# Hermite Trigonometric Interpolation

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**Abstract**

In this paper, we consider the explicit representation of Hermite interpolation by trigonometric polynomial $R_n(x)$ of order $n$ on the zeros of $\sin mx$ at the points $x_k = \frac{2\pi k}{n}$, where, $k = 0, 1, 2, ......., n-1$ and $n$ is even $(n = 2m)$. We discuss about the convergence for the same.

**Subject Classification: 42A15,42A20.**
**Keywords:** Hermite Interpolation, Trigonometric polynomial, Explicit representation, Estimation, Convergence.

## 1 Introduction

In this paper, we deal with the Hermite Interpolation by the trigonometric polynomial. Let $R_n(x)$ of order n be a unique trigonometric polynomial, when nodes are taken to be $x_k = \frac{2\pi k}{n}$, $k = 0, 1, 2.......n-1$, satisfying the given conditions

$$R_n(x_k) = a_k, \qquad R_n'(x_k) = b_k,$$

where, $a_k$ and $b_k$ are arbitrary real numbers. Here we determine the explicit representation of trigonometric polynomial $R_n(x)$ and discuss its convergence.

In 1960, O.Kiš[6] studied the $(0, 2)$ case. A. Sharma and A.K. Varma[3] generalized the above case by the Trigonometric $(0, m)$ interpolation in 1965, where, $m \geq 2$. Several other mathematicians have done many works with the higher order derivatives of the function. In 1973, A.K. Varma[2] discussed the Hermite-Birkhoff trigonometric interpolation by $(0, 1, 2, M)$ case. On inspired by this, we have taken the case $(0, 1)$.

## 2 Statement of the main theorem:

We are interested in the trigonometric polynomial $R_n(x)$ of suitable order such that

$$R_n(x_k) = a_k, \qquad R_n'(x_k) = b_k, \tag{2.1}$$

where, $x_k = \frac{2\pi k}{n}$ , $k = 0, 1, 2.......n-1$. This is called the case of Hermite Trigonometric Interpolation, when $n$ is even $(= 2m)$. We require the trigonometric polynomial $R_n(x)$ to have the form

$$d_0 + \sum_{k=1}^{2m-1} (d_k \cos kx + e_k \sin kx) + d_{2m} \cos 2mx. \tag{2.2}$$

We shall prove the following.

**Theorem 2.1.** *The trigonometric polynomial $R_n(x)$ satisfying (2.1) having form (2.2) is given by*

$$R_n(x) = \sum_{k=0}^{n-1} a_k U(x - x_k) + \sum_{k=0}^{n-1} b_k V(x - x_k) \tag{2.3}$$

*where,*

$$U(x) = \sum_{j=1}^{2m-1} \left(\frac{2}{n} - \frac{2j}{n^2}\right) \cos jx + \frac{1}{n} \tag{2.4}$$

$$V(x) = \frac{2}{n^2} \sum_{j=1}^{2m-1} \sin jx + \frac{1}{n^2} \sin 2mx. \tag{2.5}$$

**Theorem 2.2.** *Let $f(x)$ be a $2\pi$-periodic continuous function with $f(x) \in Lip\alpha, \alpha > 0$, then*

$$R_n(x) = \sum_{k=0}^{n-1} f(x_k) U(x - x_k) + \sum_{k=0}^{n-1} b_k V(x - x_k) \tag{2.6}$$

*converges uniformly to $f(x)$ on every closed finite interval on the x-axis, where, $b_k's$ are arbitrary real numbers, such that*

$$| b_k |= O\left(\frac{n}{\log n}\right), \qquad k = 0, 1, 2.......n-1. \tag{2.7}$$

## 3 Proof of Theorem (2.1):

Here we shall discuss the method of determining $U(x)$ and $V(x)$ which satisfies the given conditions

$$U(x_k) = \begin{cases} 1, & \text{for } k = 0 \\ 0, & \text{for } 1 \leq |k| \leq n-1 \end{cases}, \qquad U'(x_k) = 0 \tag{3.1}$$

$$V(x_k) = 0, \qquad V'(x_k) = \begin{cases} 1, & \text{for } k = 0 \\ 0, & \text{for } 1 \leq |k| \leq n-1 \end{cases} \tag{3.2}$$

respectively. We shall give full method of determining explicit form of $V(x)$. Similarly, one can obtain the explicit form of $U(x)$ owing to condition (3.1). For avoiding the repeatation we are omitting the method of determining $U(x)$.

Let $V(x) = g(x)\sin mx$, and $V(x)$ satisfies the given condition (3.2). where, $g(x)$ is a trigonometric polynomial of order $\leq m$.
Now,

$$V'(x) = g'(x)\sin mx + g(x)m\cos mx \tag{3.3}$$

and satisfying the given condition (3.2), which gives

$$m(-1)^k g(x_k) = \begin{cases} 1, & \text{for } k = 0 \\ 0, & \text{for } 1 \leq |k| \leq n-1 \end{cases} \tag{3.4}$$

Hence,

$$mg(x) = \frac{1}{2m}\sin mx \cot\frac{x}{2} + a\sin mx \tag{3.5}$$

where, $a$ is arbitrary constant and

$$\sin mx \cot\frac{x}{2} = 1 + 2\sum_{j=1}^{m-1}\cos jx + \cos mx, \qquad n = 2m$$

is the trigonometric series. Hence,

$$g(x) = \frac{2}{n^2}\{1 + 2\sum_{j=1}^{m-1}\cos jx + \cos mx\} + \frac{a}{n}\sin mx \tag{3.6}$$

Since, $V(x) = g(x)\sin mx$. Then

$$V(x) = \frac{2}{n^2}\sum_{j=1}^{2m-1}\sin jx + \frac{1}{n^2}\sin 2mx + \frac{a}{n}\sin^2 mx \tag{3.7}$$

where, $a$ is arbitrary constant. Since, $V(x - x_k)$ has no term having $\cos 2mx$. Hence $a = 0$.

## 4 Estimates of the fundamental polynomials:

**Lemma 4.1.** *The $V(x)$ defined in (2.5). Then*

$$\sum_{k=0}^{n-1} |V(x-x_k)| \leq C_1 n^{-1} \log n, \tag{4.1}$$

*where, $C_1$ is a numerical constant.*

**Proof.** We have $V(x)$ in (2.5). Then

$$\sum_{k=0}^{n-1}\Big|V(x-x_k)\Big| = \sum_{k=0}^{n-1}\Big|\frac{2}{n^2}\sum_{j=1}^{2m-1}\sin j(x-x_k)+\frac{1}{n^2}\sin 2m(x-x_k)\Big| \tag{4.2}$$

Since,

$$\sum_{k=0}^{n-1}\Big|V(x-x_k)\Big| \leq \sum_{k=0}^{n-1}\Big|\frac{2}{n^2}\sum_{j=1}^{2m-1}\sin j(x-x_k)\Big|+\Big|\frac{1}{n^2}\sin 2m(x-x_k)\Big| \tag{4.3}$$

By using the well known inequality (Jackson [5], page 120)

$$\sum_{k=0}^{n-1}\max_{p}\Big|\sum_{j=o}^{p}\sin j(x-x_k)\Big| \leq 4\log n \tag{4.4}$$

The lemma follows.

**Lemma 4.2.** *The fundamental polynomial $U(x-x_k)$ defined in (2.4). Then*

$$\sum_{k=0}^{n-1} |U(x-x_k)| \leq C_2 \log n, \tag{4.5}$$

*where, $C_2$ is a numerical constant.*

**Proof.** The fundamental polynomial $U(x)$ is given in (2.4).
Then

$$\sum_{k=0}^{n-1}\Big|U(x-x_k)\Big| = \sum_{k=0}^{n-1}\Big|\sum_{j=1}^{2m-1}\left(\frac{2}{n}-\frac{2j}{n^2}\right)\cos j(x-x_k)+\frac{1}{n}\Big| \tag{4.6}$$

$$\sum_{k=0}^{n-1}\Big|U(x-x_k)\Big| \leq \sum_{k=0}^{n-1}\max_{1\leq p\leq 2m-1}\Big|\sum_{j=0}^{p}a_j\cos j(x-x_k)\Big|+\sum_{k=0}^{n-1}\max\Big|\frac{1}{n}\Big| \tag{4.7}$$

where,

$$A_j = \left(\frac{2}{n}-\frac{2j}{n^2}\right),$$

$A_j$ is a decreasing function, when $n \leq j$. So that $\max A_j = \frac{2}{n}$. On using Jackson Theorem ([5], page 120), we get required result.

## 5 Proof of theorem (2.2):

In order to prove the theorem (2.2), we need the following results.
**Remark:** If $f(x)$ is a continuous $2\pi$-periodic function and satisfying $f(x) \in Lip\alpha, 0 < \alpha \leq 1$, Then there exist a trigonometric polynomial $T_n(x)$ of order $\leq n$ satisfying **Jackson condition**[5]

$$| f(x) - T_n(x) |= O(n^{-\alpha}) \tag{5.1}$$

also the condition due to **O.Kiš**[6]

$$| T_n^{(p)}(x) |= O(n^{p-\alpha}), p = 1. \tag{5.2}$$

A trigonometric polynomial $T_n(x)$ of order $n$, which satisfies (5.1), (5.2) By the uniqueness theorem we have

$$\begin{aligned} | f(x) - R_n(x) | &\leq| f(x) - T_n(x) | + \left|\sum_{k=0}^{n-1}(T_n(x_k) - f(x_k))U(x - x_k)\right. \\ &\quad \left. + \sum_{k=0}^{n-1}(T_n'(x_k) - b_k)V(x - x_k)\right| \\ &\leq \sum_{r=1}^{2} S_r + | f(x) - T_n(x) | . \end{aligned} \tag{5.3}$$

By using the (4.5) and (5.1), we have

$$S_1 = O(1), \tag{5.4}$$

as $0 < \alpha \leq 1$.
By using (2.7), (4.1), (5.2), we get

$$S_2 = O(1), \tag{5.5}$$

as $0 < \alpha \leq 1$.

By using (5.1), (5.4) and (5.5) in (5.3),the theorem as follows.

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