A Note on Frames in Fréchet Spaces

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Abstract

Motivated by the perturbation theory for Hilbert frames in Hilbert spaces, we present some perturbation results for Fréchet frames in Fréchet spaces. A necessary and sufficient condition for frames in Fréchet spaces in terms of operators on the underlying space is given.

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1 Introduction and Preliminaries

Let $\mathcal{H}$ be a complex separable Hilbert space with an inner product $\langle .,. \rangle$. A countable sequence $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H}$ is a *frame* (or *Hilbert frame*) for $\mathcal{H}$ if there exist positive scalars $A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

The scalars A and B are called *lower* and *upper frame bounds*, respectively. The sequence $\{f_k\}_{k=1}^{\infty}$ is called a *Bessel sequence* with *Bessel bound* B if the upper inequality in (1.1) holds for all $f \in \mathcal{H}$. Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$. The operator $T : \ell^2(\mathbb{N}) \to \mathcal{H}$ given by

$$T : \{c_k\}_{k=1}^{\infty} \longrightarrow \sum_{k=1}^{\infty} c_k f_k, \quad \{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N}).$$

is called *pre-frame operator*(or *synthesis operator*) of the frame $\{f_k\}_{k=1}^{\infty}$. The adjoint of T is the operator $T^* : \mathcal{H} \to \ell^2(\mathbb{N})$ given by $T^* f = \{\langle f, f_k \rangle\}_{k=1}^{\infty}, \quad f \in \mathcal{H}$ and is called the *analysis operator* of $\{f_k\}_{k=1}^{\infty}$. The frame operator $S = TT^* : \mathcal{H} \to \mathcal{H}$ is a bounded, linear and invertible operator on $\mathcal{H}$. This gives the *reconstruction* of each vector $f \in \mathcal{H}$,

$$f = SS^{-1} f = \sum_{k=1}^{\infty} \langle S^{-1} f, f_k \rangle f_k.$$

This shows that a frame for $\mathcal{H}$ allows each vector in $\mathcal{H}$ to be written as a linear combination of the elements in the frame, but the linear independence between the frame elements is not required. The basic theory of frames can be found in the books by Casazza and Kutyniok [1], Christensen [5] and beautiful research tutorials by Casazza [2] and Casazza and Lynch [3].

Gröchenig, generalized Hilbert frames to Banach spaces in [12]. Before the concept of Banach frames was formalized, it appeared in the foundational work of Feichtinger and Gröchenig [17, 18] related to the atomic decompositions.

Definition 1.1. *[12] Let $\mathcal{X}$ be a Banach space and let $\mathcal{X}_d$ be an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$. Let $\{f_k^*\} \subset \mathcal{X}^*$ and $\mathcal{S} : \mathcal{X}_d \to \mathcal{X}$ be given. The pair $(\{f_k^*\}, \mathcal{S})$ is called a* Banach frame *for $\mathcal{X}$ with respect to $\mathcal{X}_d$ if*

1. *$\{f_k^*(f)\} \in \mathcal{X}_d$, for all $f \in \mathcal{X}$.*

2. *There exist positive constants A and B with $0 < A \leq B < \infty$ such that*

$$A\|f\|_{\mathcal{X}} \leq \|\{f_k^*(f)\}\|_{\mathcal{X}_d} \leq B\|f\|_{\mathcal{X}}, \quad \text{for all } f \in \mathcal{X}. \tag{1.2}$$

3. *$\mathcal{S}$ is a bounded linear operator such that*

$$\mathcal{S}(\{f_k^*(f)\}) = f, \quad \text{for all } f \in \mathcal{X}.$$

Regarding the existence of Banach frames in Banach spaces Casazza et al. proved in [4] that every separable Banach space has Banach frame. Jain et al. improved this result in [15] and proved that "if $\mathcal{X}$ is a Banach space, not necessarily separable, such that $\mathcal{X}^*$ weak* separable, then $\mathcal{X}$ has normalized tight and exact Banach frame". It is proved in [15] that: "if a Banach space has a Banach frame then, it also has normalized tight as well a normalized tight exact Banach frame". A sufficient condition for exact Banach frames can be found in [15]. It is well known that the sum of two Banach frame need not be a Banach frame. Jain et al. proved sufficient condition for finite sum of Banach frames in [15]. Some results which connects frames in Banach spaces and eigen-value of a boundary value problem can be found in [21]. For different type of framing model in Banach spaces, we refer to [4].

It is well known that the perturbation theory is useful in both pure and applied mathematics. For example, applications of perturbation in frame theory, see [1, 2, 5, 8, 16, 17]. In this paper, we discuss mainly perturbation of Fréchet frames. A necessary and sufficient condition for Fréchet frames in terms of operators on the underlying space is given.

1.1 Background on frames in Fréchet spaces

The notion of frames in Fréchet spaces introduced in [18, 19, 20]. Let $\{Y_s, |.|_s\}_{s\in\mathbb{N}_o}$ be a sequence of separable Banach spaces such that

$$\{0\} \neq \bigcap_{n\in\mathbb{N}_o} Y_s \subseteq .... \subseteq Y_2 \subseteq Y_1 \subseteq Y_o \tag{1.3}$$

$$|.|_o \leq |.|_1 \leq |.|_2 \leq .... \tag{1.4}$$

$$Y_F = \bigcap_{n\in\mathbb{N}_o} Y_s \text{ is dense in } Y_s, \quad s \in \mathbb{N}_o. \tag{1.5}$$

Then, Y_F is a Fréchet space with the sequence of norms $|.|_s$ $(s \in \mathbb{N}_o)$.

Definition 1.2. *Let X_F be a Fréchet space determined by the separable Banach spaces X_s $(s \in \mathbb{N}_o)$, satisfying $(1.2) - (1.4)$, and let Θ_F be a Fréchet space determined by the BK-spaces Θ_s $(s \in \mathbb{N}_o)$, satisfying $(1.2) - (1.4)$. A sequence $\{g_i\}_{i=1}^{\infty} \subset X_F^*$ is called a General pre-Frechet frame (in short, General pre-F-frame) for X_F with respect to Θ_F if there exist sequences $\{s_k\}_{k\in\mathbb{N}_o}$, $\{\widetilde{s}_k\}_{k\in\mathbb{N}_o} \subset \mathbb{N}_o$, which increase to ∞ with the property $s_k \leq \widetilde{s}_k,\ k \in \mathbb{N}_o$ and there exist constants $B_k, A_k > 0, k \in \mathbb{N}_o$, satisfying*

(i) $\{g_i(f)\}_{i=1}^{\infty} \in \Theta_F$.
(ii) $A_k\|f\|_{s_k} \leq |||\{g_i(f)\}_{i=1}^{\infty}|||_k \leq B_k\|f\|_{\widetilde{s}_k},\ f \in X_F,\ k \in \mathbb{N}_o.$

Remark 1.1. *Let $\{f_i\}_{i=1}^{\infty} \subset X_F^*$ be a General pre-F-frame for X_F with respect to Θ_F. One can see that every subsequence $\{X_{p_k}\}_{k=1}^{\infty}$ of $\{X_s\}_{s=1}^{\infty}$ and every subsequence $\{\Theta_{q_k}\}_{k=1}^{\infty}$ of $\{\Theta_s\}_{s=1}^{\infty}$ have suitable subsubsequences so that conditions in Definition 1.2 holds with the same $\{f_i\}_{i=1}^{\infty}$ and corresponding subsubsequences of norms with corresponding constants. Furthermore, note that Definition 1.2 allows to combine two General pre-F-frames in appropriate way to obtain a General pre-F-frame. Let X_F and Θ_F be given as in Definition 1.2. In addition, assume that all the spaces Θ_s, $s \in \mathbb{N}_o$, have the following two (natural) properties:*

1. *Inserting zeros to a sequence in Θ_s (at any places) leads to a sequence which is also in Θ_s and has the same Θ_s-norm.*
2. *If $\{c_i\}_{i=1}^{\infty} \in \Theta_s$, then every subsequence $\{c_{n_k}\}_{k=1}^{\infty}$ also belongs to Θ_s and $|||\{c_{n_k}\}_{k=1}^{\infty}|||_s \leq |||\{c_i\}_{i=1}^{\infty}|||_s$.*

Regarding the existence of General pre-F-frames, we refer to [18, 19, 20].

2 Main Results

We start with the following result, where ith term of a General pre-F-frames is perturbed by $(i+1)$th term.

Theorem 2.1. *Let $\{g_i\}_{i=1}^{\infty} \subset X_F^*$ be a General pre-F-frame for X_F with respect to Θ_F. If there exists a $z_o \in X_F$ such that $g_i(z_o) = 1$ for all $i \in \mathbb{N}$. Then, $\{g_i - g_{i+1}\}_{i=1}^{\infty}$ is not General pre-F-frame for X_F with respect to Θ_F.*

Proof. By hypothesis, $(g_i)z_o = 1$ for all $i \in \mathbb{N}$. This gives $(g_i - g_{i+1})(z_o) = 0$ for all $i \in \mathbb{N}$. Assume that $\{g_i - g_{i+1}\}_{i=1}^{\infty}$ is a General pre-F-frame for X_F with respect to Θ_F. Then, we can find positive numbers A_k^{Φ}, B_k^{Φ} such that $\{(g_i - g_{i+1})(f)\}_{i=1}^{\infty} \in \Theta_F$ and

$$A_k^{\Phi}\|f\|_{s_k} \le |||\{(g_i - g_{i+1})(f)\}_{i=1}^{\infty}|||_k \le B_k\|f\|_{\widetilde{s}_k}, \ f \in X_F, \ k \in \mathbb{N}_o. \tag{2.1}$$

By using lower frame inequality in (2.1), we obtain $z_o = 0$, a contradiction. □

The following proposition provides sufficient conditions for perturbation of pre-F-frames.

Proposition 2.1. *Let $\{f_i\}_{i=1}^{\infty} \subset X_F^*$ be a pre-F-frame for X_F with respect to Θ_F. Let $\{g_i\}_{i=1}^{\infty} \subset X_F^*$ be such that $\{g_i(f)\}_{i=1}^{\infty} \in \Theta_F$ $(f \in X_F)$. Assume that*

$$|||\{g_i(f)\}_{i=1}^{\infty}|||_k \le M_k\|f\|_{\widetilde{s}_k}, \ f \in X_F, \ k \in \mathbb{N}_o.$$

Then, $\{f_i + \lambda g_i\}_{i=1}^{\infty}$ is pre-F-frame for X_F with respect to Θ_F, provided $|\lambda| < \frac{A_k}{M_k}$ $(k \in \mathbb{N}_o)$

Proof. For all $f \in X_F, \ k \in \mathbb{N}_o$, we have

$$\begin{aligned} |||\{(f_i + \lambda g_i)(f)\}_{i=1}^{\infty}|||_k &\le |||\{f_i(f)\}_{i=1}^{\infty}|||_k + |||\{\lambda g_i(f)\}_{i=1}^{\infty}|||_k \\ &\le (B_k + |\lambda|M_k)\|f\|_{\widetilde{s}_k}. \end{aligned}$$

Similarly

$$\begin{aligned} |||\{(f_i + \lambda g_i)(f)\}_{i=1}^{\infty}|||_k &\ge |||\{f_i(f)\}_{i=1}^{\infty}|||_k - |||\{\lambda g_i(f)\}_{i=1}^{\infty}|||_k \\ &\ge (A_k - |\lambda|M_k)\|f\|_{\widetilde{s}_k} \\ &= \gamma_k\|f\|_{\widetilde{s}_k}, \ f \in X_F, \ k \in \mathbb{N}_o \quad (\gamma_k = (A_k - |\lambda|M_k). \end{aligned}$$

The proof is complete. □

The next theorem provides necessary and sufficient conditions for General pre-F-frame for X_F in terms of operators.

Theorem 2.2. *Let $\{f_i\}_{i=1}^{\infty} \subset X_F^*$ be a General pre-F-frame for X_F with respect to Θ_F and let $\mathcal{Q} \in \mathcal{B}(X_F)$. Assume that $\mathcal{W} \in \mathcal{B}(\Theta_F)$ is such that for all $x \in X_F$, $\mathcal{W} : \{f_i(x)\} \to \{\mathcal{Q}f_i(x)\}$. Then, there exists a bounded linear operator $\widehat{\Theta}$ such that $\{\mathcal{Q}(f_n)\}$ is a General pre-F-frame for X_F with respect to Θ_F if and only if*

$$\|\mathcal{W}(\{f_n(x)\})\|_{\Theta_F} \ge c\|\mathcal{J}(\{\mathcal{Q}f_n(x))\})\|_{\mathcal{Z}_d} \text{ for all } x \in X_F, \tag{2.2}$$

where c is a positive constant and $\mathcal{J} \in \mathcal{B}(\Theta_F)$ is an operator such that for all $x \in X_F$

$$\mathcal{J} : \{\mathcal{Q}f_n(x)\} \to \{f_n(x)\}.$$

Proof. Suppose first that $\{\mathcal{Q}(f_n)\}$ is a General pre-F-frame for X_F with respect to Θ_F Let $A_{\mathcal{Q}}$ and $B_{\mathcal{Q}}$ be a choice of retro frame bounds for $\mathcal{F}_{\mathcal{Q}}$ and let $\mathcal{P} : X_F \to \Theta_F$ be the analysis operator associated with $\{\mathcal{Q}(f_n)\}$ which is given by

$$\mathcal{P} : x \to \{f_n(x)\}, x \in X_F.$$

Let $\Theta = \mathcal{P}^{-1} : R(\mathcal{P}) \to X_F$. Note that Θ is bounded linear operator.

Choose $\mathcal{J} = \mathcal{P}\widehat{\Theta}$ and $c = \frac{A_{\mathcal{Q}}}{\|\mathcal{P}\|} > 0$.

Then

$$\begin{aligned}\|\mathcal{W}(\{f_n(x)\})\|_{\Theta_F} &= \|\{(\mathcal{Q}f_n)(x)\}\|_{\Theta_F}\\ &\geq A_{\mathcal{Q}}\|x\|\\ &\geq c\ \|\{f_n(x)\}\|_{\Theta_F}\\ &= c\ \|\mathcal{J}(\{\mathcal{Q}f_n(x)\})\|_{\Theta_F} \text{ for all } x \in X_F.\end{aligned}$$

The forward part is proved.

For the reverse part, suppose that (2.2) is satisfied.

We compute

$$\begin{aligned}\|\{\mathcal{Q}f_n(x)\}\|_{\Theta_F} &= \|\mathcal{W}(\{f_n(x)\})\|_{\Theta_F}\\ &\leq \|\mathcal{W}\|\|\{f_n(x)\}\|_{\Theta_F}\\ &\leq \|\mathcal{W}\|\|\mathcal{P}\|\|x\| \text{ for all } x \in X_F.\end{aligned} \tag{2.3}$$

By using (8), we have

$$\begin{aligned}cA\|x\| &\leq c\|\{f_n(x)\}\|_{\Theta_F} \quad \text{(where } A \text{ is lower retro frame bound of } \mathcal{F})\\ &= c\|\mathcal{J}(\{\mathcal{Q}f_n(x)\})\|_{\Theta_F}\\ &\leq \|\mathcal{W}(\{f_n(x)\})\|_{\Theta_F}\ (= \|\{\mathcal{Q}f_n(x)\}\|_{\Theta_F}).\end{aligned} \tag{2.4}$$

Set $A_k = cA$ and $B_k = \|\mathcal{W}\|\|\mathcal{P}\|\ k \in \mathbb{N}_o$.

Then, by using (2.3) and (2.4), we have

$$A_k\|x\|_{s_k} \leq |||\{g_i(x)\}_{i=1}^{\infty}|||_k \leq B_k\|x\|_{\widetilde{s}_k},\ x \in X_F,\ k \in \mathbb{N}_o.$$

□

The following theorem provides a necessary condition for perturbation of General pre-F-frame for X_F in terms of an eigenvalue of a matrix associated with the perturbed sequence.

Theorem 2.3. *Let $\{f_i\}_{i=1}^{\infty} \subset X_F^*$ be a General pre-F-frame for X_F with respect to Θ_F. Let $\{g_k\}_{k=1}^{m} \subset X_F^*$ ($m \in \mathbb{N}$ is fixed) be linearly independent functional and let for each k ($1 \leq k \leq m$) there exists an $x_k \in X_F$ such that $f_i(x_k) = \alpha_k^{(i)}$ for all $i \in \mathbb{N}$. If $\{f_i + \sum_{k=1}^{m} \alpha_k^{(i)} g_k\}$ is a General pre-F-frame for X_F with respect to Θ_F, then -1 is not an eigenvalue of the matrix*

$$\begin{bmatrix} g_1(x_1) & g_2(x_1) & \dots & g_m(x_1)\\ g_1(x_2) & g_2(x_2) & \dots & g_m(x_2)\\ \dots & \dots & \dots & \dots\\ g_1(x_m) & g_2(x_m) & \dots & g_m(x_m)\end{bmatrix}$$

Proof. It is sufficient to prove the result for $m = 2$.
Let, if possible, -1 is an eigenvalue of the matrix

$$\begin{bmatrix} g_1(x_1) & g_2(x_1) \\ g_1(x_2) & g_2(x_2) \end{bmatrix}$$

Then

$$\begin{vmatrix} g_1(x_1)+1 & g_2(x_1) \\ g_1(x_2) & g_2(x_2)+1 \end{vmatrix} = 0.$$

Therefore, there exists scalars α and β not both zero such that

$$\alpha g_1(x_1) + \beta g_1(x_2) = -\alpha$$
$$\alpha g_2(x_1) + \beta g_2(x_2) = -\beta.$$

Choose $z_0 = -\alpha x_1 - \beta x_2$. Then z_0 is a non-zero vector in X_F. Indeed, if $z_0 = 0$, then $g_1(z_0) = 0$ and $g_2(z_0) = 0$. This gives $\alpha = 0$ and $\beta = 0$, which is a contradiction.
We compute

$$\begin{aligned} &\Big(f_i + \sum_{k=1}^{2} \alpha_k^{(i)} g_k\Big) z_0 \\ &= f_i(z_0) + \alpha_1^{(i)} g_1(z_0) + \alpha_2^{(i)} g_2(z_0) \\ &= f_i(-\alpha x_1 - \beta x_2) + \alpha_1^{(i)} g_1(-\alpha x_1 - \beta x_2) + \alpha_2^{(i)} g_2(-\alpha x_1 - \beta x_2) \\ &= -\alpha\alpha_1^{(n)} - \beta\alpha_2^{(n)} + \alpha_1^{(n)}\alpha + \alpha_2^{(n)}\beta \\ &= 0. \end{aligned} \tag{2.5}$$

By using lowing frame inequality of $\{f_i + \sum_{k=1}^{m} \alpha_k^{(i)} g_k\}$ and (2.5), we obtain $z_0 = 0$, a contradiction. This completes the proof. □

To conclude the paper, we give a sufficient condition for the perturbation of General pre-F-frame for X_F.

Proposition 2.2. *Let $\{f_i\}_{i=1}^{\infty} \subset X_F^*$ be a General pre-F-frame for X_F with respect to Θ_F. If there exists a vector $z_0 \in X_F$ such that $f_i(z_0) = \lambda$ for all $i \in \mathbb{N}$, where λ is a non-zero scalar, then there exists a non-zero functional $\phi_{z_0} \in X_F^*$ such that $\{f_i + \phi_{z_0}\}$ is not a General pre-F-frame for X_F with respect to Θ_F.*

Proof. Suppose that $\{f_i + \phi_{z_0}\}_{i=1}^{\infty}$ is a pre-F-frame for X_F with respect to Θ_F. Let $\psi \in X_F^*$ be such that $\psi(z_0) = \alpha$, where α is a non-zero scalar. Choose $\phi_{z_0} = \frac{-\lambda}{\alpha}\psi$. Then, ϕ_{z_0} is a non-zero functional in X_F^*. Next we show that $\{f_i + \phi_{z_0}\}_{i=1}^{\infty}$ is not a pre-F-frame for X_F with respect to Θ_F. Let A_k B_k be pre-F-frame bounds for $\{f_i + \phi_{z_0}\}_{i=1}^{\infty}$.
Then

$$A_k\|f\|_{s_k} \leq |||\{(f_i + \phi_{z_0})(f)\}_{i=1}^{\infty}|||_k \leq B_k\|f\|_{\tilde{s}_k}, \; f \in X_F, \; k \in \mathbb{N}_o. \tag{2.6}$$

But $(f_i + \phi_{z_0})(z_0) = 0$ for all $i \in \mathbb{N}$. Thus, by (2.6) $z_0 = 0$, a contradiction. This completes the proof. □

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