Steck Determinants and Parking Functions

Chanchal Kumar

IISER Mohali, Knowledge City,
Sector 81, SAS Nagar, Punjab, India.

chanchal@iisermohali.ac.in

Abstract

For an arbitrary $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_i > 0$ for all i, consider the n-dimensional polytope

$$\Pi_n(\mathbf{x}) = \{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_i \geq 0 \text{ and } \sum_{j=1}^{i} y_j \leq \sum_{j=1}^{i} x_j \text{ for all } 1 \leq i \leq n\}.$$

The volume $V_n(\mathbf{x}) = \mathrm{vol}(\Pi_n(\mathbf{x}))$ of the polytope $\Pi_n(\mathbf{x})$ has many combinatorial interpretations. In fact, $V_n(\mathbf{x})$ is a polynomial expression in $x_1, \ldots, x_n$, given by a Steck determinant [2]. If $x_1, \ldots, x_n$ are all positive integers, then $V_n(\mathbf{x})$ can also be expressed in terms of the number of λ-parking functions for $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i = x_1 + \ldots + x_{n-i+1}$ for all $1 \leq i \leq n$. For specific values of $\mathbf{x} = (x_1, \ldots, x_n)$, many formula for the volume $V_n(\mathbf{x})$ have been derived using various techniques. In this paper, these expressions for $V_n(\mathbf{x})$ are deduced simply by evaluating the corresponding Steck determinants.

Subject Classification: [2010] 05E40, 13D02
Keywords: Steck determinants; λ-parking functions.

1 Introduction

For $n \geq 1$, Pitman and Stanley [2] considered the n-dimensional polytope

$$\Pi_n(\mathbf{x}) = \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : y_i \geq 0 \text{ and } \sum_{j=1}^{i} y_j \leq \sum_{j=1}^{i} x_j \text{ for all } 1 \leq i \leq n \right\}.$$

for arbitrary $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_i > 0$ for all $1 \leq i \leq n$, and showed that the n-dimensional volume $V_n(\mathbf{x}) = \mathrm{vol}(\Pi_n(\mathbf{x}))$ of the polytope $\Pi_n(\mathbf{x})$ is a homogeneous polynomial in $x_1, \ldots, x_n$ of degree n. More precisely, they showed that

$$V_n(\mathbf{x}) = \sum_{\mathbf{a} \in K_n} \prod_{i=1}^{n} \frac{{x_i}^{a_i}}{a_i!} = \frac{1}{n!} \sum_{\mathbf{a} \in K_n} \binom{n}{a_1, \ldots, a_n} x_1^{a_1} \cdots x_n^{a_n},$$

where K_n consists of all n-tuples $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ of non-negative integers such that $\sum_{i=1}^{j} a_i \geq j$ for all $1 \leq j \leq n-1$ and $|\mathbf{a}| = \sum_{i=1}^{n} a_i = n$. It can be easily shown that the cardinality $|K_n|$ of K_n is the n^{th} Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Another interpretation of $V_n(\mathbf{x})$ is obtained by considering uniform order statistics. The cumulative distribution function of the random vector of order statistics of n independent random variables with uniform distribution on an interval is given by a determinant formula of Steck [4]. We have,

$$V_n(\mathbf{x}) = \det \Lambda \left(\sum_{j=1}^{n} x_j, \sum_{j=1}^{n-1} x_j, \ldots, x_1 + x_2, x_1 \right), \tag{1.1}$$

where the matrix $\Lambda(\lambda_1, \ldots, \lambda_n) = [m_{ij}]_{n \times n}$ is given by $m_{ij} = \begin{cases} \frac{\lambda_{n-i+1}^{j-i+1}}{(j-i+1)!} & i \leq j+1, \\ 0 & i > j+1, \end{cases}$ for $\lambda = (\lambda_1, \ldots, \lambda_n)$. The determinant $\det \Lambda(\lambda_1, \ldots, \lambda_n)$ of the matrix $\Lambda(\lambda_1, \ldots, \lambda_n)$ is called a *Steck determinant.*

Pitman and Stanley [2] obtained the volume $V_n(\mathbf{x}) = V_n(x_1, \ldots, x_n)$ for some specific values of $\mathbf{x} = (x_1, \ldots, x_n)$ using some well-known results in the theory of empirical distributions. They proved the following results in section 2 of [2].

Theorem 1.1. *For $a, b \geq 0$,*

$$V_n(a, b, \ldots, b) = \frac{a(a+nb)^{n-1}}{n!}.$$

Theorem 1.2. *For $n \geq 3$ and $a, b, c \geq 0$,*

$$V_n(a, \overbrace{b, \ldots, b}^{n-2 \text{ places}}, c) = \frac{a(a+nb)^{n-1} + na(c-b)(a+(n-1)b)^{n-2}}{n!}.$$

Theorem 1.3. *For $n \geq 3, 1 \leq m \leq n-2$ and $a, b, c \geq 0$,*

$$V_n(a, \overbrace{b, \ldots, b}^{n-m-1 \text{ places}}, c, \overbrace{0, \ldots, 0}^{m-1 \text{ places}}) = a \sum_{j=0}^{m} \frac{(c-(m+1-j)b)^j \, (a+(n-j)b)^{n-j-1}}{j!(n-j)!}.$$

We now proceed to show a relationship between $V_n(\mathbf{x})$ and the number of λ-parking functions of length n. A sequence $\mathbf{p} = (p_1, \ldots, p_n)$ of non-negative integers is called an *(ordinary) parking function of length* n if the nondecreasing rearrangement $\mathbf{q} = (q_1, \ldots, q_n)$ of $\mathbf{p}$ satisfies $q_i < i$ for all $1 \leq i \leq n$. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ with $\lambda_1 \geq \ldots \geq \lambda_n$. A sequence $\mathbf{p} = (p_1, \ldots, p_n)$ of non-negative integers is called a *λ-parking function of length*

n if the nondecreasing rearrangement $\mathbf{q} = (q_1, \ldots, q_n)$ of $\mathbf{p}$ satisfies $q_i < \lambda_{n-i+1}$ for all $1 \le i \le n$. Clearly, (ordinary) parking functions of length n are λ-parking functions for $\lambda = (n, n-1, \ldots, 2, 1)$. Let $\mathbf{P}_n(\lambda)$ be the set of λ-parking functions of length n. Then $|\mathbf{P}_n(\lambda)| = n! \det \Lambda(\lambda_1, \ldots, \lambda_n)$ (see [2], Theorem 11). The number of (ordinary) parking function of length n equals $(n+1)^{n-1}$. If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ is in arithmetic progression, that is, $\lambda_i = a + (n-i)b$ for $1 \le i \le n$, then in view of Theorem 1.1 and formula 1.1, the number of λ-parking functions equals $a(a+nb)^{n-1}$. For an (oriented) graph G on the vertex set $\{0, 1, \ldots, n\}$ rooted at 0, Postnikov and Shapiro [3] defined a notion of G-parking functions. They showed that the number of G-parking functions is same as the number of rooted oriented-spanning trees of G. Hence, another proof of Theorem 1.1 is given in Example 4.2 of [3]. A purely combinatorial proof of Theorem 1.1 is given by C. H. Yan [5]. Our aim in this paper is to demonstrate that these three results can be easily deduced from the properties of Steck determinants.

2 Properties of Steck Determinants

Let $n \ge 1$ be a positive integer. Suppose $\lambda_n = x$ is a variable and $\lambda_i = x + (n-i)b$ for $1 \le i \le n$ with $b \ge 0$. Consider the Steck determinant

$$\det \Lambda(\lambda_1, \ldots, \lambda_n) = \det \begin{bmatrix} \frac{x}{1} & \frac{x^2}{2!} & \frac{x^3}{3!} & \cdots & \frac{x^{n-1}}{(n-1)!} & \frac{x^n}{n!} \\ 1 & \frac{x+b}{1} & \frac{(x+b)^2}{2!} & \cdots & \frac{(x+b)^{n-2}}{(n-2)!} & \frac{(x+b)^{n-1}}{(n-1)!} \\ 0 & 1 & \frac{x+2b}{1} & \cdots & \frac{(x+2b)^{n-3}}{(n-3)!} & \frac{(x+2b)^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{x+(n-2)b}{1} & \frac{(x+(n-2)b)^2}{2!} \\ 0 & 0 & 0 & \cdots & 1 & \frac{x+(n-1)b}{1} \end{bmatrix}.$$

Let $f_n(x) = \det \Lambda(\lambda_1, \ldots, \lambda_n)$. Clearly, $f_n(x)$ is a polynomial in x (of degree at most n). Let C_i be the i^{th} column of $\Lambda(\lambda_1, \ldots, \lambda_n)$. Then $f_n(x) = \det[C_1, \ldots, C_n]$.

Proposition 2.1. *The derivative $f_n'(x)$ of the polynomial $f_n(x)$ is given by $f_n'(x) = f_{n-1}(x+b)$ for $n > 1$. Also, $f_n(0) = 0$.*

Proof. Since $f_n(x) = \det[C_1, \ldots, C_n]$, the derivative $f_n'(x)$ of $f_n(x)$ with respect to x is given by

$$f_n'(x) = \sum_{i=1}^{n} \det[C_1, \ldots, C_i', \ldots, C_n],$$

where C_i' is the derivative of C_i with respect to x. For $i \ge 2$, $C_i' = C_{i-1}$. Therefore, $\det[C_1, \ldots, C_i', \ldots, C_n] = 0$ for $i \ge 2$. Hence $f_n'(x) = \det[C_1', C_2, \ldots, C_n]$. As $C_1' = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e_1}$, on expanding the $\det[C_1', C_2, \ldots, C_n]$ along the first column, we get

$f_n'(x) = \det \Lambda(\lambda_1, \ldots, \lambda_{n-1})$, where $\Lambda(\lambda_1, \ldots, \lambda_{n-1})$ is the matrix obtained from the $n \times n$ matrix $\Lambda(\lambda_1, \ldots, \lambda_n)$ by deleting the first row and the first column. Clearly, $f_n'(x) = f_{n-1}(x+b)$. The second part is obvious. □

Proposition 2.2.

$$f_n(x) = \frac{x(x+nb)^{n-1}}{n!}.$$

Proof. Proof is by induction on n. Clearly, $f_1(x) = x$ and $f_2(x) = \frac{x(x+2b)}{2!}$. Assume that $f_m(x) = \frac{x(x+mb)^{m-1}}{m!}$ is valid for $1 \leq m < n$. Then

$$f_n'(x) = f_{n-1}(x+b) = \frac{(x+b)(x+nb)^{n-2}}{(n-1)!}.$$

On integration, we get

$$f_n(x) = \frac{(x+b)(x+nb)^{n-1}}{(n-1)(n-1)!} - \frac{(x+nb)^n}{(n-1)n(n-1)!} + C,$$

where C is a constant of integration. As $f_n(0) = 0$, we get $C = 0$. Hence, we obtain $f_n(x) = \frac{x(x+nb)^{n-1}}{n!}$, as desired. □

We now prove theorems stated in the introduction.

Proof of **Theorem 1.1**: Since $V_n(a, b, \ldots, b) = f_n(a)$, Theorem 1.1 trivially follows from Proposition 2.2. □

Proof of **Theorem 1.2**: Let $\lambda_i = a + (n-i)b$ for $1 \leq i \leq n$. Then

$$V_n(a, b, \ldots, b, c) = \det \Lambda(\lambda_1 + (c-b), \lambda_2, \ldots, \lambda_n) = \det[\tilde{C_1}, \ldots, \tilde{C_n}].$$

If C_i is the i^{th} column of the matrix $\Lambda(\lambda_1, \ldots, \lambda_n)$, then we see that $\tilde{C_i} = C_i$ for $1 \leq i \leq n-1$ and $\tilde{C_n} = C_n + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ c-b \end{bmatrix} = C_n + (c-b)\mathbf{e_n}$. As determinant is linear on columns, we see that $V_n(a, b, \ldots, b, c)$ equals

$$\det \Lambda(\lambda_1, \ldots, \lambda_n) + (c-b) \det \Lambda(\lambda_2, \ldots, \lambda_n) = f_n(a) + (c-b) f_{n-1}(a).$$

Thus Theorem 1.2 follows from Proposition 2.2. □

Proof of **Theorem 1.3**: Proof is by induction on $n+m$. If $m = 1$, then it follows from Theorem 1.2. Thus we assume that $m > 1$. Let $\lambda_i = x + (n-i)b$ for $m+1 \leq i \leq n$

and $\lambda_j = x + (n-m-1)b + c$ for $1 \le j \le m$. Let $g_n(x) = \det \Lambda(\lambda_1, \dots, \lambda_n)$. Then as in Proposition 2.1, we see that the derivative $g_n'(x) = g_{n-1}(x+b)$. Since $g_{n-1}(x+b) = \det \Lambda(\lambda_1, \dots, \lambda_{n-1})$ is a Steck determinant of order $n-1$, by induction assumption, we have

$$g_{n-1}(x) = \frac{x}{(n-1)!} \sum_{j=0}^{m} \binom{n-1}{j} (c-(m+1-j)b)^j (x+(n-1-j)b)^{n-j-2}.$$

Thus,

$$g_n'(x) = \frac{x+b}{(n-1)!} \sum_{j=0}^{m} \binom{n-1}{j} (c-(m+1-j)b)^j (x+(n-j)b)^{n-j-2}.$$

On integration, we obtain

$$\begin{aligned} g_n(x) &= \frac{x+b}{(n-1)!} \sum_{j=0}^{m} \binom{n-1}{j} (c-(m+1-j)b)^j \frac{(x+(n-1-j)b)^{n-j-1}}{n-j-1} \\ &\quad -\frac{1}{(n-1)!} \sum_{j=0}^{m} \binom{n-1}{j} (c-(m+1-j)b)^j \frac{(x+(n-1-j)b)^{n-j}}{(n-j)(n-j-1)} + C, \end{aligned}$$

where C is a constant of integration. As $g_n(0) = 0$, we get $C = 0$. Hence,

$$g_n(x) = \frac{x}{n!} \sum_{j=0}^{m} \binom{n}{j} (c-(m+1-j)b)^j (x+(n-j)b)^{n-j-1}.$$

Now Theorem 1.3 follows by substituting $x = a$ in $g_n(x)$. □

By an enumeration of λ-parking functions of length n for $\lambda = (a, \dots, a, b)$, it is shown that

$$\det \Lambda(a, \dots, a, b) = \frac{a^n - (a-b)^n}{n!}$$

for positive integers $b < a$ (see [1], Corollary 1.1). Let $h_n(x) = \det \Lambda(\lambda_1, \dots, \lambda_n)$, where $\lambda_n = x$ and $\lambda_i = x + (a-b)$ for $1 \le i \le n-1$. Then we see that derivative $h_n'(x) = \det(\lambda_1, \dots, \lambda_{n-1}) = \frac{(x+(a-b))^{n-1}}{(n-1)!}$. Thus $h_n(x) = \frac{(x+(a-b))^n}{n!} + C$, where C is a constant of integration. As $h_n(0) = 0$, we must have $C = -\frac{(a-b)^n}{n!}$. Thus

$$h_n(x) = \frac{(x+(a-b))^n}{n!} - \frac{(a-b)^n}{n!}.$$

On substituting $x = b$ in $h_n(x)$, we get $\det \Lambda(a, \dots, a, b) = h_n(b)$ as desired.

References

[1] Kumar A. and Kumar C., *Some integer sequences and standard monomials*, GANITA Vol.**67**(1), 2017, 33-40.

[2] Pitman J. and Stanley R., *A polytope related to empirical distributions, plane trees, parking functions, and the associahedron*, Discrete and Computational Geometry **27**(2002), 603-634.

[3] Postnikov A. and Shapiro B., *Trees, parking functions, syzygies, and deformation of Monomial ideals*, Trans. Amer. Math. Soc. **356** (2004), 3109-3142.

[4] Steck G. P., *Rectangle probabilities for uniform order statistics and the probability that the empirical distribution function lies between two distribution functions*, Ann. Math. Statist. **42** (1971), 1-11.

[5] Yan C. H., *On the enumeration of generalized parking functions*, Proceedings of the 31-st Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Rotan, FL, 2000), *Congressus Numerantium* **147** (2000), 201-209.