F_2^{\{k_l\}}\left(\alpha,\beta,\mu-\lambda;\gamma,\mu;\frac{-x}{z-1},\frac{z}{z-1};p\right)$$
$$F_2^{\{k_l\}}(\alpha,\beta,\lambda;\gamma,\mu;x,z;p) = (1-x-z)^{-\alpha} F_2^{\{k_l\}}\left(\alpha,\gamma-\beta,\mu-\lambda;\gamma,\mu;\frac{-x}{1-x-z},\frac{-z}{1-x-z};p\right)$$

And thus we obtain the above result.

Theorem 5.2. *Let $Re(\mu) > Re(\lambda) > 0$, $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\gamma) > 0$; $\left|\frac{x}{1-z}\right| < 1$, $|x| + |z| < 1$. Then we have*

$$D^{\lambda-\mu+m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}{}_2F_1^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p,q\right\}$$
$$= \frac{\Gamma(\lambda+m)}{\Gamma(\mu)} z^{\mu-1}\frac{(\mu-\lambda)_{2m}}{(\mu-\lambda)_m(\lambda)_m}\sum_{i=1}^{m}(-m)_i\frac{(\lambda)_{i+m}}{(\mu)_{i+m}i!}$$
$$F_2^{\{k_l\}}(\alpha,\beta,\lambda+m+i;\mu+i+m;x,z;p,q)$$

Proof. Using Theorem 2.4 and (1.11), we get

$$D^{\lambda-\mu+m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}{}_2F_1^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p,q\right\}$$
$$= D^{\lambda-\mu+m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}\frac{1}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_n B_{p,q}^{\{k_l\}}(\beta+n,\gamma-\beta)}{n!}\left(\frac{x}{1-z}\right)^n\right\}$$
$$D^{\lambda-\mu+m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}{}_2F_1^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p,q\right\}$$
$$= D^{\lambda-\mu+m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1+m}(1-z)^{-\alpha}\frac{1}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_n B_{p,q}^{\{k_l\}}(\beta+n,\gamma-\beta)}{n!}\left(\frac{x}{1-z}\right)^n\right\}$$
$$= \frac{1}{B(\beta,\gamma-\beta)}D^{\lambda-\mu+m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1+m}\sum_{n=0}^{\infty}(\alpha)_n\frac{B_{p,q}^{\{k_l\}}(\beta+n,\gamma-\beta)}{n!}(x)^n(1-z)^{-\alpha-n}\right\}$$
$$= \frac{1}{B(\beta,\gamma-\beta)}\sum_{m,n=0}^{\infty}B_{p,q}^{\{k_l\}}(\beta+n,\gamma-\beta)\frac{(x)^n}{n!}\frac{(\alpha)_n(\alpha+n)_m}{m!}D^{\lambda-\mu+m,p,q}_{z,(\{k_l\}_{l\in N_0})}z^{\lambda-1+2m}$$
$$= \frac{1}{B(\beta,\gamma-\beta)}\sum_{m,n=0}^{\infty}B_{p,q}^{\{k_l\}}(\beta+n,\gamma-\beta)\frac{(x)^n}{n!}\frac{(\alpha)_{m+n}}{m!}$$
$$\frac{B_{p,q}^{\{k_l\}}(\lambda+2m,\mu-\lambda-m)}{\Gamma(\mu-\lambda-m)}z^{\mu+m-1}$$
$$= \frac{z^{\mu+m-1}}{\Gamma(\mu-\lambda-m)}\sum_{m,n=0}^{\infty}(\alpha)_{m+n}\frac{B_{p,q}^{\{k_l\}}(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)}\frac{B_{p,q}^{\{k_l\}}(\lambda+2m,\mu-\lambda)}{B(\lambda+m,\mu-\lambda-m)}$$
$$\frac{(x)^n}{n!}\frac{z^m}{m!}B(\lambda+m,\mu-\lambda-m)$$
$$= \frac{z^{\mu-1}}{\Gamma(\mu-\lambda-m)}F_2^{\{k_l\}}(\alpha,\beta,\lambda+m;\gamma,\mu;x,z;p,q)B(\lambda+m,\mu-\lambda-m)$$
$$= \frac{\Gamma(\lambda+m)}{\Gamma(\mu)}z^{\mu-1}F_2^{\{k_l\}}(\alpha,\beta,\lambda+m;\gamma,\mu;x,z;p,q)$$

Now using [4, p.18, Theorem 3.6], we get

$$D^{\lambda-\mu+m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}{}_2F_1^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p,q\right\}$$
$$= \frac{\Gamma(\lambda+m)}{\Gamma(\mu)}z^{\mu-1}\frac{(\mu-\lambda)_{2m}}{(\mu-\lambda)_m(\lambda)_m}\sum_{i=1}^{m}(-m)_i\frac{(\lambda)_{i+m}}{(\mu)_{i+m}i!}$$
$$F_2^{\{k_l\}}(\alpha,\beta,\lambda+m+i;\mu+i+m;x,z;p,q).$$

Hence the proof.

Theorem 5.3. *Let* $Re(\mu) > Re(\lambda) > 0$, $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\gamma) > 0$; $\left|\frac{x}{1-z}\right| < 1$, $|x| + |z| < 1$. *Then we have*

$$D^{\lambda-\mu-m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}{}_2F_1^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p,q\right\}$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu+m)}z^{\mu+m-1}\frac{(\mu)_m}{(\mu-\lambda)_m}\sum_{k=0}^{m}(-1)^k{}_mC_k\frac{(\lambda)_k}{(\mu)_k}$$
$$F_2^{\{k_l\}}(\alpha,\beta,\lambda+k;\mu+k;x,z;p,q)$$

Proof. Using Theorem 2.4 and (1.12), we get

$$D^{\lambda-\mu-m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1+m}(1-z)^{-\alpha}{}_2F_1^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p,q\right\}$$
$$= D^{\lambda-\mu-m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}\frac{1}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_nB^{\{k_l\}}_{p,q}(\beta+n,\gamma-\beta)}{n!}\left(\frac{x}{1-z}\right)^n\right\}$$
$$= D^{\lambda-\mu-m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}\frac{1}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_nB^{\{k_l\}}_{p,q}(\beta+n,\gamma-\beta)}{n!}\left(\frac{x}{1-z}\right)^n\right\}$$
$$= \frac{1}{B(\beta,\gamma-\beta)}D^{\lambda-\mu-m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}(\alpha)_n\frac{B^{\{k_l\}}_{p,q}(\beta+n,\gamma-\beta)}{n!}(x)^n(1-z)^{-\alpha-n}\right\}$$
$$= \frac{1}{B(\beta,\gamma-\beta)}\sum_{m,n=0}^{\infty}B^{\{k_l\}}_{p,q}(\beta+n,\gamma-\beta)\frac{(x)^n}{n!}\frac{(\alpha)_n(\alpha+n)_m}{m!}D^{\lambda-\mu-m,p,q}_{z,(\{k_l\}_{l\in N_0})}z^{\lambda-1+m}$$
$$= \frac{1}{B(\beta,\gamma-\beta)}\sum_{m,n=0}^{\infty}B^{\{k_l\}}_{p,q}(\beta+n,\gamma-\beta)\frac{(x)^n}{n!}\frac{(\alpha)_{m+n}}{m!}$$
$$\frac{B^{\{k_l\}}_{p,q}(\lambda+m,\mu+m-\lambda)}{\Gamma(\mu+m-\lambda)}z^{\mu+2m-1}$$
$$= \frac{z^{\mu+m-1}}{\Gamma(\mu+m-\lambda)}\sum_{m,n=0}^{\infty}(\alpha)_{m+n}\frac{B^{\{k_l\}}_{p,q}(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)}\frac{B^{\{k_l\}}_{p,q}(\lambda+m,\mu+m-\lambda)}{B(\lambda+m,\mu+m-\lambda)}$$
$$\frac{(x)^n}{n!}\frac{z^m}{m!}B(\lambda,\mu+m-\lambda)$$
$$= \frac{z^{\mu+m-1}}{\Gamma(\mu+m-\lambda)}F_2^{\{k_l\}}(\alpha,\beta,\lambda+m;\gamma,\mu;x,z;p,q)B(\lambda+m,\mu-\lambda-m)$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu+m)}z^{\mu+m-1}F_2^{\{k_l\}}(\alpha,\beta,\lambda;\gamma,\mu+m;x,z;p,q).$$

We can easily obtain the following transformation using [4, p. 18, Theorem 3.6], we get

$$D^{\lambda-\mu-m,p,q}_{z,(\{k_l\}_{l\in N_0})}\left\{z^{\lambda-1}(1-z)^{-\alpha}{}_2F_1^{\{k_l\}}(\alpha,\beta;\gamma;\frac{x}{1-z};p,q\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu+m)}z^{\mu+m-1}\frac{(\mu)_m}{(\mu-\lambda)_m}\sum_{k=0}^{m}(-1)^{km}C_k\frac{(\lambda)_k}{(\mu)_k}$$

$$F_2^{\{k_l\}}(\alpha,\beta,\lambda+k;\mu+k;x,z;p,q)$$

Hence the proof.

References

[1] L.C. Andrews, Special Functions for Engineers and Applied Mathematicians, New York: McMillan, 1985.

[2] M.A. Chaudhry and S.M. Zubair, Generalized incomplete gamma functions with applications, J. Comput. Appl. Math. 55 (1994), 99 - 124.

[3] M.A. Chaudhry, A. Qadir, M. Rafique and S.M. Zubair, Extension of Euler's beta function, J. Comput. Appl. Math., 78 (1997), 19 - 32.

[4] L. Minjie, A Class of Extended Hypergeometric Functions and Its Applications, [math.CA], 2013.

[5] E. Ozergin, M.A. Ozarslan and A. Altin, Extension of gamma, beta and hypergeometric functions, J.Comput. Appl. Math., 235 (2011), 4601 - 4610.

[6] M.A. Ozarslan and E. Ozergin, Some generating relations for extended hypergeometric function via generalized fractional derivative operator, Math. Comput. Model. 52 (2010), 1825 - 1833.

[7] H.M. Srivastava, R.K. Parmar and P. Chopra, A Class of Extended Fractional Derivative Operators and Associated Generating Relations Involving Hypergeometric Functions, Axioms I (2012), 238 - 258.